2018 LTCC Course on **Aperiodic Order**

Solutions to Worksheet 2

Exercise 1: We have $\varrho^2(b) = \varrho(a) = abb$, so ϱ is irreducible. Furthermore, $\varrho^2(a) = \varrho(abb) = abbaa$, so both letters occur in the $\varrho^2(a)$ and $\varrho^2(b)$, and hence ϱ is primitive.

Exercise 2: We have $\varrho^2(a) = abbaa$, which contains all combinations of two letters as subwords, so all four seeds $a|a$, $a|b$, $b|a$ and $b|b$ are legal.
Applying ϱ to the four seeds gives

$$\begin{aligned}
a|a &\mapsto abb|abb \mapsto abbaa|abbaa \mapsto abbaaabbabb|abbaaabbabb \mapsto \ldots \\
a|b &\mapsto abb|a \mapsto abbaa|abb \mapsto abbaaabbabb|abbaa \mapsto \ldots \\
b|a &\mapsto a|abb \mapsto abb|abbaa \mapsto abbaa|abbaaabbabb \mapsto \ldots \\
b|b &\mapsto a|a \mapsto abb|abb \mapsto abbaa|abbaa \mapsto \ldots
\end{aligned}$$

Clearly, there is no fixed point under ϱ, as the central two-letter words in all cases alternates between $a|a$ and $b|a$. There are thus two fixed points under ϱ^2, with cores $\ldots abbaa|abbaa \ldots$ and $\ldots bbabb|abbaaa \ldots$.

Exercise 3: The substitution matrix for ϱ is

$$M = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

which is not symmetric. The characteristic polynomial is $(1-x)(-x) - 2 = x^2 - x - 2 = (x-2)(x+1)$, so the eigenvalues as $\lambda = 2$ and -1.
The relation $w^{(n+1)} = w^{(n)}w^{(n-1)}w^{(n-1)}$ is proved by induction. Setting $w^{(0)} = a$, we have $w^{(1)} = \varrho(a) = abb$ and $w^{(2)} = \varrho(abb) = abbaa$. Clearly, these satisfy $w^{(2)} = w^{(1)}w^{(0)}w^{(0)}$. Assuming that $w^{(n+1)} = w^{(n)}w^{(n-1)}w^{(n-1)}$ holds, we find

$$w^{(n+2)} = \varrho(w^{(n+1)}) = \varrho(w^{(n)}w^{(n-1)}w^{(n-1)}) = w^{(n+1)}w^{(n)}w^{(n)},$$

which completes the proof.
To compute the number of letters a and b in $w^{(8)}$ can be computed using the eighth power of the substitution matrix M. This can be calculated via

$$M^2 = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}, \qquad M^4 = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix},$$

and hence

$$M^8 = \begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix}\begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix} = \begin{pmatrix} 171 & 85 \\ 170 & 86 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \mathrm{card}_a(w^{(8)}) \\ \mathrm{card}_a(w^{(8)}) \end{pmatrix} = M^8 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 171 & 85 \\ 170 & 86 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 171 \\ 170 \end{pmatrix},$$

so there are 171 letters a and 170 letters b in $w^{(8)}$.

Exercise 4: The right eigenvector of M with eigenvaues $\lambda = 2$ is $(1,1)^t$, as

$$M \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The corresponding left eigenvector is $(2,1)$, because

$$(2,1)\, M = (2,1) \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = (4,2) = \lambda\, (2,1).$$

The letter frequencies in a fixed point word are encoded by the right eigenvector, so both letters are equally frequent. The corresponding ratio for $w^{(8)}$ was $\mathrm{card}_a(w^{(8)})/\mathrm{card}_b(w^{(8)}) = 171/170$.

Exercise 5: As the left eigenvector shows, the appropriate interval length have lengh ratio $2:1$, so in this case the longer interval has length 2. Indeed $2 \times 2 = 4 = 2 + 1 + 1$ and $2 \times 1 = 2$, so this is consistent. The inflation rule is shown below.

As both letters a and b are equally frequent, the distance between points in the set of left endpoints is equally frequent either 1 and 2, and so the mean distance is $3/2$. The density of the point set is the inverse (number of points per unit length), hence $2/3$.