Aperiodic Order

Part 4

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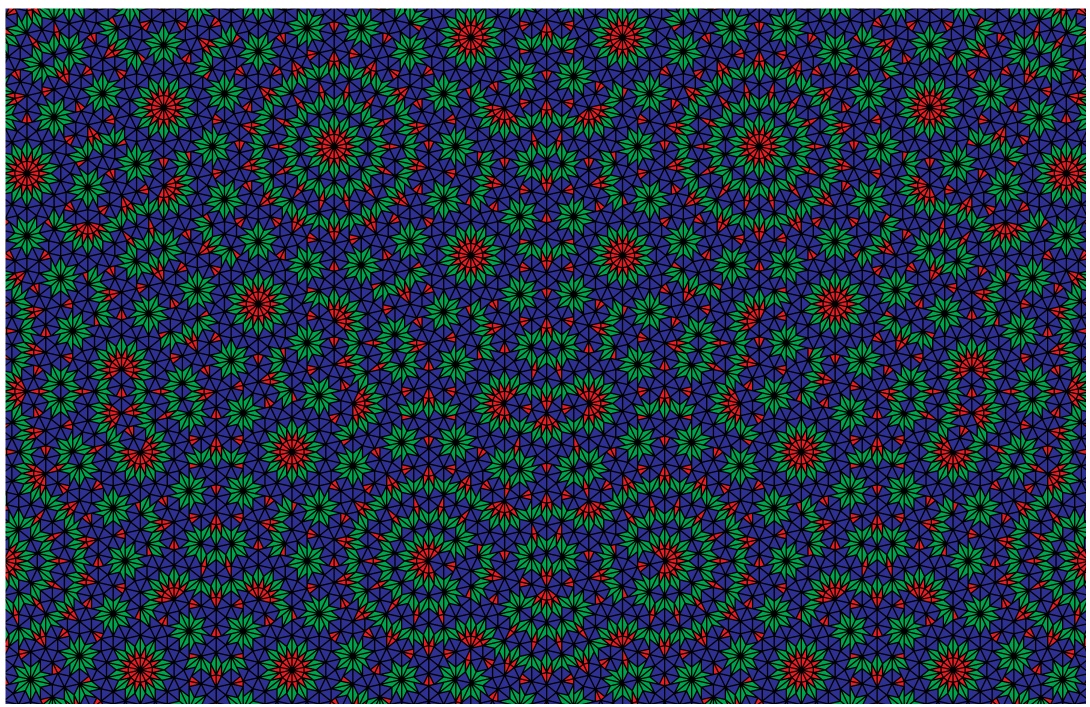
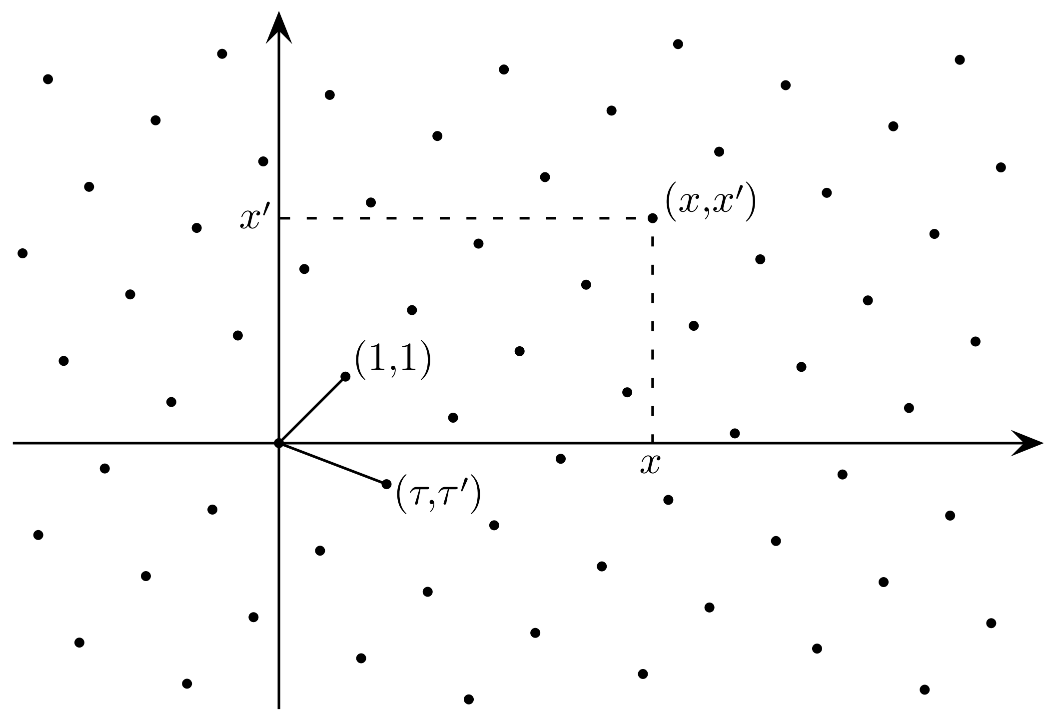


