Residue Theorem

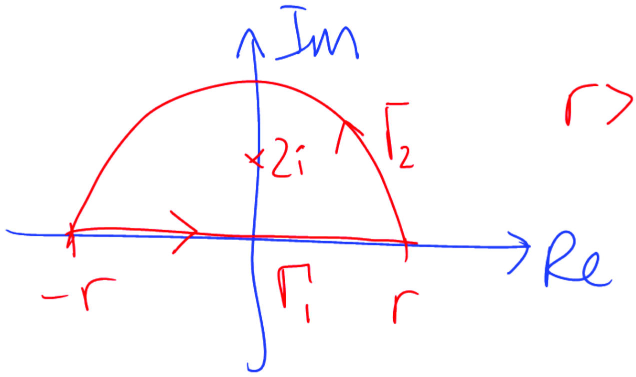
Evaluate $\int_{-\infty}^{\infty} \frac{t^2}{(t^2+4)^2}\, dt$

$$f(z) = \frac{z^2}{(z^2+4)^2}$$

$$z^2+4=0$$
$$z = \pm 2i$$

$$\boxed{I = \int_{\Gamma} \frac{z^2}{(z^2+4)^2}\, dz}$$

$$\Gamma = \Gamma_1 + \Gamma_2$$



$r > 2$

$\mathrm{Res}(f, 2i)$:

If f has pole of order k at α

then $$\mathrm{Res}(f,\alpha) = \frac{1}{(k-1)!} \lim_{z\to\alpha} \left(\frac{d^{k-1}}{dz^{k-1}} \left[(z-\alpha)^k f(z) \right] \right)$$

Here $k=2$, $\alpha = 2i$ $\quad f(z) = \dfrac{z^2}{(z^2+4)^2}$

$(z-2i)^2(z+2i)^2$

So

$$\operatorname{Res}(f,2i) = \lim_{z \to 2i} \left(\frac{d}{dz} \left(\frac{(z-2i)^2 z^2}{(z^2+4)^2} \right) \right)$$

$$= \lim_{z \to 2i} \left(\frac{d}{dz} \left(\frac{z^2}{(z+2i)^2} \right) \right)$$

$$= \lim_{z \to 2i} \left(\frac{(z+2i)^2 2z - z^2 \cdot 2(z+2i)}{(z+2i)^4} \right)$$

$$= \frac{-1}{8} i$$

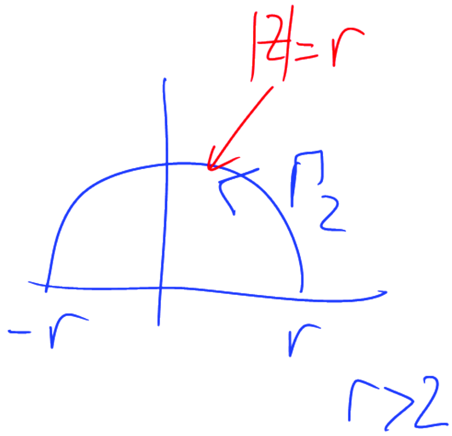
So $\quad I = \displaystyle\int_{\Gamma} \frac{z^2}{(z^2+4)^2}\,dz = 2\pi i\left(-\frac{1}{8}i\right)$

$$= \frac{\pi}{4}$$

$r \to \infty$

$$\int_{-r}^{r} \frac{t^2}{(t^2+4)^2}\,dt + \int_{\Gamma_2} \frac{z^2}{(z^2+4)^2}\,dz = \frac{\pi}{4}$$

$$\int_{\Gamma_2} \frac{z^2}{(z^2+4)^2}\,dz$$



$$|z^2+4| \geq |z|^2 - 4$$

(triangle inequality)

$$= r^2 - 4$$

Estimation Theorem

So $$\left| \int_{\Gamma_2} \frac{z^2}{(z^2+4)^2}\,dz \right| \leq \frac{r^2}{(r^2-4)^2} \times \pi r$$

Estimation Thm: $|f(z)| \le M$

$$\left| \int_{\Gamma} f(z)\,dz \right| \le ML$$

Here $f(z) = \dfrac{z^2}{(z^2+4)^2}$

so $|f(z)| = \dfrac{z^2}{|z^2+4|^2} \le \dfrac{z^2}{(z^2-4)^2}$

So $\left| \displaystyle\int_{\Gamma_2} \frac{z^2}{(z^2+4)^2}\,dz \right| \le \dfrac{\pi r^3}{(r^2-4)^2} \;\|$

$$\lim_{r \to \infty} \int_{\Gamma_2} \frac{z^2}{(z^2+4)^2}\,dz = 0$$

So $$\int_{-\infty}^{\infty} \frac{t^2}{(t^2+4)^2}\,dt = \lim_{r\to\infty} \int_{-r}^{r} \frac{t^2}{(t^2+4)^2}\,dt$$

$$= \frac{\pi}{4}$$