Aperiodic Order

Part 2

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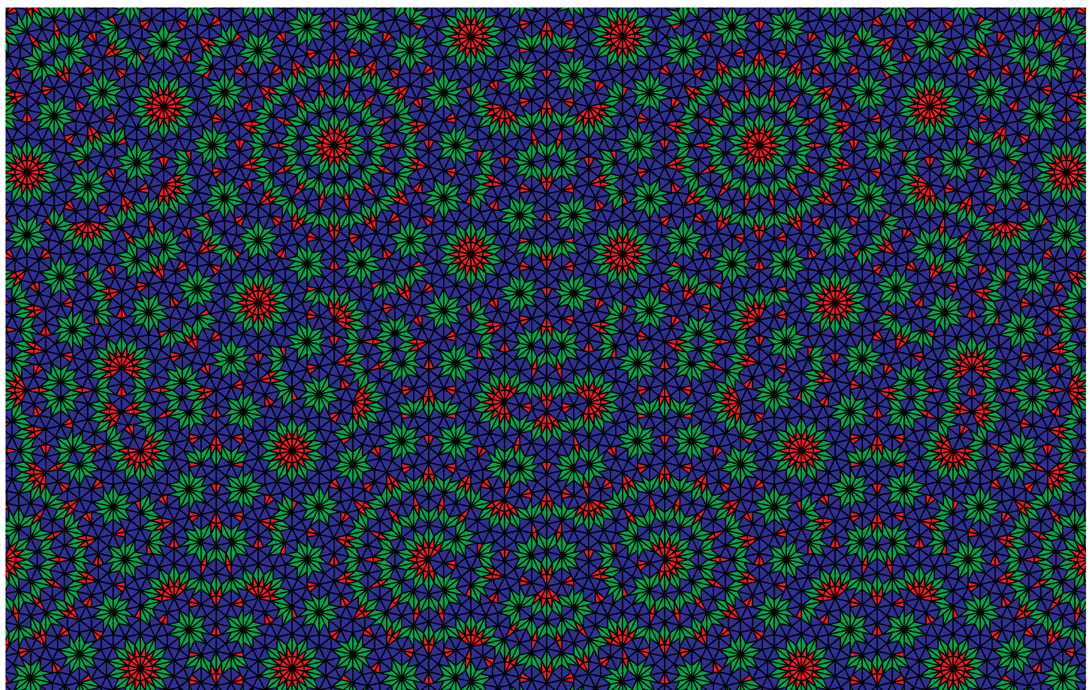


2.1 Substitution Rules

Consider n-letter alphabet $\mathcal{A}_n = \{a_i \mid 1 \leq i \leq n\}$

Free group $F_n := \langle a_1, \ldots, a_n \rangle$

A *general substitution rule* ϱ on an n-letter alphabet $\mathcal{A}_n$ is an endomorphism of the corresponding free group F_n.

This means $\varrho(uv) = \varrho(u)\varrho(v)$ and $\varrho(u^{-1}) = \big(\varrho(u)\big)^{-1}$.

Usually, we only consider substitution rules ϱ where the images $\varrho(a_i)$ of the letters contain no negative powers of the letters, but sometimes it can be advantageous to use a more general setting.

Substitution matrix $M(\varrho) \in \mathrm{Mat}(n, \mathbb{Z})$ defined by

$$\big(M(\varrho)\big)_{i,j} \;=\; \mathrm{card}_{a_i}\big(\varrho(a_j)\big).$$

2.2 Fibonacci Substitution

Example: Fibonacci's Rabbits

Liber Abaci (1202) by Leonardo of Pisa (Fibonacci)

2.2 Fibonacci Substitution

Substitution rule and matrix

$$\varrho: \quad \begin{array}{l} a \mapsto ab \\ b \mapsto a \end{array} \qquad M_\varrho = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

One-sided fixed point $w = \varrho(w)$ by iteration of ϱ on $w^{(0)} = a$:

$$a \mapsto ab \mapsto aba \mapsto abaab \mapsto abaababa \mapsto \ldots \mapsto w^{(n)} \xrightarrow{n\to\infty} w$$

Fibonacci numbers:

$|w^{(n)}| = f_{n+2}$ with $\mathrm{card}_a(w^{(n)}) = f_{n+1}$ and $\mathrm{card}_b(w^{(n)}) = f_n$

where $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$

Golden ratio:

$$\lim_{i\to\pm\infty} \frac{f_{i+1}}{f_i} = \frac{1 \pm \sqrt{5}}{2} = \begin{cases} \tau \\ \tau' \end{cases}$$

