Introduction

This is a worked example of solving a linear ordinary differential equation in terms of series about a regular singular point.

It assumes that you know about regular singular points and the Frobenius method.

Example of the Frobenius method

Example: $u'' + \frac{1}{x}u' + \left(1 - \frac{1}{x^2}\right)u = 0.$ The point 0 is a regular singular pt.

We look for a soln in Frobenius form: $x^\lambda \sum_0^\infty c_n x^n$, where $c_0 \neq 0$.

Substitute into the ODE:

$$\sum_{n=0}^{\infty} \left[(\lambda+n)(\lambda+n-1)c_n x^{\lambda+n-2} + (\lambda+n)c_n x^{\lambda+n-2} + c_n(x^{\lambda+n} - x^{\lambda+n-2})\right] = 0.$$

Simplify:

$$\sum_{n=0}^{\infty} \left([\lambda+n]^2 - 1\right) c_n x^{\lambda+n-2} + \sum_{n=0}^{\infty} c_n x^{\lambda+n} = 0.$$

Equate coefficients of powers, remembering $c_0 \neq 0$.

$x^{\lambda-2}$	$\lambda^2 - 1 = 0$	The **indicial equation**. It $\Rightarrow \lambda = 1$ or -1.	
$x^{\lambda-1}$	$([\lambda+1]^2 - 1)c_1 = 0$	$\Rightarrow c_1 = 0.$	
$x^{\lambda+r}$, $r \geq 0$	$[(\lambda+r+2)^2 - 1]c_{r+2} + c_r = 0$	$\Rightarrow c_{r+2} = -\frac{c_r}{(\lambda+r+2)^2 - 1}$	**recursn eq.**

The recursion equation implies all the odd c_r are 0.

We found $\lambda = \pm 1$, and $c_{r+2} = -\dfrac{c_r}{(\lambda + r + 2)^2 - 1}$, or $\boxed{c_r = -\dfrac{c_{r-2}}{(\lambda + r)^2 - 1}.}$

With $\lambda = 1$, $c_r = -\dfrac{c_{r-2}}{(1+r)^2 - 1} = -\dfrac{c_{r-2}}{r^2 + 2r} = -\dfrac{c_{r-2}}{(r+2)r}$

So $c_2 = -\dfrac{c_0}{4.2}$, and $c_4 = -\dfrac{c_2}{6.4} = \dfrac{c_0}{6.4.4.2}$, etc.

The general term is:

$$c_{2k} = \frac{c_0 \text{ (-1) }^k}{(2k+2)(2k)^2(2k-2)^2, \dots 4^2.2} = \frac{2c_0 \text{ (-1) }^k}{[(2k+2)\dots 2][2k\dots 2]}$$

$$= \frac{2c_0 \text{ (-1) }^k}{2^{2k+1}(k+1)!\, 2^k k!}.$$

should be k+1, not 2k+1

So we have a soln $u_1(x) = c_0 x^1 \displaystyle\sum_{k=0}^{\infty} \frac{\text{(-1) }^k x^{2k}}{2^{2k}(k+1)!k!}$ for any c_0.

The series cnvgs for all x. We call it u_1 because it's the first of two solns of the ODE.

Now look at $\lambda = -1$, the other solution of the indicial eqn.

The recursion eqn $c_r = -\dfrac{c_{r-2}}{(\lambda + r)^2 - 1}$ gives $c_r = -\dfrac{c_{r-2}}{(r-1)^2 - 1} = -\dfrac{c_{r-2}}{r^2 - 2r}$.

So $c_2 = \dfrac{c_0}{4-4}$.

The Frobenius method FAILS for $\lambda = -1$.

For this particular ODE there is only one soln of Frobenius type.

That's related to the fact that the two values of λ differ by an integer. When that happens, there may be just one or two Frobenius solutions, depending on the particular ODE.

The general theory says that when there is only one Frobenius soln, the second soln of the ODE has a logarithmic singularity at 0. Here the second solution is

$u_2(x) = u_1(x) \log x + x^{-1}\phi(x)$ where ϕ is an analytic fn.

This gives the general form of the second soln. To determine ϕ is more difficult.