Aperiodic Order
Part 3

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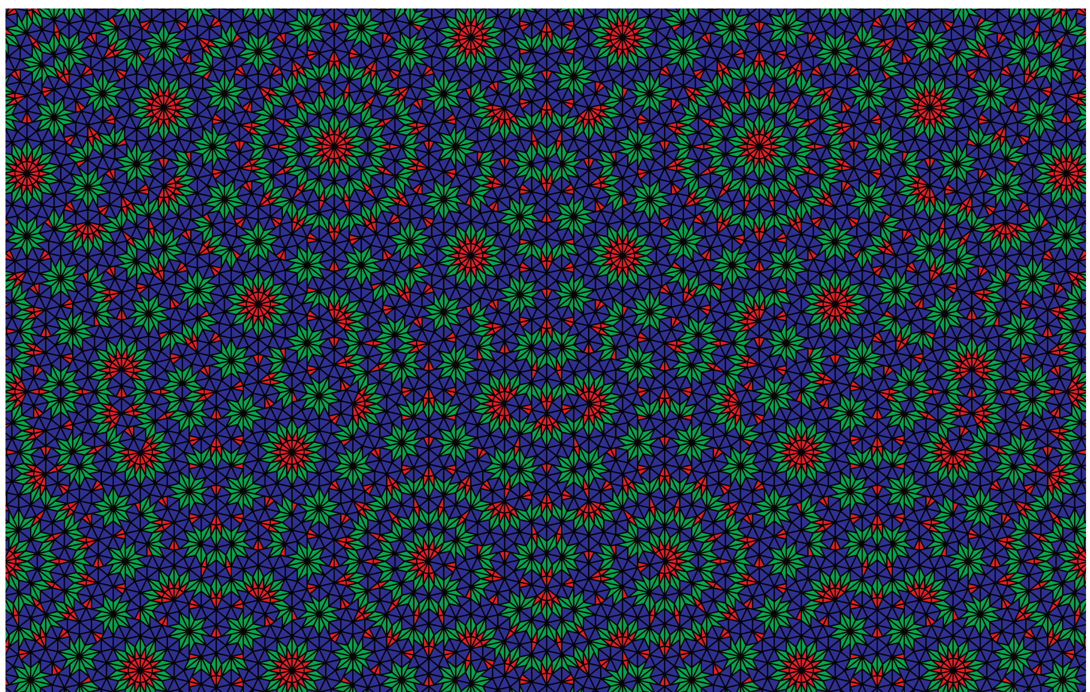
`http://mcs.open.ac.uk/ugg2/ltcc/`



3.1 Patterns

A *pattern* $\mathcal{T}$ in Euclidean space $\mathbb{R}^d$ is a non-empty set of non-empty subsets of $\mathbb{R}^d$. We refer to the elements of $\mathcal{T}$ as the *fragments* of the pattern $\mathcal{T}$.

A locally finite point set $\varLambda \subseteq \mathbb{R}^d$ can be interpreted as a pattern

$$\mathcal{T}_{\varLambda} = \big\{ \{x\} \mid x \in \varLambda \big\}$$

where we tacitly identify $\varLambda$ and $\mathcal{T}_{\varLambda}$.

A *tiling* in $\mathbb{R}^d$ is a pattern $\mathcal{T} = \{T_i \mid i \in I\} \sqsubset \mathbb{R}^d$, with (countable) index set I and non-empty closed sets $T_i \subseteq \mathbb{R}^d$, subject to the conditions $\bigcup_{i \in I} T_i = \mathbb{R}^d$ and $T_i^\circ \cap T_j^\circ = \varnothing$ for all $i \neq j$.

The fragments T_i of $\mathcal{T}$ are called the *tiles* of the tiling, and their equivalence classes up to translations (or, alternatively, up to congruence) are called *prototiles*.

3.1 Patterns

A pattern $\mathcal{T} \sqsubset \mathbb{R}^d$ is called *locally finite* if $\mathcal{T} \sqcap K$ has finite cardinality, for all compact $K \subseteq \mathbb{R}^d$.

Let $\mathcal{T} \sqsubset \mathbb{R}^d$ be a locally finite pattern. When $K \subseteq \mathbb{R}^d$ is compact, the pattern $\mathcal{T} \sqcap K$ is called a *cluster* of $\mathcal{T}$. We also speak of a *patch* when K is convex.

Two (locally finite) patterns $\mathcal{T}$ and $\mathcal{T}'$ in $\mathbb{R}^d$ are *locally indistinguishable*, or LI for short and written as $\mathcal{T} \overset{\text{LI}}{\sim} \mathcal{T}'$, when any cluster of $\mathcal{T}$ occurs also in $\mathcal{T}'$ and vice versa. This means that, for any compact $K \subseteq \mathbb{R}^d$, there are translations $t, t' \in \mathbb{R}^d$ such that $\mathcal{T} \sqcap K = (-t' + \mathcal{T}') \sqcap K$ together with $\mathcal{T}' \sqcap K = (-t + \mathcal{T}) \sqcap K$.

Here, $\mathcal{T} \sqcap A = \{T \in \mathcal{T} \mid T \cap A \neq \varnothing\}$ (all fragments in $\mathcal{T}$ that intersect A).

3.1 Patterns

Two patterns $\mathcal{T}, \mathcal{T}' \sqsubset \mathbb{R}^d$ are ε-close in the *local topology* when

$$\mathcal{T} \sqcap \overline{B_{1/\varepsilon}(0)} \; = \; (-t + \mathcal{T}') \sqcap \overline{B_{1/\varepsilon}(0)}$$

holds for some $t \in B_\varepsilon(0)$.

Lemma: If $t \neq 0$ is a period of a locally finite pattern $\mathcal{T} \sqsubset \mathbb{R}^d$, any $\mathcal{T}' \in \mathrm{LI}(\mathcal{T})$ is t-periodic as well. The group of periods is thus an invariant of an LI class. Moreover, if $\varGamma = \mathrm{per}(\mathcal{T})$ is a lattice in $\mathbb{R}^d$ ($\mathcal{T}$ is crystallographic), one has

$$\mathrm{LI}(\mathcal{T}) \; = \; \{t + \mathcal{T} \mid t \in \mathrm{FD}(\varGamma)\} \; = \; \overline{\{x + \mathcal{T} \mid x \in \mathbb{R}^d\}},$$

where $\mathrm{FD}(\varGamma)$ is a fundamental domain of $\varGamma$, and where the closure is taken in the local topology. In particular, $\mathrm{LI}(\mathcal{T})$ is compact in the local topology, and one has $\mathrm{LI}(\mathcal{T}) \simeq \mathbb{R}^d/\varGamma$ as topological spaces.

3.2 Limit translation module

For a pattern $\mathcal{T} \sqsubset \mathbb{R}^d$ and a compact set $K \subseteq \mathbb{R}^d$, define $\Delta_K(\mathcal{T})$ to be

$$\left\langle t \;\middle|\; \mathcal{T} \sqcap (x+K) = (-t+\mathcal{T}) \sqcap (x+K) \text{ for some } x \in \mathbb{R}^d \right\rangle_{\mathbb{Z}},$$

the $\mathbb{Z}$-module generated by all translations between occurrences of some K-cluster in $\mathcal{T}$.

The *limit translation module* (LTM) $\Delta(\mathcal{T})$ is defined as the inverse limit of the $\Delta_K(\mathcal{T})$ over all compact subsets $K \subseteq \mathbb{R}^d$, ordered according to inclusion.

Proposition: The limit translation module of a (locally finite) pattern $T \sqsubset \mathbb{R}^d$ is an invariant of $\mathrm{LI}(\mathcal{T})$.

The LTM of a crystallograhic pattern is its lattice of periods. The LTM of the geometric realisation of the Fibonacci sequence by intervals of lengths τ and 1 is $\mathbb{Z}[\tau]$.

3.3 Local derivability

A pattern $\mathcal{T}' \sqsubset \mathbb{R}^d$ is said to be *locally derivable* from a pattern $\mathcal{T} \sqsubset \mathbb{R}^d$, written as $\mathcal{T} \overset{\text{LD}}{\rightsquigarrow} \mathcal{T}'$, when a compact neighbourhood $K \subseteq \mathbb{R}^d$ of 0 exists such that, whenever $(-x+\mathcal{T}) \sqcap K = (-y+\mathcal{T}) \sqcap K$ holds for $x, y \in \mathbb{R}^d$, one also has $(-x+\mathcal{T}') \sqcap \{0\} = (-y+\mathcal{T}') \sqcap \{0\}$.