2.2 Fibonacci Substitution

Recursion: $w^{(n+1)} = w^{(n)}w^{(n-1)}$

Proof:

We have $w^{(2)} = aba = w^{(1)}w^{(0)}$

Induction: assume $w^{(n+1)} = w^{(n)}w^{(n-1)}$ and apply ϱ:

$$\varrho(w^{(n+1)}) = w^{(n+2)} = \varrho(w^{(n)}w^{(n-1)})$$
$$= \varrho(w^{(n)})\varrho(w^{(n-1)}) = w^{(n+1)}w^{(n)} \qquad \blacksquare$$

Two-sided Fibonacci sequence

$$a|a \xmapsto{\varrho} \underline{\underline{ab}}|ab \xmapsto{\varrho} a\underline{ba}|aba$$
$$\xmapsto{\varrho} aba\underline{\underline{ab}}|abaab \xmapsto{\varrho} abaaba\underline{ba}|abaababa$$
$$\xmapsto{\varrho} abaababaaba\underline{\underline{ab}}|abaababaabaab \xmapsto{\varrho} \cdots$$

limiting 2-cycle ▷ two fixed points under ϱ^2

2.3 Substitution Sequences

A (non-negative) substitution rule ϱ on a finite alphabet $\mathcal{A}_n$ is called

- *irreducible* when, for each index pair (i,j), there exists some $k \in \mathbb{N}$ such that a_j is a subword of $\varrho^k(a_i)$;
- *primitive* when some $k \in \mathbb{N}$ exists such that every a_j is a subword of each $\varrho^k(a_i)$.

⟶ ϱ is irreducible or primitive if and only if M_ϱ is an irreducible or a primitive non-negative integer matrix.

Number of letters under ℓ-fold substitution

$$\begin{pmatrix} \mathrm{card}_a(\varrho^\ell(u)) \\ \mathrm{card}_b(\varrho^\ell(u)) \end{pmatrix} = M_\varrho^\ell \begin{pmatrix} \mathrm{card}_a(u) \\ \mathrm{card}_b(u) \end{pmatrix},$$

⟶ right eigenvector for leading eigenvalue of M_ϱ encodes letter frequencies in the limit as $\ell \to \infty$

2.3 Substitution Sequences

Let ϱ be a substitution rule on a finite alphabet $\mathcal{A}_n$. A finite word is called *legal* for ϱ, if it occurs as a subword of $\varrho^k(a_i)$ for some $1 \le i \le n$ and some $k \in \mathbb{N}$.

Infinite (one-sided) $w = w_0 w_1 w_2 w_3 \ldots \in \mathcal{A}_n^{\mathbb{N}_0}$ and
bi-infinite (two-sided) $w = \ldots w_{-2} w_{-1} | w_0 w_1 w_2 \ldots \in \mathcal{A}_n^{\mathbb{Z}}$
sequences (words)

Local topology: two sequences w, w' are close when they agree on a large region around index 0

Convergence of a sequence of finite words (of increasing length) is implicitly considering them as embedded objects in $\mathcal{A}_n^{\mathbb{N}_0}$ or $\mathcal{A}_n^{\mathbb{Z}}$.

2.3 Substitution Sequences

Shift operator S acts on $\mathcal{A}_n^{\mathbb{N}_0}$ or $\mathcal{A}_n^{\mathbb{Z}}$ by $(Sw)_i := w_{i+1}$

An S-invariant closed subset $X \subseteq \mathcal{A}_n^{\mathbb{N}_0}$ or $X \subseteq \mathcal{A}_n^{\mathbb{Z}}$ is called a one-sided or a two-sided *shift space*.

A bi-infinite word w a *fixed point* of a primitive substitution ϱ if $\varrho(w) = w$ and $w_{-1}|w_0$ is a legal word of ϱ.

Given $w \in \mathcal{A}_n^{\mathbb{Z}}$, the shift space $\mathbb{X}(w) := \overline{\{S^i w \mid i \in \mathbb{Z}\}}$ is called the (two-sided, symbolic or discrete) *hull* of w.

$\bigl(\mathbb{X}(w), \mathbb{Z}\bigr)$ is a *topological dynamical system* with the continuous $\mathbb{Z}$-action of the shift on the compact space $\mathbb{X}(w)$, and the additional action of the substitution ϱ on $\mathbb{X}(w)$, which is continuous as well.

