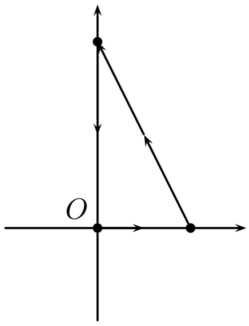
Example

Find $\oint_C \boldsymbol{F}.d\boldsymbol{r}$ where $\boldsymbol{F} = (2x+y)\boldsymbol{i} + (2x-y)\boldsymbol{j}$ and C is the triangle $y = 0, y = 1-2x$ and $x = 0$ traversed in an anticlockwise direction.

Solution

We start by drawing the triangle C, putting the arrows on to indicate the direction of integration.



We now rewrite the line integral using $r = x\boldsymbol{i} + y\boldsymbol{j}$ so that $dr = dx\boldsymbol{i} + dy\boldsymbol{j}$. The properties of a scalar product then mean that:

$$\oint_C \boldsymbol{F}.d\boldsymbol{r} = \int_C (2x+y)dx + (2x-y)dy.$$

We now integrate along each side of the triangle in turn:

- $y = 0$: we need to change the variable of integration in the second part. We do this by by multiplying by $\frac{dy}{dx}$, in this case 0.

$$\begin{aligned}\int_C (2x+y)dx + (2x-y)dy &= \int_{x=0}^{x=\frac{1}{2}} 2xdx + 2x\frac{dy}{dx}dx \\ &= \int_{x=0}^{x=\frac{1}{2}} 2xdx + 2x \times 0dx \\ &= \left[x^2\right]_0^{\frac{1}{2}} \\ &= \frac{1}{4}\end{aligned}$$

- $y = 1-2x$: we need to change the variable of integration in the second part. We do this by by multiplying by $\frac{dy}{dx}$, in this case $-2x$.

$$\begin{aligned}\int_C (2x+y)dx + (2x-y)dy &= \int_{x=\frac{1}{2}}^{x=0} (2x+(1-2x))dx + (2x-(1-2x))\frac{dy}{dx}dx \\ &= \int_{x=\frac{1}{2}}^{x=0} 1dx + (4x-1) \times (-2)dx \\ &= \int_{\frac{1}{2}}^{0} (3-8x)dx \\ &= \left[3x - 4x^2\right]_{\frac{1}{2}}^{0} \\ &= \frac{1}{2}\end{aligned}$$

- $x = 0$: we need to change the variable of integration in the first part this time. We do this by by multiplying by $\frac{dx}{dy}$, in this case 0.

$$\begin{aligned}\int_C (2x+y)dx + (2x-y)dy &= \int_{y=1}^{y=0} y\frac{dx}{dy}dy + (-y)dy \\ &= \int_{y=1}^{y=0} y \times 0dy - ydy \\ &= \left[-\frac{y^2}{2}]\right]_1^0 \\ &= -\frac{1}{2}\end{aligned}$$

To find the line integral we just add up these parts to obtain

$$\oint_C \boldsymbol{F}.d\boldsymbol{r} = \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = \frac{1}{4}.$$