4.1 Minkowski embedding revisited

We consider the example of $\mathbb{Z}[\tau] = \{m + n\tau \mid m, n \in \mathbb{Z}\}$.

Algebraic conjugation $x \mapsto x'$ in $\mathbb{Q}(\sqrt{5}\,)$ is defined by $\sqrt{5} \mapsto -\sqrt{5}$ and its extension to a field automorphism.

The *diagonal embedding* $\mathcal{L} = \big\{(x, x') \mid x \in \mathbb{Z}[\tau]\big\}$ defines a lattice in $\mathbb{R}^2$, generated by the vectors $(1,1)$ and (τ, τ'), so $\mathcal{L} = \langle \binom{1}{1}, \binom{\tau}{1-\tau} \rangle_{\mathbb{Z}}$.



4.2 Fibonacci chain revisited

The Fibonacci sequence was defined by the substitution rule

$$\varrho : \begin{array}{l} a \mapsto ab \\ b \mapsto a \end{array}$$

on the two-letter alphabet $\{a, b\}$, with bi-infinite fixed point (under ϱ^2)

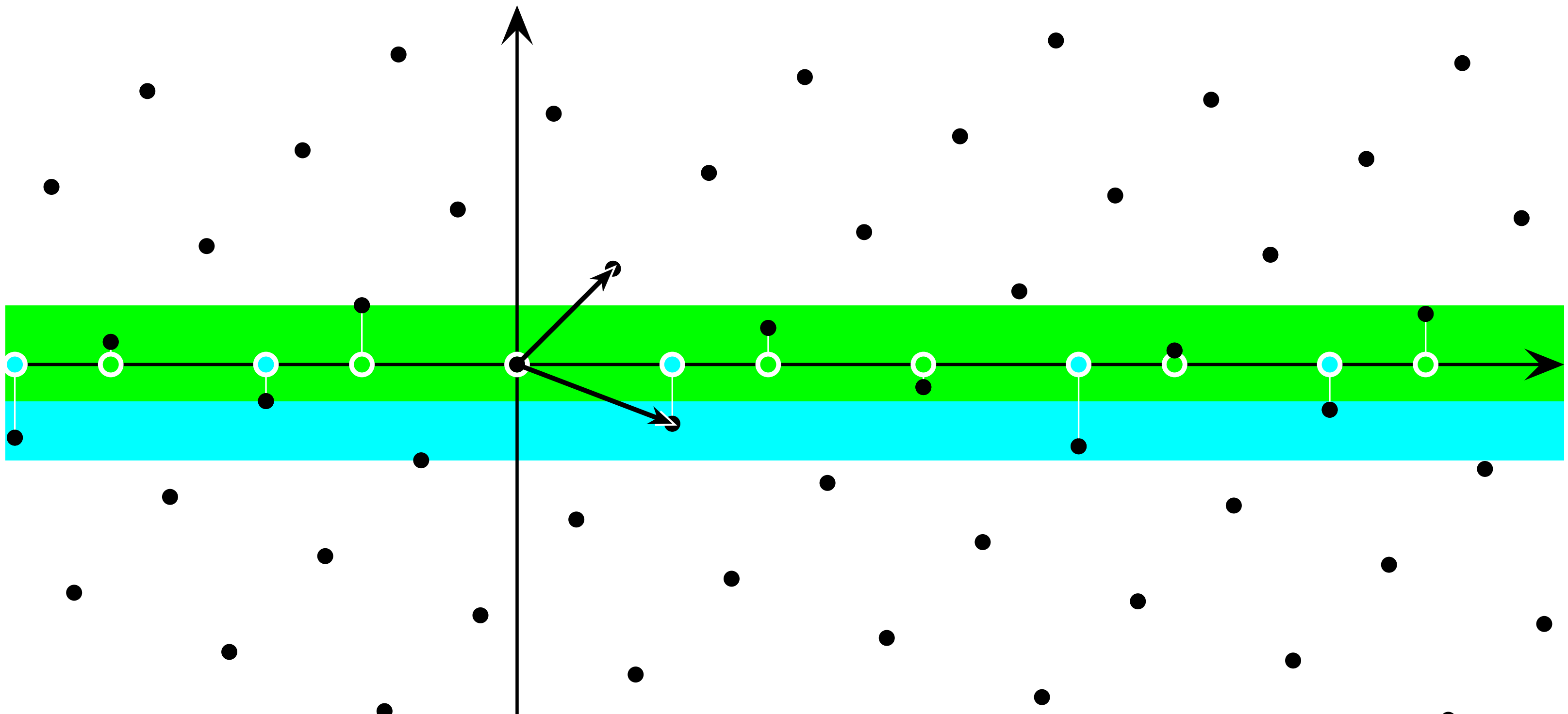
$$w = \ldots abaababaabaababaababa | abaababaabaababaababa \ldots$$