Lemma: Let the pattern $\mathcal{T}_1'$ be locally derivable from $\mathcal{T}_1 \sqsubset \mathbb{R}^d$, and let $\mathcal{T}_2 \in \mathrm{LI}(\mathcal{T}_1)$. Then, there exists some $\mathcal{T}_2' \in \mathrm{LI}(\mathcal{T}_1')$ which is locally derivable from $\mathcal{T}_2$.

Two patterns $\mathcal{T}_1, \mathcal{T}_2 \sqsubset \mathbb{R}^d$ are called *mutually locally derivable* (MLD) from each other when $\mathcal{T}_1 \overset{\text{LD}}{\rightsquigarrow} \mathcal{T}_2$ and $\mathcal{T}_2 \overset{\text{LD}}{\rightsquigarrow} \mathcal{T}_1$. Similarly, two LI classes are MLD when they are locally derivable from each other.

3.3 Local derivability

Proposition: For $\mathcal{T}, \mathcal{T}' \sqsubset \mathbb{R}^d$ with $\mathcal{T} \overset{\text{LD}}{\rightsquigarrow} \mathcal{T}'$, one has $\Delta(\mathcal{T}) \subseteq \Delta(\mathcal{T}')$.

Proof:

Local derivation $\mathcal{T} \overset{\text{LD}}{\rightsquigarrow} \mathcal{T}'$ with compact $K \subseteq \mathbb{R}^d$

$\Longrightarrow$ $\Delta_{K+K'}(\mathcal{T}) \subseteq \Delta_{K'}(\mathcal{T}')$ for all compact $K' \subseteq \mathbb{R}^d$

$\Longrightarrow$ $\Delta(\mathcal{T}) \subseteq \Delta(\mathcal{T}')$ inclusion preserved by limit. ∎

Corollary: The LTM $\Delta(\mathcal{T})$ of a pattern $\mathcal{T} \sqsubset \mathbb{R}^d$ is an invariant of the entire MLD class of $\mathrm{LI}(\mathcal{T})$.

Corollary: Two crystallographic, locally finite point sets $\varLambda, \varLambda' \subseteq \mathbb{R}^d$ are MLD if and only if they have the same lattice of periods.

3.4 Repetitivity

A pattern $\mathcal{T} \sqsubset \mathbb{R}^d$ is called (translationally) *repetitive* when, for every compact $K \subseteq \mathbb{R}^d$, there is a compact $K' \subseteq \mathbb{R}^d$ such that, for every $x, y \in \mathbb{R}^d$, the relation $\mathcal{T} \sqcap (x + K) = (-t + \mathcal{T}) \sqcap (y + K)$ holds for some $t \in K'$.

Choose $K = K_r := \overline{B_r(0)}$ and $K' = K_R$ such that $R \geq r$ is minimal. The function $R = R(r)$ is called the *repetitivity function*.

A repetitive pattern $\mathcal{T} \sqsubset \mathbb{R}^d$ is called *linearly repetitive* when its repetitivity function satisfies $R(r) = \mathcal{O}(r)$ as $r \to \infty$.

Proposition: Let $\mathcal{T} \sqsubset \mathbb{R}$ be a tiling that emerges via the geometric interpretation of a primitive substitution rule on a finite alphabet. Then, $\mathcal{T}$ is linearly repetitive.

3.5 Continuous hulls

A pattern $\mathcal{T} \sqsubset \mathbb{R}^d$ is *FLC* when, for every compact set $K \subseteq \mathbb{R}^d$, the set of K-clusters $\{(t+K) \sqcap \mathcal{T} \mid t \in \mathbb{R}^d\}$ consists of finitely many equivalence classes up to $\mathbb{R}^d$-translations.

If $\varLambda \subseteq \mathbb{R}^d$ is an FLC set, its geometric or *continuous hull* is $\mathbb{X}(\varLambda) = \overline{\{t + \varLambda \mid t \in \mathbb{R}^d\}}$, where the closure is taken in the local topology. If the $\mathbb{R}^d$-orbit of every element $\varLambda' \in \mathbb{X}(\varLambda)$ is dense, the hull $\mathbb{X}(\varLambda)$ is called *minimal*.

The subset

$$\mathbb{X}_0(\varLambda) \;=\; \{\varLambda' \in \mathbb{X}(\varLambda) \mid 0 \in \varLambda'\},$$

is sometimes called the *discrete hull* or the *transversal*.

3.6 Symmetry

A point set $\varLambda \subseteq \mathbb{R}$ is called *reflection symmetric* in the point x if $r_x(\varLambda) = \varLambda$, where $r_x \colon \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $r_x(y) = 2x - y$.

Lemma: Let $\varLambda \subseteq \mathbb{R}$ be a point set that is reflection symmetric in the distinct points x and y. Then, $\varLambda$ is periodic with period $2|x - y|$.

Proposition: Let $\varLambda \subseteq \mathbb{R}^2$ be a uniformly discrete point set with an exact n-fold rotational symmetry. If n is non-crystallographic, which means $n = 5$ or $n \geq 7$, there can only be one such rotation centre. When $n \in \{3, 4, 6\}$, the existence of more than one rotation centre is possible, and then implies lattice periodicity of $\varLambda$. When $n = 2$, the existence of another rotation centre means that $\varLambda$ is at least rank-1 periodic.

3.6 Symmetry

Let R be a linear or affine transformation of $\mathbb{R}^d$. A pattern $\mathcal{T} \sqsubset \mathbb{R}^d$ is *symmetric* under the action of R when $R(\mathcal{T}) \overset{\text{LI}}{\sim} \mathcal{T}$. Moreover, the hull $\mathbb{X}(\mathcal{T})$ is *symmetric* under the action of R when $R(\mathbb{X}(\mathcal{T})) \subseteq \mathbb{X}(\mathcal{T})$.

A discrete structure $\mathcal{T}$ in $\mathbb{R}^d$ is said to have a *local scaling property* with respect to the homothety $x \mapsto \lambda x$ for some $0 \neq \lambda \in \mathbb{R}$, if $\lambda\mathcal{T} \overset{\text{LD}}{\rightsquigarrow} \mathcal{T}$. When $M\mathcal{T} \overset{\text{LD}}{\rightsquigarrow} \mathcal{T}$ for some $M \in \mathrm{GL}(d)$, one speaks of a local scaling property relative to the linear map defined by M.

A discrete structure $\mathcal{T}$ in $\mathbb{R}^d$ is said to have a *local inflation deflation symmetry* (LIDS) relative to the linear map L if $\mathcal{T} \overset{\text{MLD}}{\leftrightsquigarrow} L(\mathcal{T})$. When $L(x) = \lambda x$, or when $L(x) = \lambda R x$ with $R \in \mathrm{O}(d, \mathbb{R})$, the number λ is called the *inflation multiplier* of the LIDS.

3.7 Inflation

Consider a finite set $\{T_1, T_2, \ldots, T_n\}$ of tiles, where each $T_i \subseteq \mathbb{R}^d$ is a compact set with non-empty interior and $\overline{T_i^\circ} = T_i$, so that we also have $0 < \mathrm{vol}(T_i) < \infty$. An *inflation rule* with inflation multiplier $\lambda > 1$ (and an extension map $x \mapsto \lambda x$) consists of the mappings

$$\lambda T_i \longmapsto \bigcup_{j=1}^{n} T_j + A_{ji}$$

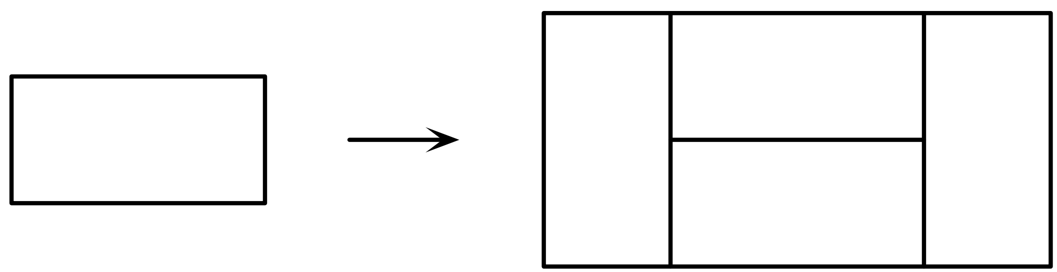
with finite sets $A_{ji} \subseteq \mathbb{R}^d$, subject to the mutual disjointness of the interiors of the sets on the right hand side and to the (individual) volume consistency conditions $\mathrm{vol}(\lambda T_i) = \sum_{j=1}^{n} \mathrm{vol}(T_j)\,\mathrm{card}(A_{ji})$, both for each $1 \leq i \leq n$.

