Introduction

This is a brief outline of the Frobenius method
for solving linear ODEs in series about a regular singular point.

It assumes familiarity with power series
and elementary linear ODEs.

David Griffel

Regular singular points

We consider the differential equation $$\boxed{u'' + p(z)u' + q(z)u = 0} \qquad (*)$$

Any 2nd order linear homogeneous ODE can be put into this form.

A **regular point** of $(*)$ is a point where p and q are analytic.

A **singular point** of $(*)$ is one that is not regular.

A **regular singular point** of $(*)$ is an isolated singular point z_0 such that $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ are both analytic at z_0. In other words, . . .

Examples:

1. $(1 - x^2)u'' - 2xu' + n(n+1)u = 0$, or $u'' - \left(\dfrac{2x}{1 - x^2}\right)u' + \left(\dfrac{n(n+1)}{1 - x^2}\right)u = 0$

   0 is a regular point, 1 and -1 are regular singular points.

2. $z^3u'' + z^2u' + u = 0$, or $u'' + z^{-1}u' + z^{-3}u = 0$.

   0 is an irregular singular point.

Near a regular pt a, a soln behaves like $(z-a)^k$ for integer $k \geq 0$.

Georg Frobenius, 1842-1917

Near a singular pt, we'd expect a soln. to be singular.

The simplest type of singular behaviour at a is $(z-a)^\lambda$ for some λ.

So we look for a soln of the form $(z-a)^\lambda f(z)$ where f is analytic at a.

Expanding f in a Taylor srs about a, we have $(z-a)^\lambda \sum_0^\infty c_n (z-a)^n$.

We can always take $c_0 \neq 0$, since if $c_0 = 0$ we can write the series as

$(z-a)^\lambda \sum_1^\infty c_n (z-a)^n = (z-a)^{\lambda+1} [c_1 + c_2(z-a) + \ldots]$, and if $c_1 = 0$, repeat...

A **Frobenius series** about a is a series of the form $(z-a)^\lambda \sum_0^\infty c_n (z-a)^n$ with $c_0 \neq 0$.

Fuchs's Theorem: If a is a regular singular point of a LHODE, there is at least one solution in the form of a Frobenius series about a. It converges inside a disk whose radius is the distance from a to the nearest other singular point of the ODE.

How to find the power λ and the coefficients in a Frobenius solution?

1. If the singularity is at a, it's convenient to change variables $z - a = z'$, and then drop the dash; the ODE changes, and the Frobenius solution becomes $\sum_0^\infty c_n z^{\lambda+n}$ with $c_0 \neq 0$
2. Substitute the series into the ODE, getting an equation of the form $\sum_n [\ldots] z^{\lambda+n} = 0$ for all z.
3. Equate the coefficient of the lowest power to zero, using $c_0 \neq 0$. It gives a quadratic eqn for λ, the **indicial equation**.
4. Equate the coefficient of the general term to zero. Get a recurrence relation for c_n. The relation contains λ.
5. Take a root λ of the indicial eqn and plug into the recurrence relation.
   If the singularity is regular, then for at least one root, the recurrence relation can be solved, giving all the c_n in terms of c_0.

The complete solution at regular singular pts

If a 2nd-order LHODE has a reg. sing. pt at $z = 0$, then:

- There is always a FS $z^\lambda \sum_0^\infty c_n z^n$. We can write it $z^\lambda \phi(z)$ where ϕ is anal. at 0.
- If the indicial eqn has two roots λ_1, λ_2 where $\lambda_1 - \lambda_2$ is not an integer, then every soln of the ODE is a combination of FS's for λ_1 and λ_2.
- If the indicial eqn has two roots λ_1, λ_2 where $\lambda_1 - \lambda_2$ is a positive integer, there is a FS $u_1(z) = z^{\lambda_1}\phi_1(z)$, and the second soln is

  either a FS $\quad u_2(z) = z^{\lambda_2}\phi_2(z)$

  or of the form $\quad u_1(z)\ln z + z^{\lambda_2}\phi_2(z)$

  where ϕ_1, ϕ_2 are analytic at 0.
- If the indicial equation has a double root λ, then there is a FS $u_1(z) = z^\lambda \phi_1(z)$ and a second soln of the form
  $$u_1(z)\ln z + z^\lambda \phi_2(z)$$
  where ϕ_1, ϕ_2 are analytic at 0.

Other forms of the logarithmic solutions

The solutions containing logarithms can be written in various different ways. For example, the functions ϕ analytic at 0 can be written as power series in z.

Another alternative form: the second solution when $\lambda_1 - \lambda_2$ is an integer is often written
$$C u_1(z)\ln z + z^{\lambda_2}\phi_2(z) \qquad (*)$$
where ϕ is some function analytic at 0, and C is an arbitrary constant.

How does this relate to the form on the previous page?

- If $C = 0$ then $(*)$ is simply a Frobenius solution using the second root λ_2 of the indicial equation.
- If $C \neq 0$, we can divide $(*)$ by C giving $\quad u_1(z)\ln z + z^{\lambda_2}\psi_2(z)$

  where $\psi_2 = (1/C)\,\phi_2$.

This is the solution given on the previous page. Many textbooks give the form $(*)$; it is more compact, but . . .