In the geometric interpretation

in terms of intervals of length τ and 1, define two point sets $\varLambda_a \subseteq \mathbb{Z}[\tau]$ and $\varLambda_b \subseteq \mathbb{Z}[\tau]$ as the set of left endpoints of intervals of type a and type b.

4.3 Fibonacci projection

The point sets Λ_a and Λ_b lift to two strips in the lattice $\mathcal{L}$:



- $\Lambda_{a,b} = \{x \in L \mid x^\star \in W_{a,b}\}$ with the windows

$$W_a = (\tau - 2, \tau - 1] \quad \text{and} \quad W_b = (-1, \tau - 2]$$

- $\Lambda = \Lambda_a \cup \Lambda_b = \{x \in L \mid x^\star \in W\}$
- window $W = W_a \cup W_b = (-1, \tau - 1]$

4.4 Euclidean model sets

Cut and project scheme (CPS):

$$
\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times \mathbb{R}^m & \xrightarrow{\;\pi_{\mathrm{int}}\;} & \mathbb{R}^m \\
\cup & & \cup & & \cup \ \text{dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\mathrm{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^\star
\end{array}
$$

Model set: $\varLambda = \{ x \in L \mid x^\star \in W \}$

Window W is a relatively compact subset of $\mathbb{R}^m$ with non-empty interior ⟹ $\varLambda$ is Meyer set

Regular model set: ∂W has zero Lebesgue measure

Generic (non-singular) model set: $L^\star \cap \partial W = \varnothing$

4.5 Fibonacci model set

Windows satisfy
$$\tau^2 W_a = W_a \,\dot\cup\, W_b \,\dot\cup\, (W_a + 1),$$
$$\tau^2 W_b = (W_a - \tau) \,\dot\cup\, (W_b - \tau), \quad \text{as}$$
$$\begin{aligned}\tau^2 (\tau - 2, \tau - 1] &= (-1, \tau] \\ &= (-1, \tau - 2) \,\dot\cup\, (\tau - 2, \tau - 1] \,\dot\cup\, (\tau - 1, \tau) \\ \tau^2 (-1, \tau - 2] &= (-\tau - 1, -1] \\ &= (-\tau - 1, -2] \,\dot\cup\, (-2, -1]\end{aligned}$$

Algebraic conjugation ($\tau \mapsto -\tau^{-1} = 1 - \tau$) gives
$$\varLambda_a = \tau^2 \varLambda_a \,\dot\cup\, \tau^2 \varLambda_b \,\dot\cup\, (\tau^2 \varLambda_a + \tau^2),$$
$$\varLambda_b = (\tau^2 \varLambda_a + \tau) \,\dot\cup\, (\tau^2 \varLambda_b + \tau),$$
which corresponds to the fixed point equations of the substitution $\varrho^2 : \; a \mapsto aba, b \mapsto ab$ with inflation multiplier τ^2.