More generally, one can equally well work with an extension map of the form $x \mapsto \lambda R x$ with $R \in \mathrm{O}(d, \mathbb{R})$, or with an expanding linear map.

3.7 Inflation

The matrix M defined by $M_{k\ell} = \mathrm{card}(A_{k\ell})$ is called the *inflation matrix*. The consistency conditions mean that λ^d is the leading eigenvalue of M and that $\big(\mathrm{vol}(T_1), \dots, \mathrm{vol}(T_n)\big)$ is a corresponding left eigenvector of M.

Example: Table tiling

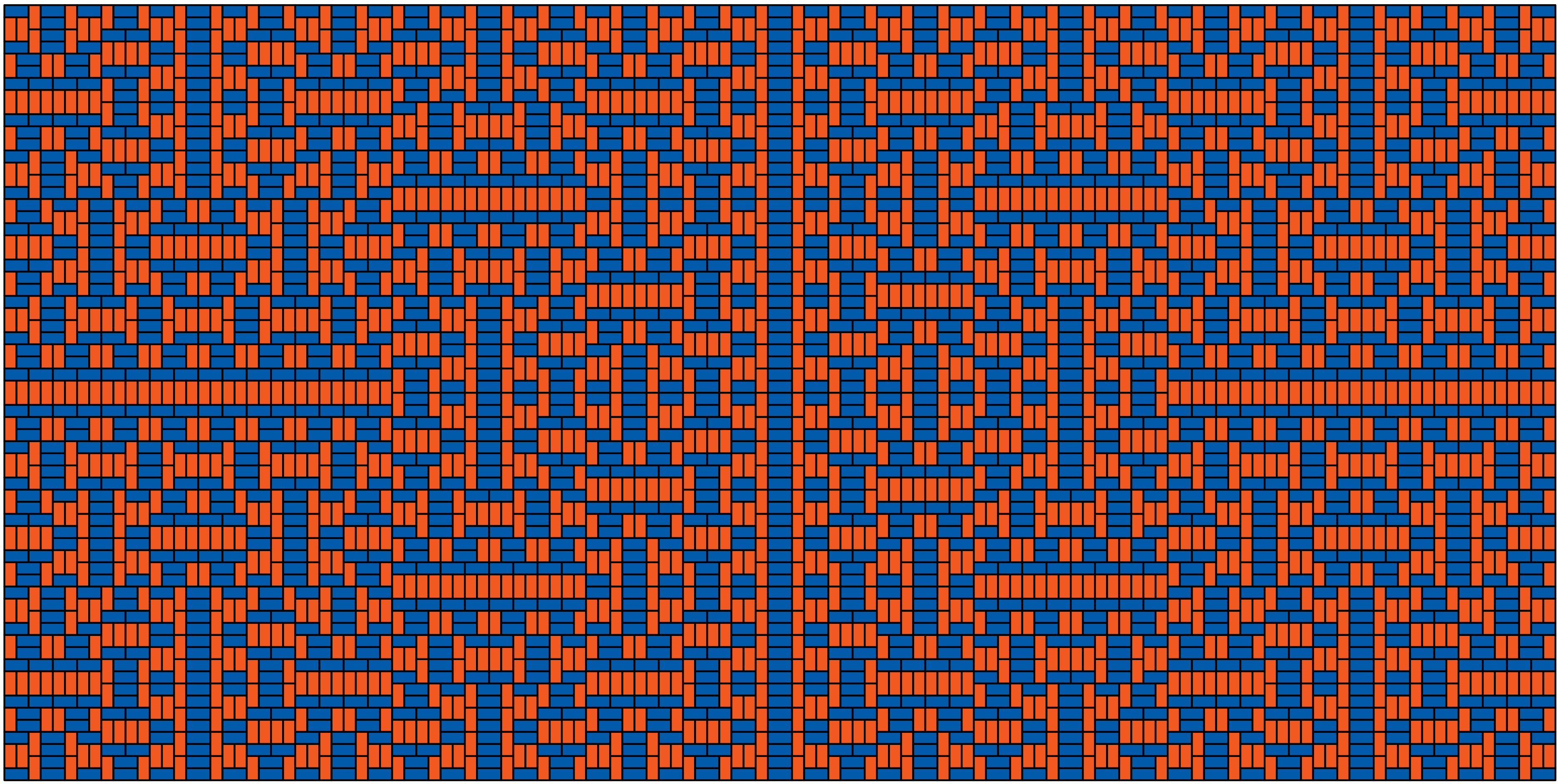


Inflation matrix (distringuishing orientation)

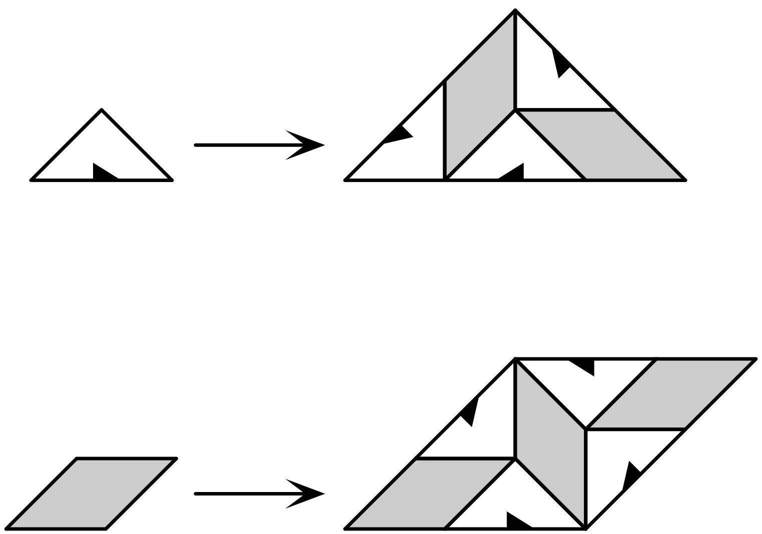
$$M = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

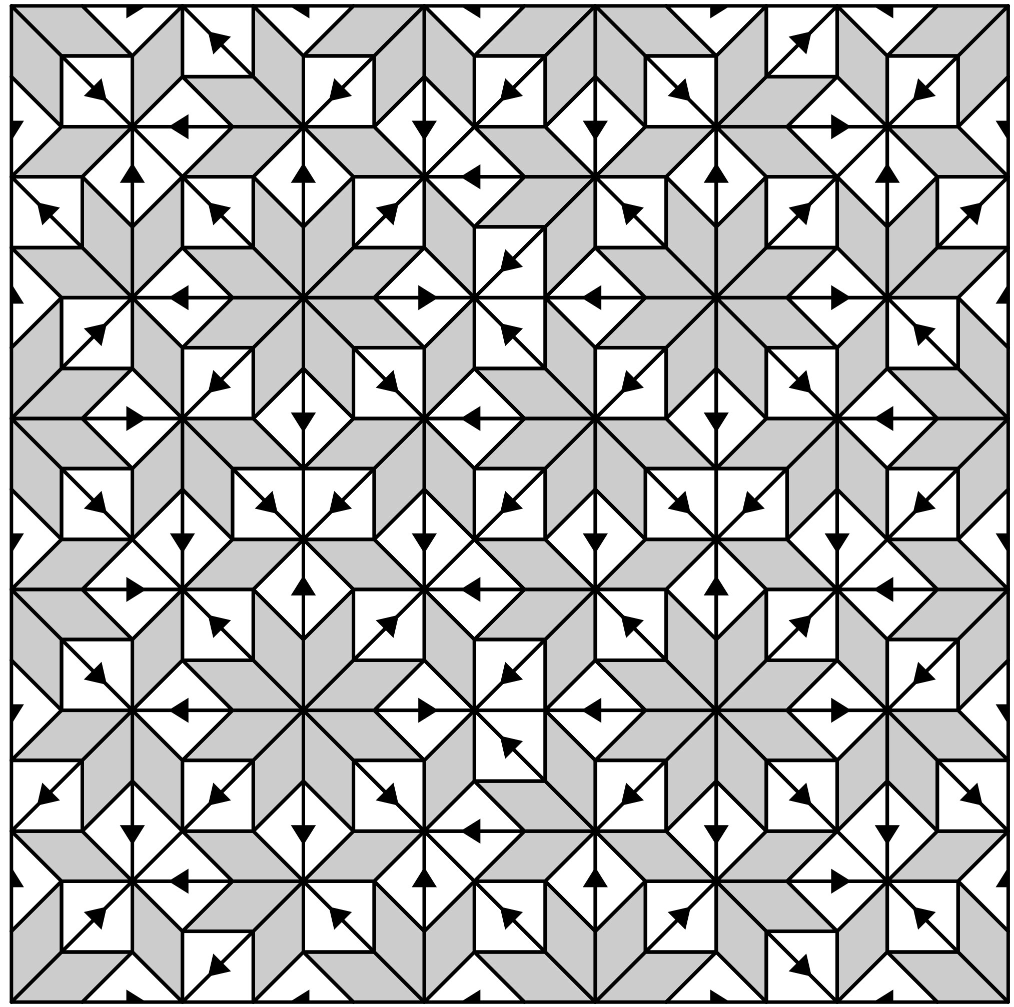
with leading eigenvalue $\lambda^2 = 4$ and eigenvector $(1, 1)$.