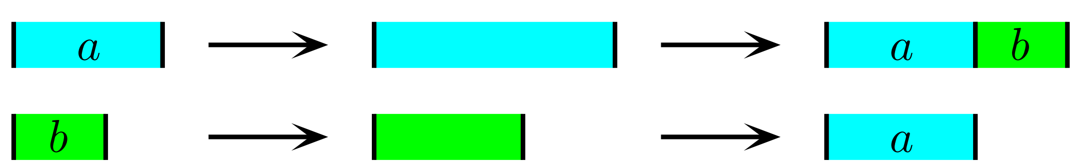
2.4 Geometric Inflation

For a primitive substitution ϱ on a finite alphabet with substitution matrix M_ϱ and PF eigenvalue λ, the associated *geometric inflation rule* with inflation multiplier λ is obtained by turning the letters a_i into closed intervals (the *prototiles*) with lengths proportional to the entries of the left PF eigenvector of M_ϱ, and by dissecting the λ-inflated prototiles into copies of the original ones, respecting the order specified by ϱ.

Example: Fibonacci substitution $\lambda = \tau = \frac{1}{2}(1+\sqrt{5})$

Left and right PF eigenvectors are proportional to $(\tau, 1)$

- ⟹ frequency of a is τ times frequency of b
- ⟹ geometric realisation by intervals of length ratio $\tau : 1$.



2.5 Local indistinguishability

Two words u and v in the same alphabet are *locally indistinguishable* (LI), denoted by $u \overset{\mathrm{LI}}{\sim} v$, when each finite subword of u is also a subword of v and vice versa.

The *LI class* of a word $w \in \mathcal{A}^{\mathbb{Z}}$ is $\mathrm{LI}(w) := \{z \in \mathcal{A}^{\mathbb{Z}} \mid z \overset{\mathrm{LI}}{\sim} w\}$

Lemma: If w is a bi-infinite word, its LI class is contained in the hull of w, and one has $\mathbb{X}(w) = \overline{\mathrm{LI}(w)}$. In particular, $\mathbb{X}(u) = \mathbb{X}(v)$ holds for any two bi-infinite words $u \overset{\mathrm{LI}}{\sim} v$.

Proof: Let $z \in \mathrm{LI}(w)$, so $z \overset{\mathrm{LI}}{\sim} w$.

For $m \in \mathbb{N}$, the subword $z_{[-m,m]}$ of length $2m+1$ occurs in w.

Define shifts j_m such that $\left(S^{j_m} w\right)_{[-m,m]} = z_{[-m,m]}$.

$\left(S^{j_m} w\right)_{m\in\mathbb{N}}$ converges to $z \implies z \in \mathbb{X}(w) \implies \mathrm{LI}(w) \subseteq \mathbb{X}(w)$.

$\mathbb{X}(w) = \overline{\{S^i w \mid \in \mathbb{Z}\}} \subseteq \overline{\mathrm{LI}(w)} \subseteq \overline{\mathbb{X}(w)} = \mathbb{X}(w)$. ∎

2.6 Hulls

A two-sided shift space $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}}$ is called *minimal* when, for all $w \in \mathbb{X}$, the shift orbit $\{S^i w \mid i \in \mathbb{Z}\}$ is dense in $\mathbb{X}$.

Proposition: If w is a bi-infinite word in the finite alphabet $\mathcal{A}$, with LI class $\mathrm{LI}(w)$ and hull $\mathbb{X}(w)$, the following assertions are equivalent.

(1) $\mathbb{X}(w)$ is minimal;

(2) $\mathrm{LI}(w)$ is closed;

(3) $\mathbb{X}(w) = \mathrm{LI}(w)$.

Proof: (2) $\Longleftrightarrow$ (3) follows from previous lemma.

For remaining statements, consider additional elements

prove proceeds by contradiction. ∎

2.7 Fixed Points

Lemma: If ϱ is a primitive substitution rule on a finite alphabet $\mathcal{A}_n$ with $n \geq 2$, there exists some $k \in \mathbb{N}$ and some $w \in \mathcal{A}_n^{\mathbb{Z}}$ such that w is a fixed point of ϱ^k (which means that $w_{-1}w_0$ is legal and $\varrho^k(w) = w$).

Proof: Assume that $|\varrho(a_i)| > 1$ for all $1 \leqslant i \leqslant n$.

Define $g \colon \mathcal{A}^2 \to \mathcal{A}^2$ by $g(xy) = \varrho(x)_{|\varrho(x)|-1}\varrho(y)_0$.