4.6 Uniform distribution

Sequence $(x_i)_{i\in\mathbb{N}}$ of points in a compact interval I of length $|I|$ is *uniformly distributed* in I if

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i) \xrightarrow{N\to\infty} \frac{1}{|I|}\int_I f(x)\,\mathrm{d}x$$

holds for all continuous functions f on I.

Projections in internal space uniformly distributed

⟹ frequencies proportional to volume of window

4.7 Frequencies

Example: Consider occurrence of points in $\varLambda$ at distance 1.

Take $x \in \varLambda$, so $x^\star \in W = (-1, \tau - 1]$. For $x + 1 \in \varLambda$, we require $(x+1)^\star = x^\star + 1 \in W$, which holds if $x^\star \in (-1, \tau - 2]$.

Relative frequency (frequency per point) of distance 1:

$$\frac{\mathrm{vol}\big((-1, \tau - 2]\big)}{\mathrm{vol}\big((-1, \tau - 1]\big)} = \frac{\tau - 1}{\tau} = (\tau - 1)^2 = 2 - \tau$$

which is the frequency of the letter b in w.

Absolute frequency (frequency per volume) is given by the product of the relative frequency with the volume density $\mathrm{dens}(\varLambda)$, which is

$$\mathrm{dens}(\varLambda) = \mathrm{dens}(\mathcal{L})\,\mathrm{vol}(W) = \frac{\mathrm{vol}(W)}{\mathrm{vol}\big(\mathrm{FD}(\mathcal{L})\big)} = \frac{\tau + 2}{5},$$

because $\mathrm{vol}(W) = \tau$ and $\mathrm{vol}\big(\mathrm{FD}(\mathcal{L})\big) = \tau - \tau' = 2\tau - 1$.

4.8 General CPS

A *cut and project scheme* (CPS) is a triple $(\mathbb{R}^d, H, \mathcal{L})$ with a (compactly generated) locally compact Abelian group (LCAG) H, a lattice $\mathcal{L}$ in $\mathbb{R}^d \times H$ and the two natural projections $\pi\colon \mathbb{R}^d \times H \longrightarrow \mathbb{R}^d$ and $\pi_{\mathrm{int}}\colon \mathbb{R}^d \times H \longrightarrow H$, subject to the conditions that $\pi|_{\mathcal{L}}$ is injective and that $\pi_{\mathrm{int}}(\mathcal{L})$ is dense in H.

$$
\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times H & \xrightarrow{\;\pi_{\mathrm{int}}\;} & H \\
\cup & & \cup & & \cup \ \text{dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\mathrm{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^{\star}
\end{array}
$$

Star-map: $\star\colon\ L \longrightarrow H$ with $x \mapsto x^{\star} := \pi_{\mathrm{int}}\big((\pi|_{\mathcal{L}})^{-1}(x)\big)$

4.8 General CPS

Let $(\mathbb{R}^d, H, \mathcal{L})$ be a CPS. If $W \subseteq H$ is a relatively compact set with non-empty interior, the projection set

$$\curlywedge(W) := \{x \in L \mid x^\star \in W\}$$

or any translate $t + \curlywedge(W)$ with $t \in \mathbb{R}^d$, is called a *model set*.

A model set is termed *regular* when $\mu_H(\partial W) = 0$, where μ_H is the Haar measure of H.

If $L^\star \cap \partial W = \varnothing$, the model set is called *generic*.

4.9 Cluster frequencies

Let $\varLambda$ be a regular model set for the general CPS $(\mathbb{R}^d, H, \mathcal{L})$, with a compact window $W = \overline{W^\circ}$, and let $P \subseteq \varLambda$ be a finite cluster.

