Introduction to Geometric Measure Theory — Take-home exam

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1 Questions

Do four questions for maximum credit. Email your solutions in pdf format to me by Monday 7 November 2016. My email address is **t.c.oneil@open.ac.uk**.

1. Find an example of a compact set $E \subset \mathbb{R}$ for which

$$\underline{\dim}_B(E) < \overline{\dim}_B(E).$$

2. Determine the Hausdorff, packing and box dimensions of

$$\bigcup_{p,q\in\mathbb{Q},\, r\in\mathbb{Q}^+} S((p,q),r).$$

   (Where $S((p,q),r)$ denotes the circle centre (p,q) with radius r.)

3. Prove that Hausdorff dimension is monotonic, countably stable, and, for bounded set, is. bounded above by lower box dimension.

4. Prove that Hausdorff measure $\mathcal{H}^s$ is a Borel regular outer measure for each $s \geq 0$.

5. Provide the missing details in the proof of the Vitali Covering Theorem.

6. Let μ and ν be Radon measures on $\mathbb{R}^n$, $0 < t < \infty$ and $A \subseteq \mathbb{R}^n$. Prove:

   (a) If $\liminf_{r\searrow 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \leq t$ for all $x \in A$, then $\mu(A) \leq t\nu(A)$.

(b) If $\limsup_{r\searrow 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \geq t$ for all $x \in A$, then $\mu(A) \geq t\nu(A)$.

7. Working in $\mathbb{R}^n$, show that:

   (a) Every m-rectifiable set has σ-finite $\mathcal{H}^m$-measure.

   (b) Any subset of an m-rectifiable set is m-rectifiable.

   (c) The countable union of m-rectifiable sets is m-rectifiable.

   (d) If E is m-rectifable, then there is an m-rectifiable set Borel set B that contains E and for which $\mathcal{H}^m(B) = \mathcal{H}^m(E)$.