Introduction

This is an example of working with Legendre polynomials.
It also introduces the idea of a generating function.

You don't need to know much about Legendre Polynomials,
just that for each integer $n \geq 0$, there is a function called $P_n(x)$.

David Griffel

Generating function for P_n (1)

Problem: Let $$g(u,x) = \sum_{0}^{\infty} u^n P_n(x) \text{ for } |x| \leq 1 \text{ and } |u| < 1$$

where P_n are the Legendre polynomials. (The series converges . . .)

Calculate $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial x}$, and show that $u\frac{\partial g}{\partial u} = (x-u)\frac{\partial g}{\partial x}$, given that

$$(x^2-1)\,P'_n(x) = n\,[\,xP_n(x) - P_{n-1}(x)\,]$$

$$nP_n - (2n-1)\,xP_{n-1} + (n-1)\,P_{n-2} = 0.$$

Solution: $\frac{\partial g}{\partial u} = \sum_{1}^{\infty} nu^{n-1}P_n(x)$, and $\frac{\partial g}{\partial x} = \sum_{0}^{\infty} u^n P'_n(x) = \sum_{1}^{\infty} u^n P'_n(x)$

Using the given formula for P'_n, we get $$\frac{\partial g}{\partial x} = \sum_{1}^{\infty} \frac{nu^n\,[xP_n(x) - P_{n-1}(x)]}{x^2-1},$$

So $$(x-u)\frac{\partial g}{\partial x} = \sum_{1}^{\infty} \frac{nu^n x^2 P_n - nu^{n+1}xP_n - nu^n xP_{n-1} + nu^{n+1}P_{n-1}}{x^2-1}$$

We calculated

$$u\frac{\partial g}{\partial u} = \sum_1^\infty n u^n P_n(x) \tag{1}$$

and

$$(x-u)\frac{\partial g}{\partial x} = \sum_1^\infty \frac{nu^n x^2 P_n - nu^{n+1} x P_n - nu^n x P_{n-1} + nu^{n+1} P_{n-1}}{x^2-1} \tag{2}$$

We want . . .

The coeff. of u^n in (2) is $\left[nx^2P_n - (n-1)xP_{n-1} - nxP_{n-1} + (n-1)P_{n-2}\right]/(x^2-1)$

$$= \left[nx^2P_n - (2n-1)xP_{n-1} + (n-1)P_{n-2}\right]/(x^2-1).$$

Now use the given recurrence relation $nP_n - (2n-1)\,xP_{n-1} + (n-1)\,P_{n-2} = 0$.

Then the coefficient of u^n in (2) becomes

$$\left[nx^2P_n - nP_n\right]/(x^2-1) \;=\; nP_n.$$

This is the coefficient of u^n in (1), which proves that $\quad u\frac{\partial g}{\partial u} = (x-u)\frac{\partial g}{\partial x}$.

Problem, part 2: Show that $\;g(u,x) = (1-2xu+u^2)^{-1/2}\;$ for $-1 \le x \le 1$

given that: if a solution of

$$u\frac{\partial g}{\partial u} = (x-u)\frac{\partial g}{\partial x} \tag{3}$$

is zero when $x = 1$ then it is zero when $|x| < 1$.

You're also given that $P_n(1) = 1$ for all n.

Solution: Let $\gamma(u,x) = (1-2xu+u^2)^{-1/2}$. Easy calculus shows γ satisfies (3).

Also $\gamma(u,1) = (1-2u+u^2)^{-1/2} = (1-u)^{-1}$ for all u,

and $g(u,1) = \sum u^n P_n(1) = \sum u^n = (1-u)^{-1}$.

Since the PDE is linear, $g-\gamma$ is a solution, and $g-\gamma = 0$ when $x = 1$. Therefore $g-\gamma = 0$ everywhere, as required. We have shown that

$$(1-2xu+u^2)^{-1/2} = \sum_0^\infty u^n P_n(x)$$

$(1-2xu+u^2)^{-1/2}$ is a called a **generating function** for the Legendre polynomials.

The eqn

$$(1-2xu+u^2)^{-1/2} = \sum_0^\infty u^n P_n(x) \tag{4}$$

can be derived in other ways. Legendre *defined* the P_n in 1782 essentially by (4).

The generating fn is useful in various ways. For example, it can be used to obtain recurrence relations for the Legendre pols – assuming the gen. fn. wasn't derived using those relations.

Another application: (4) implies a nice integral formula for P_n

$$P_n(x) = \frac{1}{2\pi i} \oint \frac{u^{-n-1}}{\sqrt{1-2xu+u^2}}\, du \tag{5}$$

integrated round any anticlockwise oriented closed path round the origin.

Note: (5) suggests extending $P_n(x)$ to non-integer n (cf Rodrigues)

There are generating functions for Bessel fns and many other families of functions.