3.7 Inflation



3.8 Ammann–Beenker tiling





3.8 Ammann–Beenker tiling

Inflation matrix (distinguishing triangles and rhombuses)

$$M = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

PF eigenvalue $\lambda^2 = (1+\sqrt{2})^2 = 3 + 2\sqrt{2}$.

Left eigenvector $\frac{1}{2}(1, \sqrt{2}\,)$, so areas of the two prototiles are $\frac{1}{2}$ (triangle) and $\frac{1}{2}\sqrt{2}$ (rhombus), when choosing edge length of rhombus as 1.

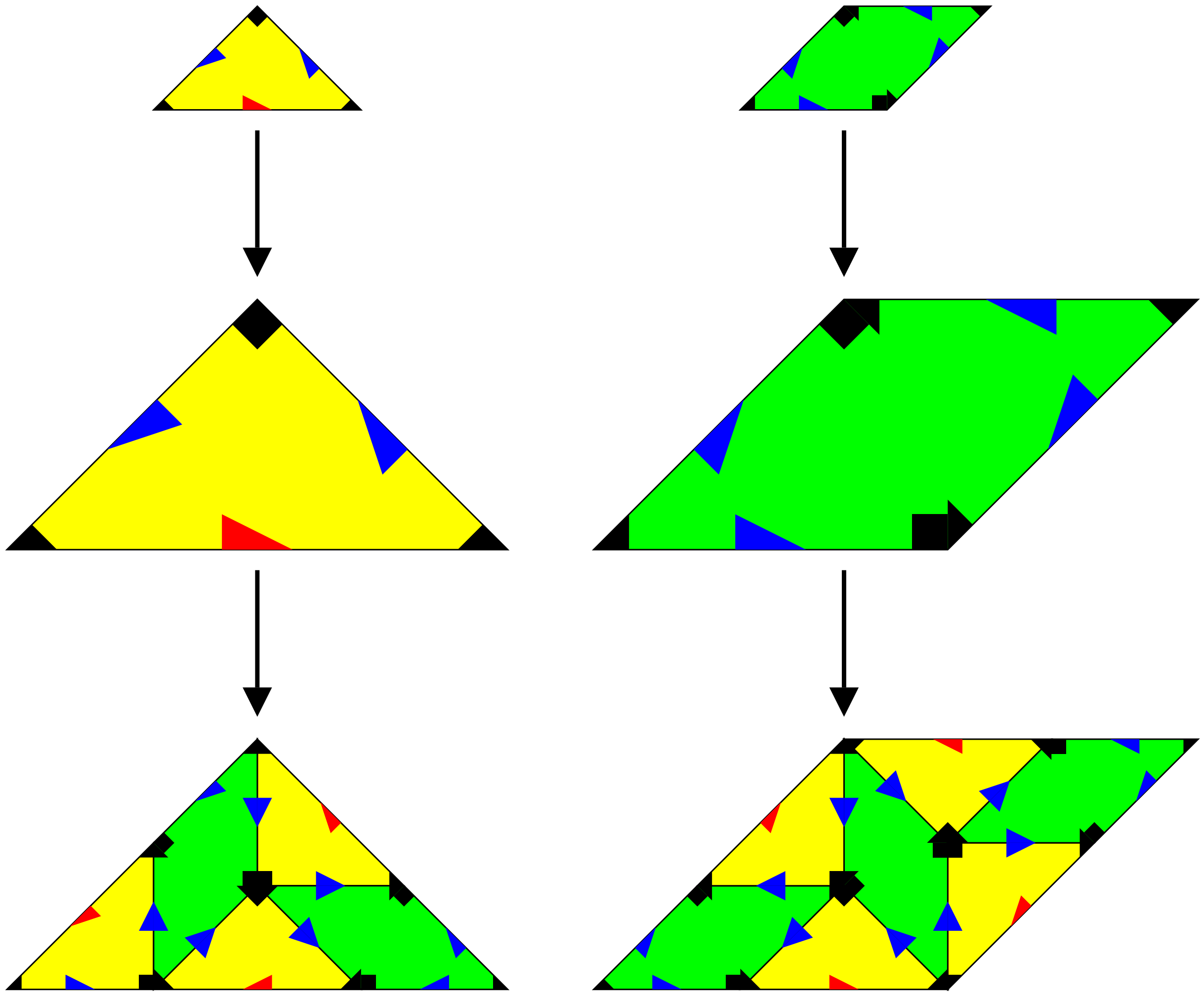
Right eigenvector in statistical normalisation

$$\nu = \left(2-\sqrt{2}, \sqrt{2}-1\right)^T$$

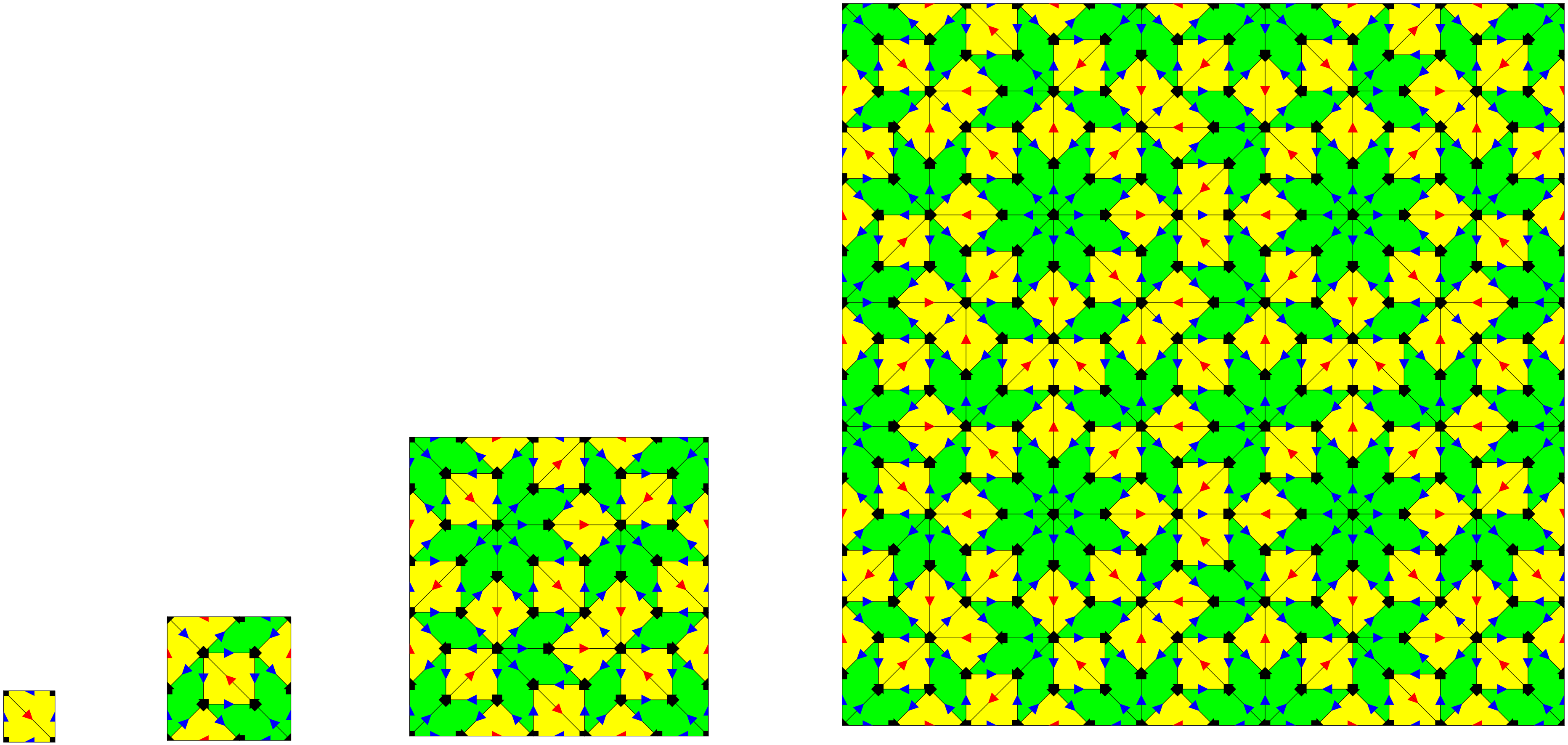
gives frequencies of triangles and rhombuses.

