Surface Area

Find the area of the surface

$$z = \overbrace{y^2 + 2xy - x^2 + 2}^{f}$$

lying over the annulus $\frac{3}{8} \le x^2 + y^2 \le 1$.

$$S.A. = \int_S \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ \underset{r\,dr\,d\theta}{\overset{dx\,dy}{dA}}$$

where $z = f(x, y)$.

$$\frac{\partial f}{\partial x} = 2y - 2x \quad \& \quad \frac{\partial f}{\partial y} = 2y + 2x$$

So $$SA = \int_S \sqrt{1 + (2y-2x)^2 + (2y+2x)^2}\ dx\,dy$$

$$= \int_S \sqrt{1 + 8y^2 + 8x^2}\ dx\,dy$$

$$= \int_S \sqrt{1 + 8\underbrace{(x^2 + y^2)}_{r^2}}\ dx\,dy$$

$$\frac{3}{8} \le x^2 + y^2 \le 1 \Rightarrow \sqrt{\frac{3}{8}} \le r \le 1$$

So $$SA = \int_{\theta=-\pi}^{\pi}\int_{r=\sqrt{3/8}}^{1} \sqrt{1+8r^2}\, r\,dr\,d\theta$$

$$= \int_{r=\sqrt{3/8}}^{1} \sqrt{1+8r^2}\, r \Big[\theta\Big]_{-\pi}^{\pi}\, dr$$

$$= 2\pi \int_{\sqrt{3/8}}^{1} (\underbrace{1+8r^2}_{})^{1/2}\, r\, dr \qquad u = 1+8r^2$$

$$= \frac{2\pi}{16}\int_{\sqrt{3/8}}^{1} (\underbrace{1+8r^2}_{})^{1/2}\, 16r\, dr$$

$$= \frac{2\pi}{16}\left[\frac{(1+8r^2)^{3/2}}{3/2}\right]_{\sqrt{3/8}}^{1}$$

$$= \frac{\pi}{8}\times\frac{2}{3}\Big[(1+8r^2)^{3/2}\Big]_{\sqrt{3/8}}^{1}$$

$$= \frac{\pi}{12}\left(9^{3/2} - 4^{3/2}\right)$$

$$9^{3/2} = \left(9^{1/2}\right)^3$$

$$= \frac{\pi}{12}\left(27 - 8\right) = \frac{19\pi}{12}$$

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