The *repetition set* of P,

$$\begin{aligned} \operatorname{rep}(P) \; &:= \; \bigl\{ t \in L \mid t + P \subseteq \varLambda \bigr\} \; = \; \bigl\{ t \in L \mid t^\star + P^\star \subseteq W \bigr\} \\ &= \; \Bigl\{ t \in L \mid t^\star \in \Bigl(\bigcap_{x \in P} (W - x^\star) \Bigr) \Bigr\} \end{aligned}$$

is itself a regular model set.

The relative frequency (per point of $\varLambda$) of P is given by

$$\operatorname{rel\,freq}_{\varLambda}(P) \; = \; \frac{\operatorname{vol}\bigl(\bigcap_{x \in P} (W - x^\star) \bigr)}{\operatorname{vol}(W)} .$$

This is related to the absolute frequency of P by

$$\operatorname{abs\,freq}_{\varLambda}(P) \; = \; \operatorname{dens}(\mathcal{L}) \operatorname{rel\,freq}_{\varLambda}(P) .$$

4.10 Cyclotomic model sets

$$
\begin{array}{ccccc}
\mathbb{R}^2 & \xleftarrow{\;\pi\;} & \mathbb{R}^2 \times \mathbb{R}^{\phi(n)-2} & \xrightarrow{\;\pi_{\text{int}}\;} & \mathbb{R}^{\phi(n)-2} \\
\cup & & \cup & & \cup \quad \text{dense} \\
\pi(\mathcal{L}_n) & \xleftarrow{\;1-1\;} & \mathcal{L}_n & \longrightarrow & \pi_{\text{int}}(\mathcal{L}_n) \\
\| & & & & \| \\
\mathbb{Z}[\xi_n] & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & \mathbb{Z}[\xi_n]^\star
\end{array}
$$

ξ_n: primitive nth root of unity

ϕ: Euler's totient function

$\star$-map: $\quad x \mapsto (\sigma_2(x), \ldots, \sigma_{\frac{1}{2}\phi(n)}(x))$

σ_i: $\quad$ Galois automorphisms of $\mathbb{Q}(\xi_n)$

$\mathcal{L}_n$: Minkowski embedding of $\mathbb{Z}[\xi_n]$, given by

$$\mathcal{L}_n = \left\{ (x, \sigma_2(x), \ldots, \sigma_{\frac{1}{2}\phi(n)}(x)) \;\middle|\; x \in \mathbb{Z}[\xi_n] \right\} \subseteq \mathbb{C}^{\frac{1}{2}\phi(n)} \simeq \mathbb{R}^{\phi(n)}$$

