2018 LTCC Course on **Aperiodic Order**

Examination Worksheet

Consider the substitution rule

$$\varrho : \quad \begin{array}{l} a \mapsto abb \\ b \mapsto abbb \end{array}$$

on the two-letter alphabet $\{a, b\}$.

Question 1: Show that ϱ is irreducible and primitive. Show that the iteration of ϱ on a suitable legal two-letter seed leads to a bi-infinite fixed point word $w = \varrho(w)$.

Question 2: Determine the substitution matrix M for ϱ and compute its leading eigenvalue λ. Calculate the right and left eigenvector of M to the eigenvalue λ. What is the frequency of letters a and b in a fixed point word? What are the natural lengths of intervals in the geometric realislation of ϱ as an inflation rule on intervals, when choosing the longer interval to have length 1?

Question 3: Consider the point set $\varLambda = \varLambda_a \dot{\cup} \varLambda_b$ of all left interval endpoints (of type a and b) in the interval tiling corresponding to the fixed point word w, and calculate the average distance of points. What is the density of $\varLambda$?

Question 4: Consider the Minkowski embedding of $\mathbb{Z}[\sqrt{3}\,]$ in $\mathbb{R}^2$, and show that it is a rectangular lattice $\mathcal{L}$ generated by $(1,1)^T$ and $\bigl(\sqrt{3}, -\sqrt{3}\,\bigr)^T$. Compute the density of $\mathcal{L}$.

Question 5: Let $\curlywedge(W) := \{x \in \pi(\mathcal{L}) \mid x^\star \in W\}$ denote a model set with an interval as window W, where π denotes the projection on the first coordinate (the physical space) and the $\star$-map is given by algebraic conjugation $\sqrt{3} \mapsto -\sqrt{3}$. What volume (length) must W have to produce a model set of the same density as $\varLambda$? Given the frequency of letters a and b in w computed in Question 2, what would the length ratio of the corresponding windows W_a and W_b for $\varLambda_a$ and $\varLambda_b$ have to be, assuming these were model sets?

Question 6: From the substitution rule ϱ, derive scaling relations for the point sets $\varLambda_a$ and $\varLambda_b$ under scaling by λ. Assuming that these are model sets with windows W_a and W_b, what are the corresponding relations that the windows have to satisfy? Check that these equations are satisfied by the windows $W_a = (-1, 0]$ and $W_b = (-\lambda, -1]$.

Question 7: Using the windows of Question 6, calculate the frequency of the interval sequence corresponding to the word *bab*. Without doing any calculation, what would you expect the frequencies for *bbab*, *babb* and *bbabb* to be, and why?

Question 8: As $\varLambda = \curlywedge(W)$ is a model set, it is pure point diffractive, with Bragg peaks supported on the projection $\pi(\mathcal{L}^*)$ of the dual lattice $\mathcal{L}^*$. Calculate a lattice basis for the dual lattice, and determine the set $\pi(\mathcal{L}^*)$.