2018 LTCC Course on **Aperiodic Order**

Worksheet 4

Consider the square lattice $\mathbb{Z}^2 = \{(m, n) \mid m, n \in \mathbb{Z}\}$, and define a family of linear functions by $f_s(x) = \frac{x}{\tau} + s$, where $\tau = (1 + \sqrt{5})/2$ is the golden ratio.

Exercise 1: Which lattice points of $\mathbb{Z}^2$ do the graphs of f_0, f_1 and $f_{-\frac{1}{\tau}}$ meet? In other words, which pairs $(m, n) \in \mathbb{Z}^2$ satisfy $n = f_s(m)$ for $s \in \{0, 1, -\frac{1}{\tau}\}$?

Exercise 2: For an arbitrary lattice point $(m, n) \in \mathbb{Z}^2$, calculate its orthogonal projection onto the line $y = f_0(x)$. Compute the distance of the projected point from the origin, and show that this distance is an element of $\mathbb{Z}[\tau]/\sqrt{\tau + 2}$.

Exercise 3: Consider the strip

$$S := \{(x, y) \in \mathbb{R}^2 \mid f_{-\frac{1}{\tau}}(x) \le y < f_1(x)\}.$$

Argue that, for any given $p \in \mathbb{Z}$, there are at least one and at most two lattice points $(m, n) \in S \cap \mathbb{Z}^2$ with $m = p$. Find all lattice points $(m, n) \in S \cap \mathbb{Z}^2$ with $0 \le m \le 6$.

Exercise 4: Consider the 'staircase' obtained by connecting the lattice points in the list from Exercise 3 that differ by $(1, 0)$ (horizontal step) or $(0, 1)$ (vertical step). Encode each horizontal step by a letter a and each vertical step by a letter b. Show that the resulting word is a legal word for the Fibonacci substitution $a \mapsto ab$, $b \mapsto a$.

Exercise 5: Argue that the projection of the set $\mathbb{Z}^2 \cap S$ onto the line $y = f_0(x)$ is a model set. What is the corresponding window?