4.11 Ammann–Beenker model set

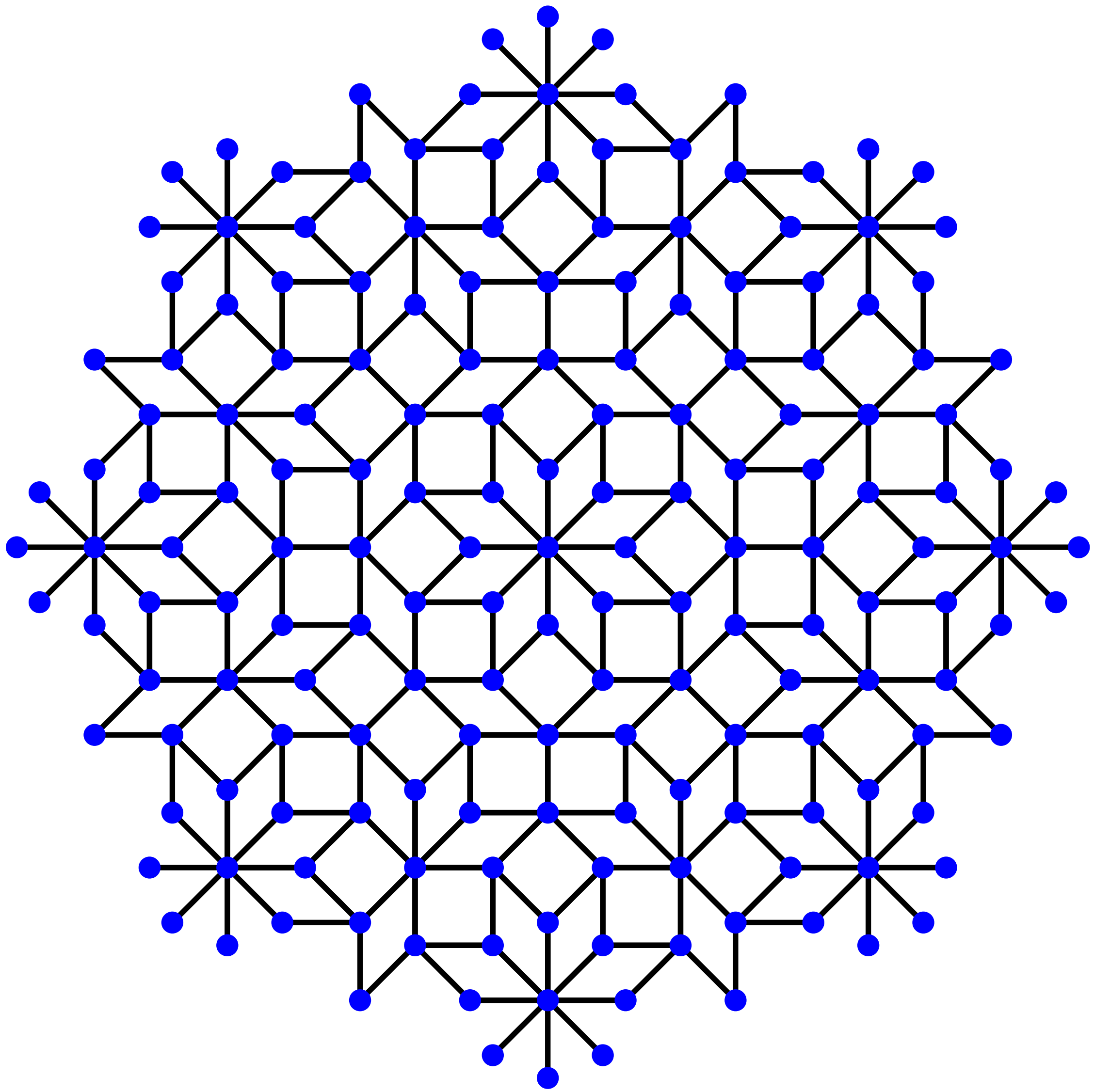
We use the Minkowski embedding of $\mathbb{Z}[\xi]$ with the explicit choice $\xi = \mathrm{e}^{2\pi\mathrm{i}/8}$ and the conjugation map defined by $\xi \mapsto \xi^3$.

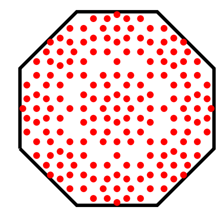
This leads to the lattice $\mathcal{L} = \sqrt{2}\, R_8\, \mathbb{Z}^4$, with the rotation matrix

$$R_8 \; = \; \frac{1}{2} \begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \\ \sqrt{2} & -1 & 0 & 1 \\ 0 & 1 & -\sqrt{2} & 1 \end{pmatrix} .$$

Ammann–Beenker model set obtained with centred regular octagon of unit edge length as its window