Start with any legal two-letter word

$\Longrightarrow$ all images are legal

$\Longrightarrow$ after n^2 steps one word xy must be repeated

$\Longrightarrow$ there is $1 \leqslant k \leqslant n^2 - 1$ s.t. $g^k(xy) = xy$

$\Longrightarrow$ $\varrho^k(x|y) = \dots x|y \dots$

$\Longrightarrow$ seed xy produces fixed point under ϱ^k. ∎

2.7 Fixed Points

Proposition: Let ϱ be a primitive substitution rule on a finite alphabet $\mathcal{A}$. Then, any two bi-infinite fixed points u and v of ϱ are LI. The same conclusion holds if u and v are fixed points of possibly different positive powers of ϱ.

Proof: Two fixed points with $\varrho^k(u) = u$ and $\varrho^\ell(v) = v$.

$\implies \varrho^{\mathrm{lcm}(k,\ell)}(u) = u$ and $\varrho^{\mathrm{lcm}(k,\ell)}(v) = v$. Let $\widetilde{\varrho} = \varrho^{\mathrm{lcm}(k,\ell)}$.

Choose $a \in \mathcal{A}$ and let w be a finite subword of u

$\implies w$ contained in $\widetilde{\varrho}^p(u_{-1}|u_0)$ for some $p \in \mathbb{N}$.

$u_{-1}u_0$ legal

$\implies$ subword of $\widetilde{\varrho}^q(a)$ for some $q \in \mathbb{N}$ (primitivity).

a is subword of v (primitivity)

$\implies \widetilde{\varrho}^{p+q}(a)$ subword of $v \implies w$ subword of v.

$\implies u \overset{\text{LI}}{\sim} v$. ∎

2.8 Repetitivity

A bi-infinite word w (over a finite alphabet) is *repetitive* when every finite subword of w reappears in w with bounded gaps.

Proposition: If w is a bi-infinite word in a finite alphabet, the hull $\mathbb{X}(w)$ is minimal if and only if w is repetitive.

Lemma: Any bi-infinite fixed point of a primitive substitution on a finite alphabet is repetitive.

Proof: Alphabet $\mathcal{A} = \{a_1, \dots, a_n\}$.

w bi-infinite fixed point of ϱ (primitive)

$\Longrightarrow$ $k \in \mathbb{N}$ s.t. a_1 is subword of $\varrho^k(a_i)$ for all i

$\Longrightarrow$ a_1 occurs in w with bounded gaps

u subword of w $\Longrightarrow$ u subword of $\varrho^\ell(a_1)$ for some ℓ

$\Longrightarrow$ u appears in w with bounded gaps. ∎

2.8 Repetitivity

Theorem: Every primitive substitution rule on a finite alphabet possesses a unique hull. This hull consists of a single, closed LI class.

Proof:

ϱ primitive $\implies$ bi-infinite fixed point w of ϱ^k.

Hull $\mathbb{X}(w)$ is independent of choice of fixed point.

w repetitive

$$\implies \mathbb{X}(w) = \overline{\mathrm{LI}(w)} = \mathrm{LI}(w)$$

$\implies \mathbb{X}(w)$ minimal ∎

A bi-infinite sequence w in a finite alphabet is called (topologically) *aperiodic* when $\mathbb{X}(w)$ contains no periodic sequence. A primitive substitution rule ϱ is *aperiodic* when its unique hull contains no periodic element.

2.9 Invariant Measures

Consider a two-sided shift space $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}}$ for a finite alphabet $\mathcal{A}$. For any $w \in \mathbb{X}$, the point (or Dirac) measure δ_w is an element of the set of probability measures $\mathbb{P}(\mathbb{X})$ on $\mathbb{X}$. It is defined by $\delta_w(A) = 1$ if w is an element of $A \subseteq \mathcal{A}^{\mathbb{Z}}$, and $\delta_w(A) = 0$ otherwise. Clearly,

$$\mu_N := \frac{1}{2N+1} \sum_{i=-N}^{N} \delta_{S^i w}$$

defines a sequence in $\mathbb{P}(\mathbb{X})$. It has a converging subsequence, whose limit μ is then a *shift invariant* element of $\mathbb{P}(\mathbb{X})$, which means that $S.\mu(A) := \mu(S^{-1}(A)) = \mu(A)$ for all Borel sets A.

A shift invariant probability measure μ on $\mathbb{X}$ is called *ergodic* (with respect to the $\mathbb{Z}$-action of the shift) if the measure $\mu(A)$ of any invariant Borel set A is either 0 or 1.

2.9 Invariant Measures

A system with a unique invariant probability measure is called *uniquely ergodic*. If a uniquely ergodic system is also minimal, it is called *strictly ergodic*.

Theorem: Let ϱ be a primitive substitution on a finite alphabet. Its hull $\mathbb{X}$ is then strictly ergodic under the $\mathbb{Z}$-action of the shift.

