2018 LTCC Course on **Aperiodic Order**

Solutions to Worksheet 1

Exercise 1:

(a) Clearly $\varLambda_a = 2\,\mathbb{Z}$ is uniformly discrete and relatively dense, so also Delone. All points are equivalent by translation so $\varLambda_a$ is obviously FLC. As $\varLambda_a - \varLambda_a = \varLambda_a$ it is also Meyer.

(b) The set $\varLambda_b = \{n + 1/n \mid n \in \mathbb{Z} \setminus \{0\}\}$ is relatively dense as the maximum distance between points does not exceed 4. It is also unformly discrete as points have minimum distance 1/2. The set $\varLambda_b$ is not FLC because the distances between neighbouring points take on infinitely many values, and neither is it Meyer, because the difference set $\varLambda_b - \varLambda_b$ fails to be uniformly discrete. This can be seen, for instance, by looking at the distances between neighbouring points,

$$(n+1) + \frac{1}{n+1} - \Big(n + \frac{1}{n}\Big) \;=\; 1 + \frac{1}{n+1} - \frac{1}{n} \;=\; 1 - \frac{1}{n(n+1)},$$

so $\varLambda_b - \varLambda_b$ has an accumulation point at 1 (which itself is neither in $\varLambda_b$ nor in $\varLambda_b - \varLambda_b$).

(c) The set $\varLambda_c = -\mathbb{N} \cup \{0\} \cup \sqrt{3}\,\mathbb{N}$ is clearly uniformly discrete and relatively dense, as point have minimum distance 1 and maximum distance $\sqrt{3}$, and hence also Delone. Moreover, it is clearly FLC as there are only finitely many local surrounding for any given finite radius. The difference set

$$\varLambda_c - \varLambda_c \;=\; \{m + n\sqrt{3} \mid m, n \in \mathbb{Z} \text{ and } mn \geq 0\}$$

fails to be uniformly discrete because $\sqrt{3}$ is irrational, so that

$$\inf\big\{|u - v| \,\big|\, u, v \in \varLambda_c - \varLambda_c,\, u \neq v\big\} \;=\; \inf\big\{|k - \ell\sqrt{3}| \,\big|\, k, \ell \in \mathbb{N}\big\} \;=\; 0,$$

so $\varLambda_c$ is not Meyer.

(d) Finally, consider $\varLambda_d = \mathbb{Z} \setminus S$, where S is an arbitrary subset of $2\,\mathbb{Z}$. Since all odd numbers are in $\varLambda_d$, it is clearly relatively dense. The minimum distance is still 1, so it is also uniformly discrete, and hence Meyer. For any given radius, there are only finitely many ways in which points can be present or removed, so it is FLC. The difference set $\varLambda_d - \varLambda_d \subseteq \mathbb{Z}$, and actually equal to $\mathbb{Z}$ unless all odd points have been removed, in which case it is $2\,\mathbb{Z}$. So it is always uniformly discrete, and hence $\varLambda_d$ is Meyer.

Exercise 2: Using the relation $x^n - 1 = \prod_{\ell | n} Q_\ell(x)$ successively for $n = 1, 2, \dots, 6$ gives

$$\begin{aligned}
Q_1(x) &= x - 1 \\
Q_2(x) &= \frac{x^2 - 1}{x - 1} = x + 1 \\
Q_3(x) &= \frac{x^3 - 1}{x - 1} = x^2 + x + 1 \\
Q_4(x) &= \frac{x^4 - 1}{(x-1)(x+1)} = x^2 + 1 \\
Q_5(x) &= \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1 \\
Q_6(x) &= \frac{x^6 - 1}{(x-1)(x+1)(x^2+x+1)} = x^2 - x + 1
\end{aligned}$$

Exercise 3: Rotation by $\pi/5$ in $\mathbb{R}^2$ corresponds to multiplication by $\exp(\pi\mathrm{i}/5)$ in $\mathbb{C}$, so the point set in $\mathbb{R}^2$ is invariant under rotation by $\pi/5$ if $\exp(\pi\mathrm{i}/5)\,\mathbb{Z}[\xi] = \mathbb{Z}[\xi]$.
Observing that $-\exp(\pi\mathrm{i}/5) = \exp(\pi\mathrm{i} + \pi\mathrm{i}/5) = \exp(6\pi\mathrm{i}/5) = \xi^3$, and using that $\xi^5 = 1$ and $1 + \xi + \xi^2 + \xi^3 + \xi^4 = 0$, we find

$$\begin{aligned}
-\xi^3\,\mathbb{Z}[\xi] &= \bigl\{-a_0\xi^3 - a_1\xi^4 - a_2 - a_3\xi \mid a_0, a_1, a_2, a_3 \in \mathbb{Z}\bigr\} \\
&= \bigl\{-a_0\xi^3 - a_1(-1 - \xi - \xi^2 - \xi^3) - a_2 - a_3\xi \mid a_0, a_1, a_2, a_3 \in \mathbb{Z}\bigr\} \\
&= \bigl\{(a_1 + a_3) + (a_1 - a_3)\xi + a_1\xi^2 + (a_1 - a_0)\xi^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{Z}\bigr\} \\
&= \mathbb{Z}[\xi],
\end{aligned}$$

where the final equality is true because the coefficients take all possible values in $\mathbb{Z}$.

Exercise 4: By definition, we have

$$\begin{aligned}
\tau\,\mathbb{Z}[\tau] &= \bigl\{m\tau + n\tau^2 \mid m, n \in \mathbb{Z}\bigr\} \\
&= \bigl\{m\tau + n\tau + n \mid m, n \in \mathbb{Z}\bigr\} \\
&= \bigl\{n + (m+n)\tau \mid m, n \in \mathbb{Z}\bigr\} \\
&= \bigl\{n + \ell\tau \mid \ell, n \in \mathbb{Z}\bigr\} = \mathbb{Z}[\tau],
\end{aligned}$$

because $\tau^2 = \tau + 1$.

Exercise 5: The diagonal embedding is

$$\begin{aligned}
\mathcal{L} &= \bigl\{(x, x') \mid x \in \mathbb{Z}[\sqrt{2}\,]\bigr\} \\
&= \bigl\{(m + n\sqrt{2}, m - n\sqrt{2}\,) \mid m, n \in \mathbb{Z}\bigr\} \\
&= \bigl\{m(1,1) + n(\sqrt{2}, -\sqrt{2}\,) \mid m, n \in \mathbb{Z}\bigr\}
\end{aligned}$$

which is a planar lattice with basis vectors $v_1 = (1,1)$ and $v_2 = (\sqrt{2}, -\sqrt{2}\,)$.
Since $|v_1| = \sqrt{2}$, $|v_2| = 2$ and $v_1 \cdot v_2 = 0$, $\mathcal{L}$ is a rotated rectangular lattice.