# Some integration examples

## Useful identities

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 & \tan^2 x + 1 &= \sec^2 x \\
\sin 2x &= 2\sin x\cos x & \cos 2x &= 2\cos^2 x - 1 = 1 - 2\sin^2 x \\
\cos(a+b) + \cos(a-b) &= 2\cos a\cos b & \sin(a+b) + \sin(a-b) &= 2\sin a\cos b \\
\int \cos x dx &= \sin x + C & \int \sin x dx &= -\cos x + C \\
\int \sec^2 dx &= \tan x + C & \int \tan x dx &= \ln|\sec x| + c
\end{aligned}$$

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1 & 1 - \tanh^2 x &= \mathrm{sech}^2 x \\
\sinh 2x &= 2\sinh x\cosh x & \cosh 2x &= 2\cosh^2 x - 1 = 2\sinh^2 x + 1 \\
\int \cosh x dx &= \sinh x + C & \int \sinh x dx &= \cosh x + C
\end{aligned}$$

## Examples

1. Find $\int_0^{\frac{\pi}{4}} 2\sin^2 2x dx$.

   We re-arrange the integrand using the formula $\cos 2x = 1 - 2\sin^2 x$. This is because we can integrate a double angle directly.

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} 2\sin^2 2x dx &= \int_0^{\frac{\pi}{4}} (1 - \cos 4x) dx \\
&= \left[ x - \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{4}.
\end{aligned}$$

2. Find $\int_0^{\frac{\pi}{2}} \sin^3 x dx$.

   We re-arrange the integrand using the formula $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^3 x dx &= \int \sin x(1 - \cos^2 x) dx \\
&= \int_0^{\frac{\pi}{2}} (\sin x - \sin x\cos^2 x) dx.
\end{aligned}$$

   Now we observe that $-\sin x$ is the derivative of $\cos x$ so that the second term is a sight integral of the form $\int f'(x) f^n(x) dx$.

$$\begin{aligned}
\text{Thus } \int_0^{\frac{\pi}{2}} \sin^3 x dx &= \int_0^{\frac{\pi}{2}} (\sin x - \sin x\cos^2 x) dx \\
&= \left[ -\cos x + \frac{\cos^3 x}{3} \right]_0^{\frac{\pi}{2}} \\
&= -\left( -1 + \frac{1}{3} \right) \\
&= \frac{2}{3}.
\end{aligned}$$

3. Find $\int_0^{\frac{\pi}{3}} 2\tan^3 x dx$.

We re-arrange the integrand using the formula $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}\int_0^{\frac{\pi}{3}} 2\tan^3 x dx &= \int_0^{\frac{\pi}{3}} (2\tan x)(\sec^2 x - 1)dx \\ &= \int_0^{\frac{\pi}{3}} (2\tan x \sec^2 x - 2\tan x)dx.\end{aligned}$$

Now we observe that $\sec^2 x$ is the derivative of $\tan x$ so that the first term is a sight integral of the form $\int f'(x)f^n(x)dx$.

$$\begin{aligned}\int_0^{\frac{\pi}{3}} 2\tan^3 x dx &= \int_0^{\frac{\pi}{3}} (2\sec^2 x \tan x - 2\tan x)dx \\ &= \Big[\tan^2 x - 2\ln|\sec x|\Big]_0^{\frac{\pi}{3}} \\ &= 3 - 2\ln 2.\end{aligned}$$

4. Find $\int_0^{\frac{\pi}{3}} 2\sin 2x \cos 3x dx$.

We re-arrange the integrand using the formula $\sin(a+b) + \sin(a-b) = 2\sin a \cos b$.

$$\begin{aligned}\int_0^{\frac{\pi}{3}} 2\sin 2x \cos 3x dx &= \int_0^{\frac{\pi}{3}} (\sin(5x) + \sin(-x))dx \\ &= \int_0^{\frac{\pi}{3}} (\sin(5x) - \sin(x))dx \\ &= \left[\frac{-\cos(5x)}{5} + \cos x\right]_0^{\frac{\pi}{3}} \\ &= -\frac{1}{10} + \frac{1}{2} + \frac{1}{5} - 1 = \frac{-2}{5}.\end{aligned}$$

5. Find $\int 2\sqrt{x^2-1}dx$, for $x \geq 1$.

It is often a good idea to try the substitutions $x = \cos u$ or $x = \cosh u$ for integrals containing the terms $\sqrt{1-x^2}$ or $\sqrt{x^2-1}$ respectively (where there is no factor $x$).

Since $x \geq 1$ let $x = \cosh u$. Then $\sqrt{x^2-1} = \sqrt{\cosh^2 u - 1} = \sqrt{\sinh^2 u} = \sinh u$ and $\frac{dx}{du} = \sinh u$.

$$\begin{aligned}\int 2\sqrt{x^2-1}dx &= \int 2\sinh^2 u du \\ &= \int (\cosh 2u - 1)du \text{ using the `double angle' formula} \\ &= \frac{\sinh 2u}{2} - u + C \\ &= \sinh u \cosh u - u + C \\ &= x\sqrt{x^2-1} - \cosh^{-1}(x) + C.\end{aligned}$$

Note that the integral is only defined if $x \leq -1$ or $x \geq 1$. In the former case we would let $x = -\cosh u$ and then proceed in the same way.