Lagrange Multipliers

Find the minimum distance from the origin to

the intersection of $xy = 6$ with $7x + 24z = 0$.

Minimise $x^2+y^2+z^2$

subject to $xy=6$ (4) and $7x+24z=0$ (5)

Let $F = x^2+y^2+z^2+\lambda_1 xy+\lambda_2(7x+24z)$

$6/x$

$$\frac{\partial F}{\partial x} = 2x+\lambda_1 y+7\lambda_2 = 0 \quad (1)$$

$$\frac{\partial F}{\partial y} = 2y+\lambda_1 x = 0 \quad (2) \Rightarrow \lambda_1 = -\frac{2y}{x} = -\frac{2}{x}\cdot\frac{6}{x}$$

$$\frac{\partial F}{\partial z} = 2z+24\lambda_2 = 0 \Rightarrow z+12\lambda_2=0 \quad (3)$$

$$\Rightarrow \lambda_2 = -z/12 = -\frac{1}{12}\left(-\frac{7x}{24}\right)$$

So (1) becomes

$$2x-\frac{12}{x^2}\cdot\frac{6}{x}+7\cdot\frac{7x}{12\cdot 24}=0$$

Multiply through by $12\cdot 24\cdot x^3$:

$$24^2x^4-12\cdot 6\cdot 12\cdot 24+7^2x^4=0$$

$$x^4(\underbrace{24^2+7^2}_{7,24,25}) = 12\cdot 6\cdot 12\cdot 24$$

$$= 12\cdot 6\cdot 12\cdot 2\cdot 12 = 12^4$$

$$25^2x^4 = 12^4 \Rightarrow x^4 = \frac{12^4}{25^2}$$

$$\sqrt{\frac{12^4}{25^2}} = \frac{12^2}{25}$$

$$x^2 = \pm\frac{12^2}{25} \quad \text{ie} \quad x^2=\frac{12^2}{25} \quad (x \text{ real})$$

$$x = \pm \frac{12}{5}$$

$$x = \frac{12}{5} \quad y = \frac{5}{2} \quad z = -\frac{7}{10}$$

$$x = -\frac{12}{5} \quad y = -\frac{5}{2} \quad z = \frac{7}{10}$$

So the minimum distance is

$$d = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{5}{2}\right)^2 + \left(-\frac{7}{10}\right)^2}$$

$$= \sqrt{\frac{12^2 \times 4 + 5^2 \times 25 + 49}{100}}$$

$$= \sqrt{\frac{1250}{100}} = \sqrt{\frac{25}{2}} = \underline{\underline{\frac{5}{\sqrt{2}}}} \quad (\approx 3.54)$$

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