Taylor Series

Find the Taylor series about 0 for the following, up to z^6:

$$f(z) = \mathrm{Log}(\cosh z).$$

Hence find the Taylor series of $f(z) = \tanh z$.

$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z \qquad z \in \mathbb{C} - \{0\}$

$$\boxed{\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots} \qquad z \in \mathbb{C}$$

$\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots \qquad |z| < 1$

Let $w = 1 + z$ so $z = w - 1$

$\cosh z$

So $$\boxed{\mathrm{Log}(w) = (w-1) - \frac{(w-1)^2}{2} + \frac{(w-1)^3}{3} - \ldots}$$

$|w-1| < 1$

$$\operatorname{Log}(\cosh z) = \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots$$

$\frac{\sinh z}{\cosh z}$

$$-\frac{1}{2}\left(\frac{z^2}{2!} + \frac{z^4}{4!} + \ldots\right)^2 + \frac{1}{3}\left(\frac{z^2}{2!} + \ldots\right)^3 - \ldots$$

$$= \frac{z^2}{2} - \frac{z^4}{12} + \frac{z^6}{45} - \ldots \qquad |z| < r$$

$$f(z) = \tanh z$$

$$\tanh z = \frac{2z}{2} - \frac{4z^3}{12} + \frac{6z^5}{45} - \ldots$$

$$= z - \frac{z^3}{3} + \frac{2z^5}{15} \ldots \qquad |z| < r$$

$\tanh z$ is analytic on $\{z : |z| < \pi/2\}$

so $r = \pi/2$