Proposition: Let$\mathcal{A}$ be a finite alphabet and let $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}}$ be a two-sided shift space. Then, the dynamical system of $\mathbb{X}$ under the shift action is uniquely ergodic if and only if the frequencies of all finite words exist uniformly, for each element of $\mathbb{X}$. Moreover, it is strictly ergodic if and only if all frequencies exist uniformly and are positive.

2.10 Aperiodic Sequences

Theorem: Let w be a bi-infinite word in a finite alphabet that is repetitive and non-periodic. Then, the hull $\mathbb{X}(w)$ is uncountable, and even contains uncountably many pairwise disjoint translation orbits.

Corollary: The symbolic hull of a repetitive word over a finite alphabet consists of either one (periodic case) or uncountably many (non-periodic case) pairwise disjoint $\mathbb{Z}$-orbits under the shift action. In the periodic case, the hull itself is a finite set.

Proposition: The symbolic hull of a repetitive, bi-infinite word over a finite alphabet is either finite or a Cantor set.

2.10 Aperiodic Sequences

Theorem: Let ϱ be a primitive substitution rule on a finite alphabet with substitution matrix M_ϱ, and let w be a bi-infinite fixed point of ϱ. If the PF eigenvalue of M_ϱ is irrational, the sequence w is aperiodic.

Proof:

Assume to the contrary that w has a non-trivial finite period

$\Longrightarrow$ all letter frequencies are rational

$\Longrightarrow$ entries of right PF eigenvector are rational

$\Longrightarrow$ PF eigenvalue is rational — contradiction

$\mathbb{X}(w) = \mathrm{LI}(w)$

$\Longrightarrow$ hull cannot contain periodic elements ■

2.11 Further Examples

Noble means substitutions:

$$\varrho_p : \quad \begin{array}{l} a \mapsto a^p b \\ b \mapsto a \end{array}$$

with $\lambda = \frac{1}{2}\big(p + \sqrt{p^2 + 4}\,\big) = [p; p, p, p, \ldots]$, a PV unit.

Period doubling substitution:

$$\varrho_{\rm pd} : \quad \begin{array}{l} a \mapsto ab \\ b \mapsto aa \end{array}$$

with $\lambda = 2$.

Thue–Morse substitution:

$$\varrho_{\rm TM} : \quad \begin{array}{l} a \mapsto ab \\ b \mapsto ba \end{array}$$

with $\lambda = 2$.

2.12 Thue–Morse Sequence

Bi-infinite fixed point sequences:

$$a|a \xmapsto{\varrho^2} abba|abba \xmapsto{\varrho^2} abbabaabbaababba|abbabaabbaababba$$
$$\xmapsto{\varrho^2} \cdots \longrightarrow w$$

$$b|a \xmapsto{\varrho^2} baab|abba \xmapsto{\varrho^2} baababbaabbabaab|abbabaabbaababba$$
$$\xmapsto{\varrho^2} \cdots \longrightarrow w'$$

forming a two-cycle $w \xmapsto{\varrho} w' \xmapsto{\varrho} w$ with $w \overset{\mathrm{LI}}{\sim} w'$

$w = u|v$ and $w' = \bar{u}|v$ ($\bar{a} = b$, $\bar{b} = a$), v satisfies

$$v_{2i} = v_i \quad \text{and} \quad v_{2i+1} = \bar{v}_i$$

$$v = v_0 v_2 v_4 \ldots \quad \text{and} \quad \bar{v} = v_1 v_3 v_5 \ldots$$

$$v_i = \begin{cases} a, & \text{if the binary digits of } i \text{ sum to an even number,} \\ b, & \text{otherwise.} \end{cases}$$

2.12 Thue–Morse Sequence

Proposition: The one-sided fixed point v of the Thue–Morse substitution does not contain any subword of the form zzz_0 with a non-empty finite word z.

Proof: $v_i = v_{i+1} \implies i$ odd $\implies v$ cannot contain aaa or bbb.

Also, $ababa$ or $babab$ cannot occur
$\implies$ any subword of length $\geqslant 5$ contains aa or bb.

Proof by contradiction: assume z subword s.t. zzz_0 subword of v, of minimal length $|z| = \ell$.

If ℓ is odd, $|zzz_0| > 5$ contains aa or bb at least twice, starting at odd positions, so at even distances. One distance must be $\ell \implies$ contradiction.

If ℓ is even, then $z' = z_0 z_2 \ldots z_{\ell-2}$ would be a shorter word, in contradiction to ℓ being minimal. ■

As v is (strongly) cubefree (or overlap-free), the Thue–Morse substitution is aperiodic.