⟹ triangles and rhombuses cover same area fraction

3.8 Ammann–Beenker tiling



3.8 Ammann–Beenker tiling



two cycle ⟶ fixed point tilings ⟶ continuous hull

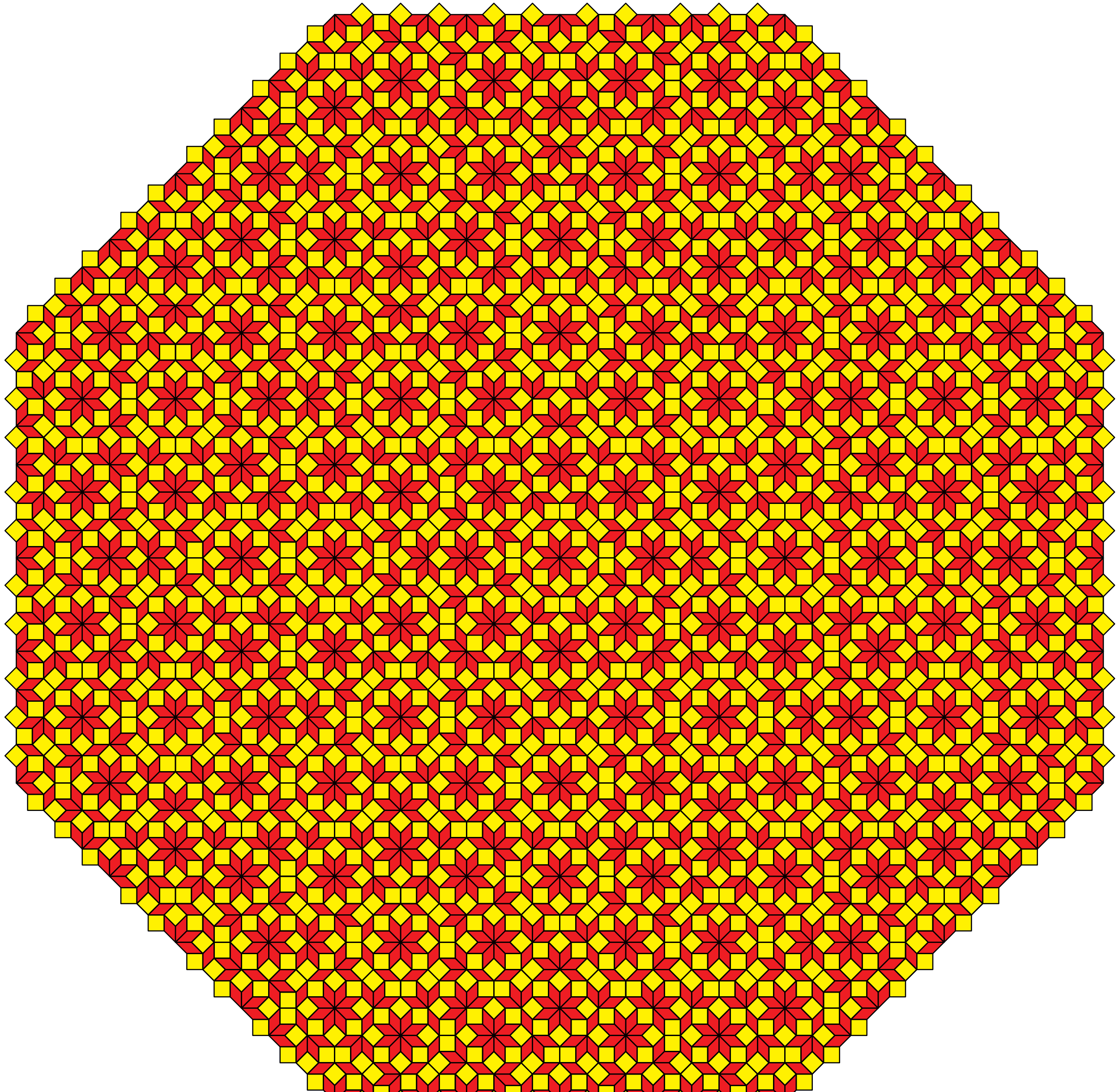
3.8 Ammann–Beenker tiling

Proposition: The Ammann–Beenker tiling $\mathcal{T}$ is a linearly repetitive FLC tiling that is aperiodic. It possesses an LIDS with inflation multiplier $1+\sqrt{2}$. The continuous hull is compact and satisfies $\mathbb{X}(\mathcal{T}) = \mathrm{LI}(\mathcal{T})$. The corresponding dynamical system $(\mathbb{X}(\mathcal{T}), \mathbb{R}^2, \mu)$ is strictly ergodic.

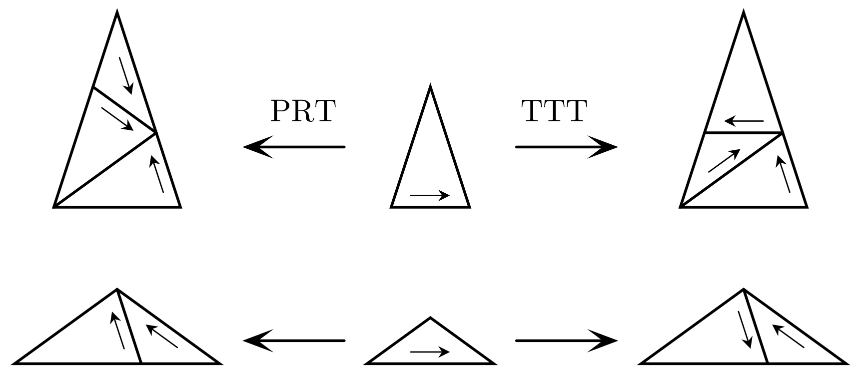
Note that the two inflation rules define two LI classes that are *not* MLD.

While the decorated version of the Ammann–Beenker tiling can be reduced to the undecorated one by simply removing all markings except for the arrows on the hypotenuses of the triangles, the converse is not true because the decorated version of the tiling contains information that cannot be locally derived from the undecorated tiling. There is no local way to decide upon the position of the symmetry-breaking (house-shaped) vertex markers.

