2018 LTCC Course on **Aperiodic Order**

Solutions to Worksheet 4

Exercise 1: For $s = 0$, the relation $n = f_0(m) = \frac{m}{\tau} = m\tau - m$ shows that the only integer value for n is obtained for $m = 0$, so the only lattice point on the line $y = f_0(x)$ is $(x, y) = (0, 0)$.
For $s = 1$, the equation becomes $n = f_1(m) = \frac{m}{\tau} + 1 = m\tau - m + 1$, so again $m = 0$ is the only possiblity, thus $(x, y) = (1, 0)$.
Finally, for $s = -\frac{1}{t}$, we get $n = f_{-\frac{1}{\tau}}(m) = \frac{m-1}{\tau}$, so in this case $m = 1$ is the only integer that makes n integer as well, and hence $(x, y) = (0, 1)$.

Exercise 2: As $f_0(x)$ has slope $\frac{1}{\tau}$, an orthogonal line must have slope $-\tau$. Hence the line $y = g(x)$ through a lattice point $(m, n) \in \mathbb{Z}^2$ has the form $y = -\tau x + b$ with $n = -\tau m + b$, which implies that $b = n + m\tau$, so $g(x) = -\tau x + n + m\tau = \tau(m - x) + n$.
The intersection point (x, y) of the two lines is given by

$$y = f_0(x) = \frac{x}{\tau} = x(\tau - 1) = g(x) = \tau(m - x) + n$$

which gives $x = (n + m\tau)/(2\tau - 1)$ and $y = (n + m\tau)/(\tau + 2)$.
The distance d of (x, y) from the origin is thus given by

$$\begin{aligned} d^2 &= \left(\frac{n + m\tau}{2\tau - 1}\right)^2 + \left(\frac{n + m\tau}{\tau + 2}\right)^2 = (n + m\tau)^2 \left(\frac{1}{4\tau + 4 - 4\tau + 1} + \frac{1}{\tau + 1 + 4\tau + 4}\right) \\ &= (n + m\tau)^2 \left(\frac{1}{5} + \frac{1}{5\tau + 5}\right) = (n + m\tau)^2 \, \frac{\tau + 2}{5\tau + 5} \end{aligned}$$

Now, using $\tau\tau' = -1$ and $\tau + \tau' = 1$, we find

$$\frac{\tau + 2}{5\tau + 5} = \frac{(\tau + 2)(\tau' + 2)}{(5\tau + 5)(\tau' + 2)} = \frac{\tau\tau' + 2(\tau + \tau') + 4}{5\tau\tau' + 10\tau + 5\tau' + 10} = \frac{5}{5\tau + 10} = \frac{1}{\tau + 2},$$

so

$$d^2 = \frac{1}{\tau + 2} \left(n + m\tau\right)^2$$

and hence

$$d = \sqrt{\frac{1}{\tau + 2}} \, \big|n + m\tau\big|,$$

which shows that d is in $\mathbb{Z}[\tau]/\sqrt{\tau + 2}$.

Exercise 3: In y direction, the strip has width $1 + \frac{1}{\tau} = 1 + \tau - 1 = \tau$. As $1 < \tau < 2$, there can be at most two lattice points (x, y) within the strip for any given values of x, and there must be at least one. Clearly $(0, 0)$ is in the strip, while $(0, 1)$ is just outside.

The condition $(x, y) \in S$ can be expressed as $x - 1 \leq \tau y < x + \tau$. Explicit checking then gives the list of points as

$$\big\{(0,0),(1,0),(1,1),(2,1),(2,2),(3,2),(4,2),(4,3),(5,3),(5,4),(6,4)\big\}.$$

Exercise 4: Denoting the horizontal and vertical steps by a and b, one obtains the sequence $ababaababa$, which is a legal Fibonacci word because it occurs at the end of $\varrho^6(a) = abaababaabaababaababa$.

Exercise 5: We have a lattice $\mathcal{L} = \mathbb{Z}^2 \subseteq \mathbb{R}^2$, together with projections onto the line $y = f_0(x)$ (which plays the role of physical space) and its orthogonal complement $y = -\tau x$ (the internal space). The window W corresponds to the cross-section of the strip S along the internal space, which is a half-open interval whose endpoints are the projections of the of the lattice points $(1, 0)$ (included) and $(0, 1)$ (excluded). The set of projected points on the line $y = f_0(x)$ is thus a regular model set.