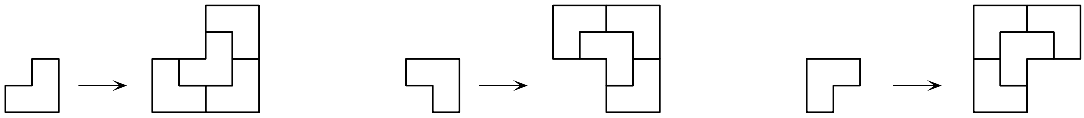
2018 LTCC Course on **Aperiodic Order**

Solutions to Worksheet 3

Exercise 1: The three rotated inflation rules are



Exercise 2: As there are four different prototiles (up to translations), the inflation matric is a 4×4 matrix. By inspection, it is given by

$$M = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

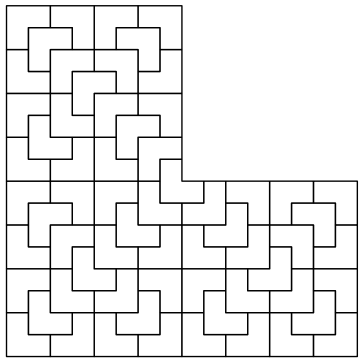
To calculate the number of tiles obtained in 8 inflation steps, we need to compute the eighth power of this matrix,

$$\begin{aligned}
M^2 &= \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} = 2 \begin{pmatrix} 3 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 3 \end{pmatrix} \\
M^4 &= 4 \begin{pmatrix} 3 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 3 \end{pmatrix} = 8 \begin{pmatrix} 9 & 8 & 7 & 8 \\ 8 & 9 & 8 & 7 \\ 7 & 8 & 9 & 8 \\ 8 & 7 & 8 & 9 \end{pmatrix} \\
M^8 &= 64 \begin{pmatrix} 9 & 8 & 7 & 8 \\ 8 & 9 & 8 & 7 \\ 7 & 8 & 9 & 8 \\ 8 & 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 9 & 8 & 7 & 8 \\ 8 & 9 & 8 & 7 \\ 7 & 8 & 9 & 8 \\ 8 & 7 & 8 & 9 \end{pmatrix} = 128 \begin{pmatrix} 129 & 128 & 127 & 128 \\ 128 & 129 & 128 & 127 \\ 127 & 128 & 129 & 128 \\ 128 & 127 & 128 & 129 \end{pmatrix}
\end{aligned}$$

so there are $128 \times 129 = 16\,512$ tiles in the same orientation, $128^2 = 16\,384$ tiles rotated by $\pm\pi/2$, and $128 \times 127 = 16\,256$ tiles rotated by π.

Exercise 3: The matrix M is symmetric, so the left and right eigenvectors are transposes of each other. The leading eigenvalue is 4, with left eigenvector $(1, 1, 1, 1)$ and right eigenvector $(1, 1, 1, 1)^T$. As all entries are the same, all orientations occur with equal frequency in a fixed point tiling.

Exercise 4: The symmetric patch occurs in the threefold inflation of a single chair tile



so it is legal for the inflation rule.

Exercise 5: The initial patch as fourfold rotation symmetry, and the inflation is compatible with rotations by multiples of $\pi/2$ by construction, so the image of the patch under any power of the chair inflation rule also has fourfold rotational symmetry. As the centre patch of the tiling stabilises under inflation (due to the fact that the inflated tile retains the original tile at its position), iteration of the chair inflation rule on the symmetric patch converges to a fixed point tiling with individual fourfold symmetry. In fact, as the patch is also reflection symmetric, and the reflection symmetry is also preserved by the inflation rule, the tiling has individual D_4 symmetry. A patch of the tiling is shown below.