3.8 Ammann–Beenker tiling



3.9 Penrose and TTT tilings



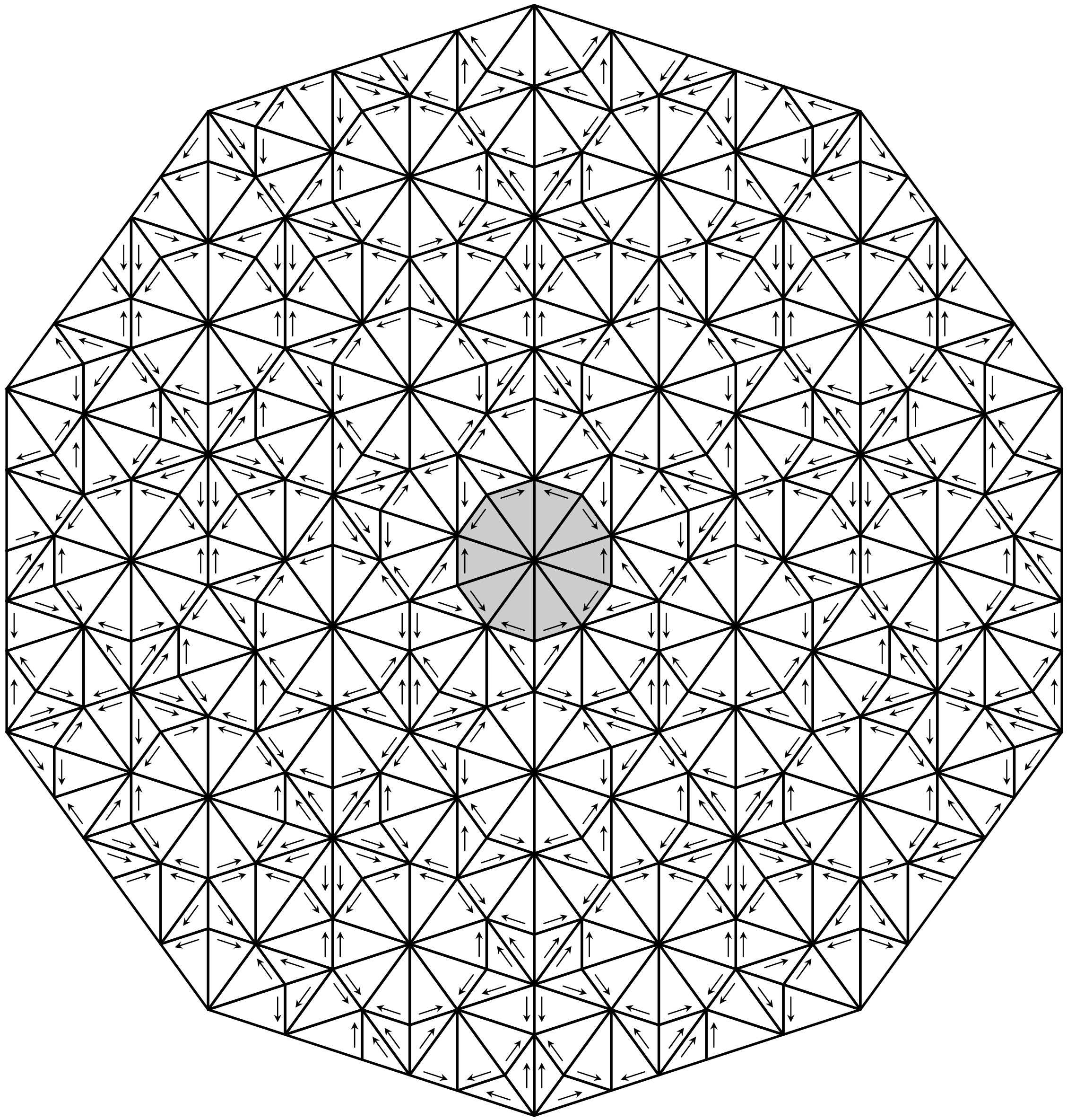
generating the Penrose–Robinson tiling and the Tübingen triangle tiling

Inflation matrix

$$M_{\mathrm{T}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 ,$$

with PF eigenvalue τ^2. The area ration of the prototiles is τ, and the frequency vector is $\nu = (\tau - 1, 2 - \tau)^T$. The larger triangles cover $\frac{1}{5}(2 + \tau) \approx 0.7236$ of the plane.

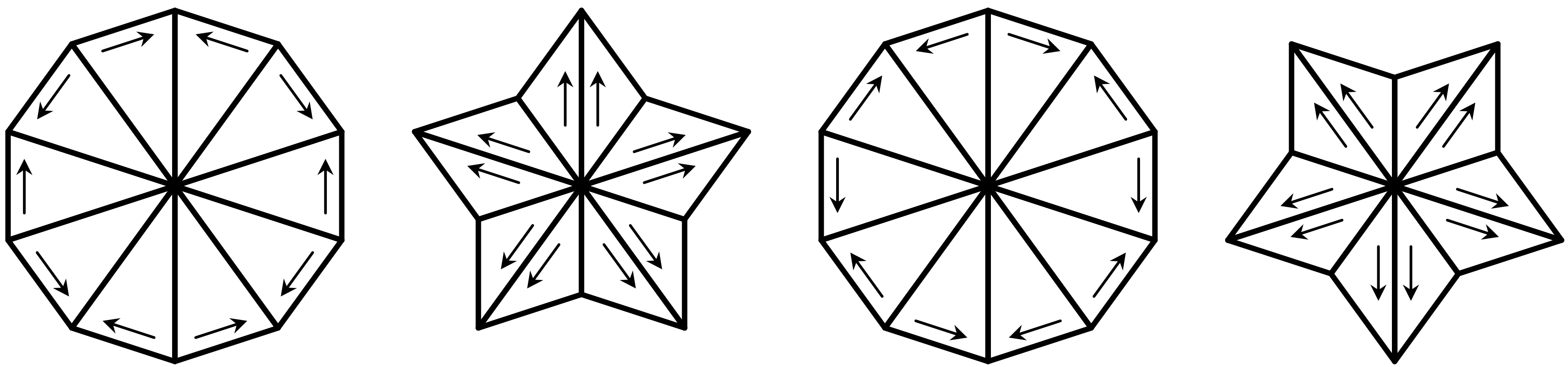
3.9 Penrose and TTT tilings



3.9 Penrose and TTT tilings

central patch of 4-cycle under the PRT inflation

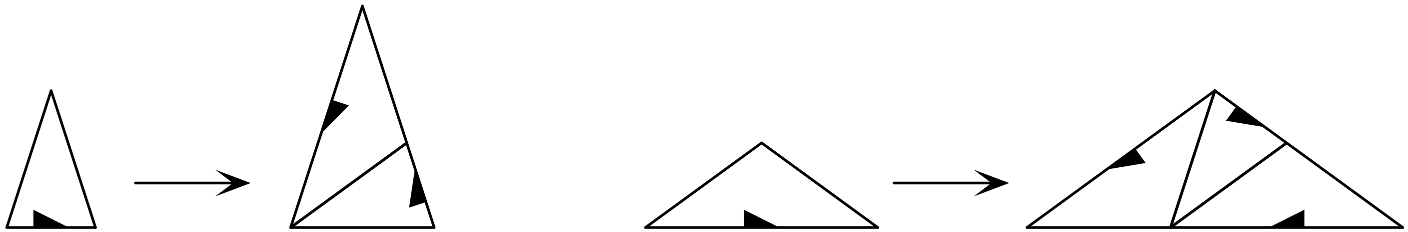
⟹ 4 fixed point tilings with individual fivefold symmetry



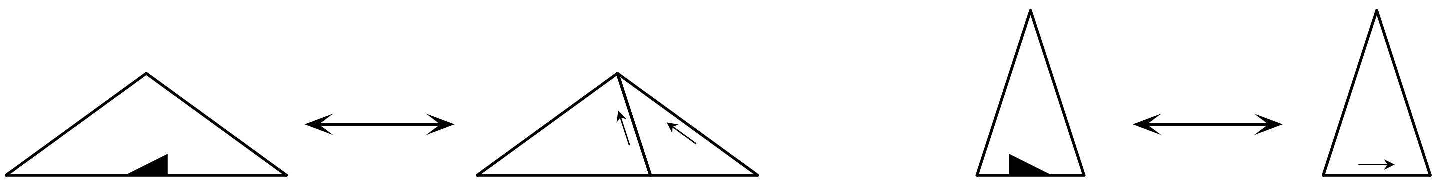
Proposition: The Penrose–Robinson tiling $\mathcal{T}$ is a linearly repetitive FLC tiling of $\mathbb{R}^2$ that is aperiodic, with an LIDS with inflation multiplier τ. The continuous hull is compact and satisfies $\mathbb{X}(\mathcal{T}) = \mathrm{LI}(\mathcal{T})$. The corresponding dynamical system $(\mathbb{X}(\mathcal{T}), \mathbb{R}^2, \mu)$ is strictly ergodic.