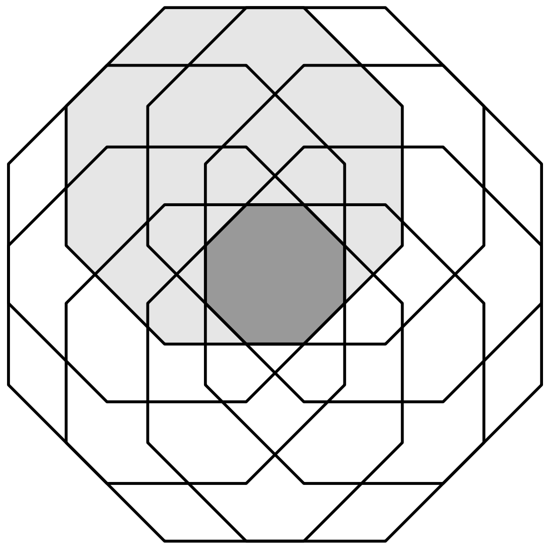
4.11 Ammann–Beenker model set

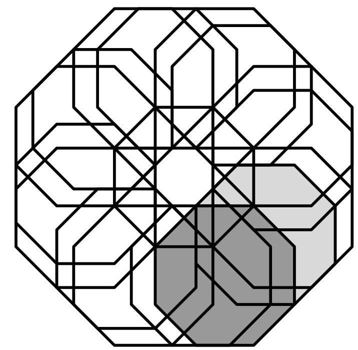




4.11 Ammann–Beenker model set

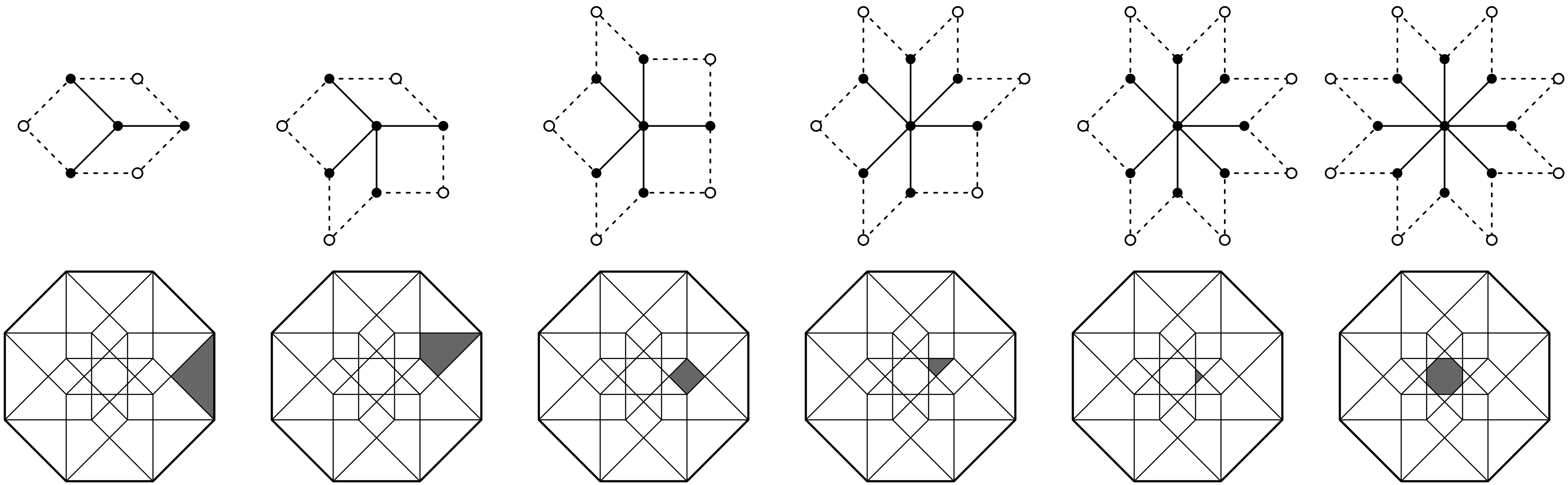
Local inflation/deflation symmetry (LIDS)





The inflation multiplier (in direct space) is $\lambda = 1 + \sqrt{2}$. The corresponding action on the window is multiplication (scaling) by $\lambda^\star = -1/\lambda$. The rescaled octagon can be expressed as the intersection of eight translated copies of the original window, with translations that are elements of $\mathbb{Z}[\xi]$. Likewise, W can be written as a union of translated copies of the rescaled window $\lambda^\star W$, implying LIDS.

4.11 Ammann–Beenker model set



vertex	coordination	orbit length	relative frequency
1	3	8	$-1+\sqrt{2} = \lambda^{-1} \approx 0.41421$
2	4	8	$6-4\sqrt{2} = 2\lambda^{-2} \approx 0.34315$
3	5	8	$-14+10\sqrt{2} = 2\lambda^{-3} \approx 0.14214$
4	6	8	$34-24\sqrt{2} = 2\lambda^{-4} \approx 0.05887$
5	7	8	$-41+29\sqrt{2} = \lambda^{-5} \approx 0.01219$
6	8	1	$17-12\sqrt{2} = \lambda^{-4} \approx 0.02944$