3.9 Penrose and TTT tilings

Robinson's triangle inflation:

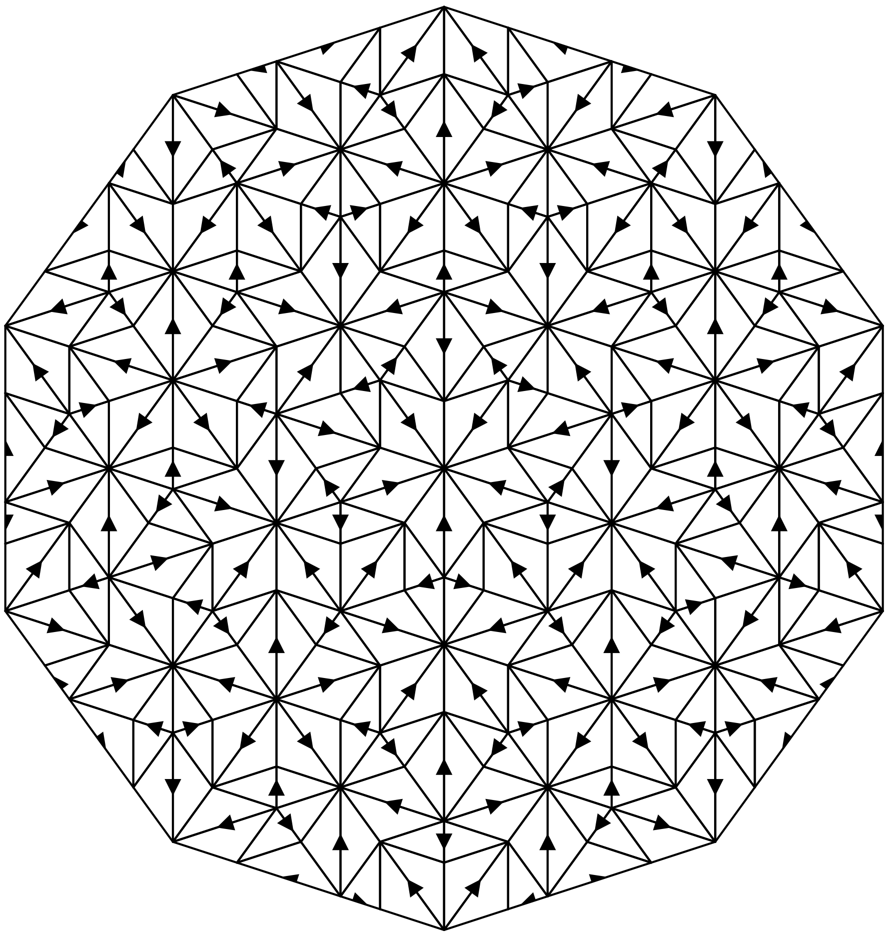


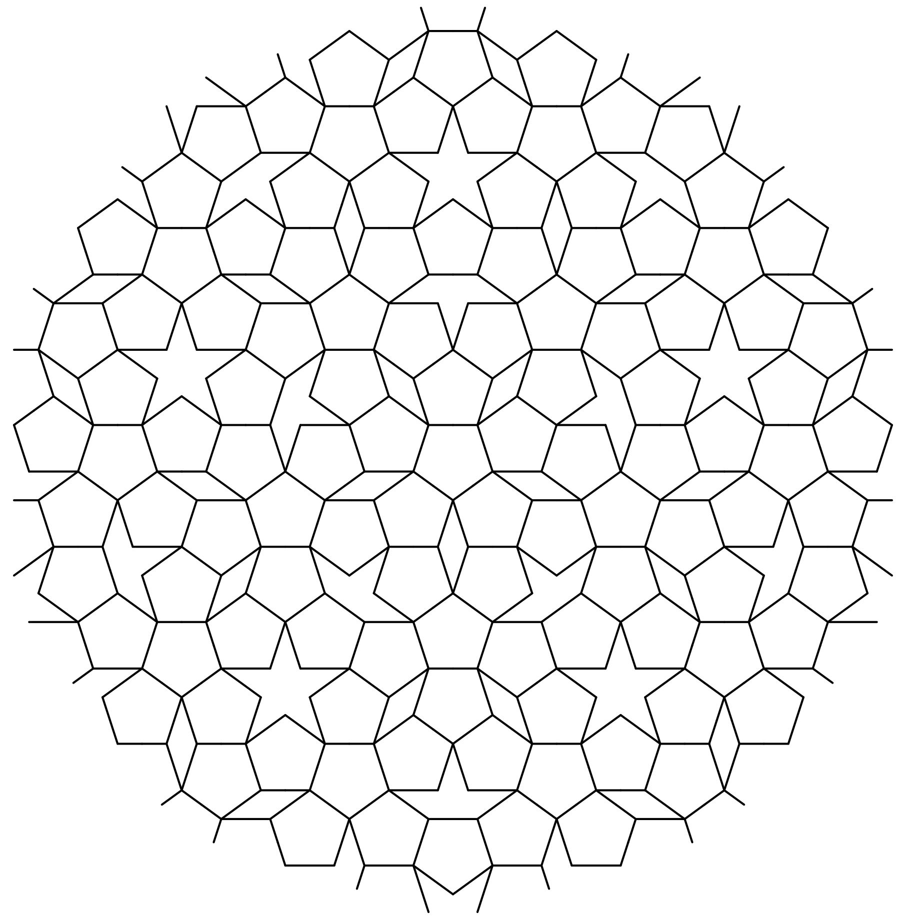
Relation to PRT tiling:



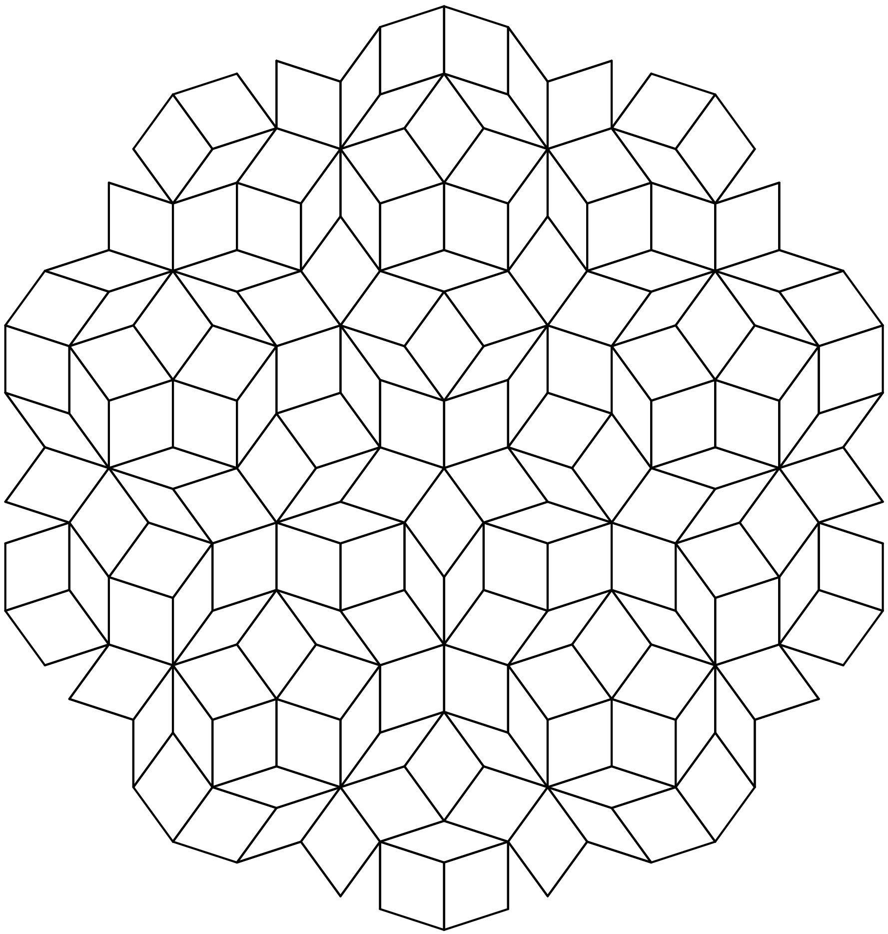
⟹ the two tilings are MLD

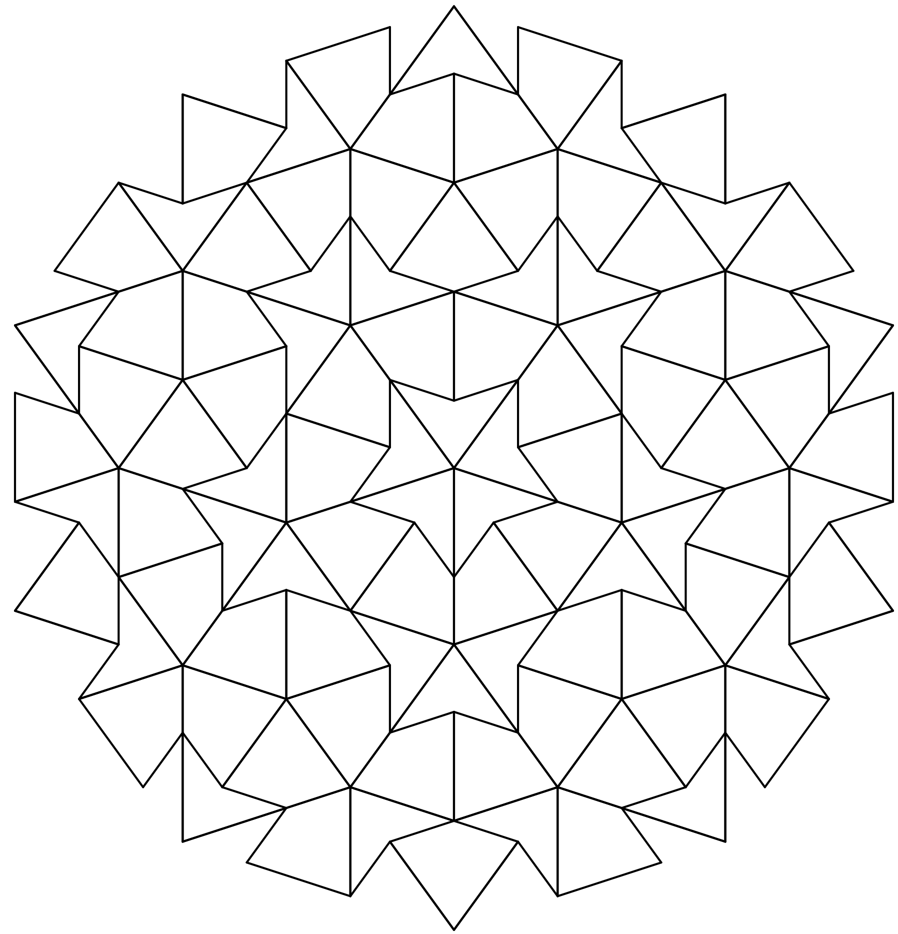
3.9 Penrose and TTT tilings





3.9 Penrose and TTT tilings





3.9 Penrose and TTT tilings

Theorem: The LI classes of the following planar tilings (in appropriate scale and relative orientation) belong to the same MLD class:

(1) Robinson's triangular version of the Penrose tiling;

(2) the Penrose–Robinson tiling (PRT);

(3) the rhombic Penrose tiling;

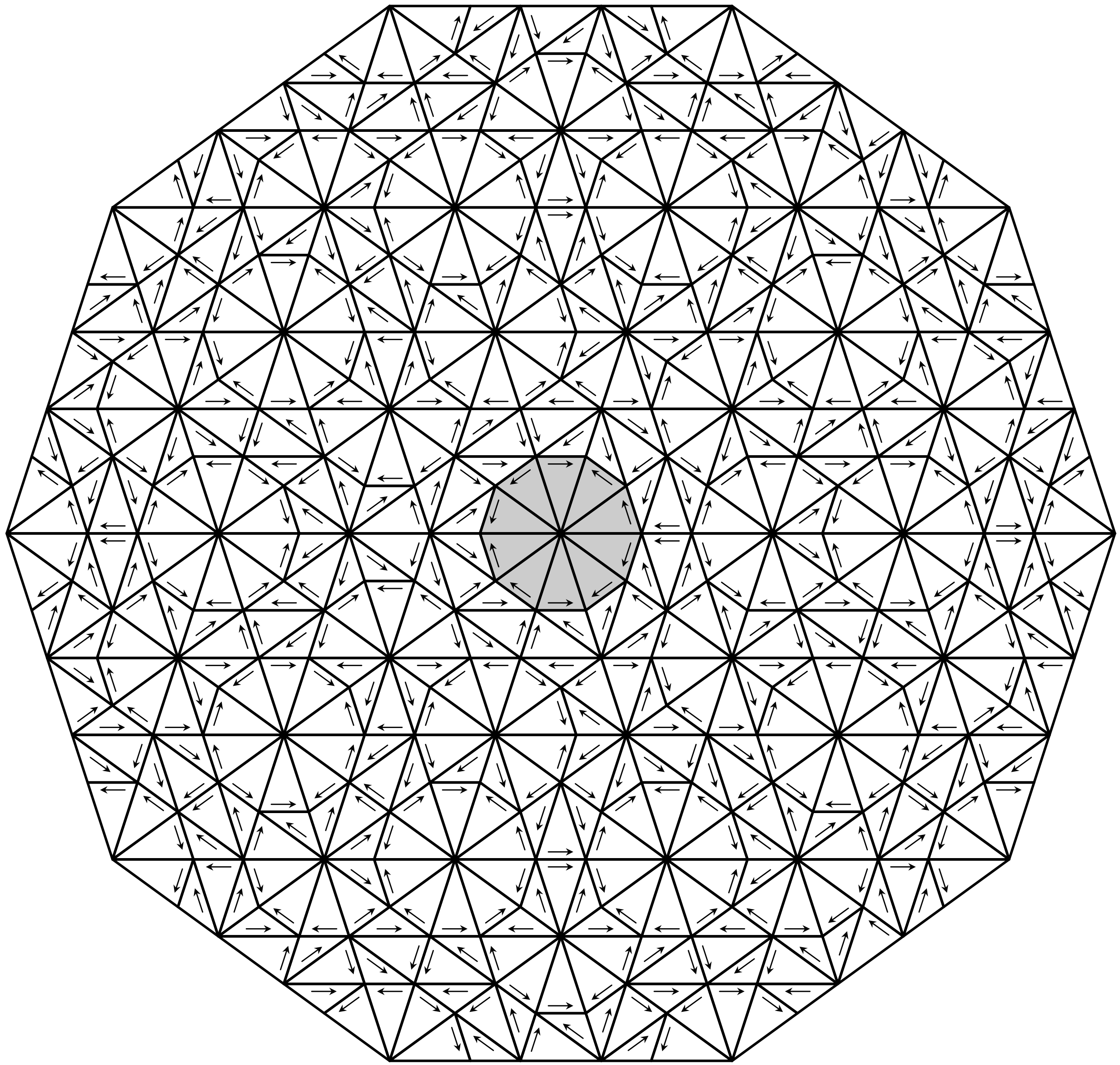
(4) the Penrose pentagon tiling;

(5) the kites and darts version of the Penrose tiling;

(6) the vertex point set of the rhombic Penrose tiling.

3.9 Penrose and TTT tilings

TTT:



3.9 Penrose and TTT tilings

The hulls of the rhombic Penrose tilings and the TTT are both D_{10}-symmetric, but neither hull contains tilings with individual tenfold symmetry.

There are individual D_5-symmetric Penrose tilings, but no such tiling exists in the TTT hull.

One can construct a local rule to turn each TTT element into a rhombic Penrose tiling, but the converse is not possible.

Consequently, the TTT and the Penrose tilings define distinct MLD classes.

3.10 Inflation tilings and periodicity

Theorem: Let $\mathcal{T} \subseteq \mathbb{R}^d$ be an FLC pattern that satisfies $\lambda\mathcal{T} \overset{\text{LD}}{\rightsquigarrow} \mathcal{T}$ for some $\lambda > 1$. If λ is irrational, the pattern $\mathcal{T}$ is non-periodic.

Proof:

Assume there there is a nontrivial period t,
so $t + \mathcal{T} = \mathcal{T}$ for some $0 \neq t \in \mathbb{R}^d$

$\implies \lambda t + \lambda\mathcal{T} = \lambda\mathcal{T}$

$\lambda\mathcal{T} \overset{\text{LD}}{\rightsquigarrow} \mathcal{T} \implies \lambda t$ is also a period of $\mathcal{T}$.

λ irrational

$\implies$ group of translations generated by t and λt is dense in $\mathbb{R}t$

$\implies$ contradiction. ∎