An introduction to Geometric Measure Theory
Part 4: Rectifiability

Toby O'Neil, 24 October 2016

Last week. . .

1. Finished off the energy discussion.
2. Talked about covering theorems.
3. Gave a classical application of covering theorems: the Lebesgue density theorem

This week

1. Define rectifiability and (pure) unrectifiability
2. Discuss their different properties and some useful theorems
3. State area and coarea formulae

Lipschitz maps

Definition

For metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is Lipschitz if there is $L \geq 0$ such that

$$d_Y(f(a), f(b)) \leq L d_X(a, b) \text{ for each } a, b \in X.$$

Theorem

If $f\colon A \to \mathbb{R}^m$ is Lipschitz for $A \subseteq \mathbb{R}^n$, then there is a Lipschitz map $g\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $f = g|_A$.

Extending Lipschitz $f: A \to \mathbb{R}$ where $A \subseteq \mathbb{R}^n$

Define $\tilde{f}: \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{f}(x) = \inf \{ f(a) + \underbrace{\mathrm{Lip}(f)|x-a|}_{\geq 0} : a \in A \}$$

If $b \in A$, then $\tilde{f}(b) = f(b)$, so $\tilde{f}|_A = f$

[$\tilde{f}(b) \leq f(b)$ – obvious, and if $a \in A$, then $f(b) \leq f(a) + \mathrm{Lip}(f)|b-a|$]

To see that $\tilde{f}$ is Lipschitz with $\mathrm{Lip}(\tilde{f}) = \mathrm{Lip}(f)$, for $x, y \in \mathbb{R}^n$

$$\tilde{f}(x) \leq \inf_A \{ f(a) + \mathrm{Lip}(f)(|y-a| + |x-y|) \}$$
$$= \tilde{f}(y) + \mathrm{Lip}(f)|x-y|$$

now interchange x and y ☐

Rademacher's Theorem

Theorem (Rademacher)

Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *be a Lipschitz mapping, then* f *is differentiable at* $\mathcal{H}^n$*-almost every point in* $\mathbb{R}^n$*.*

Theorem

If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *is Lipschitz, then for each* $\epsilon > 0$*, there is a continuously differentiable map* $g\colon \mathbb{R}^n \to \mathbb{R}^m$ *such that*

$$\mathcal{H}^n(\{x : f(x) \neq g(x)\}) < \epsilon.$$

Sard's Theorem (basic form)

Theorem

If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *is a Lipschitz map, then*

$$\mathcal{H}^n(\{f(x) : \dim(Df(x)(\mathbb{R}^n)) < n\}) = 0.$$

(The set of points where the derivative is singular has an image of zero n-measure.)

Rectifiability

Definition

A set $E \subseteq \mathbb{R}^n$ is *m*-rectifiable if there are Lipschitz maps $f_i \colon \mathbb{R}^m \to \mathbb{R}^n$ for $i = 1, 2, \ldots$ such that

$$\mathcal{H}^m\left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m)\right) = 0.$$

A set $F \subseteq \mathbb{R}^n$ is purely *m*-unrectifiable if $\mathcal{H}^m(E \cap F) = 0$ for every *m*-rectifiable set E.

Geometric characterisations

Notation

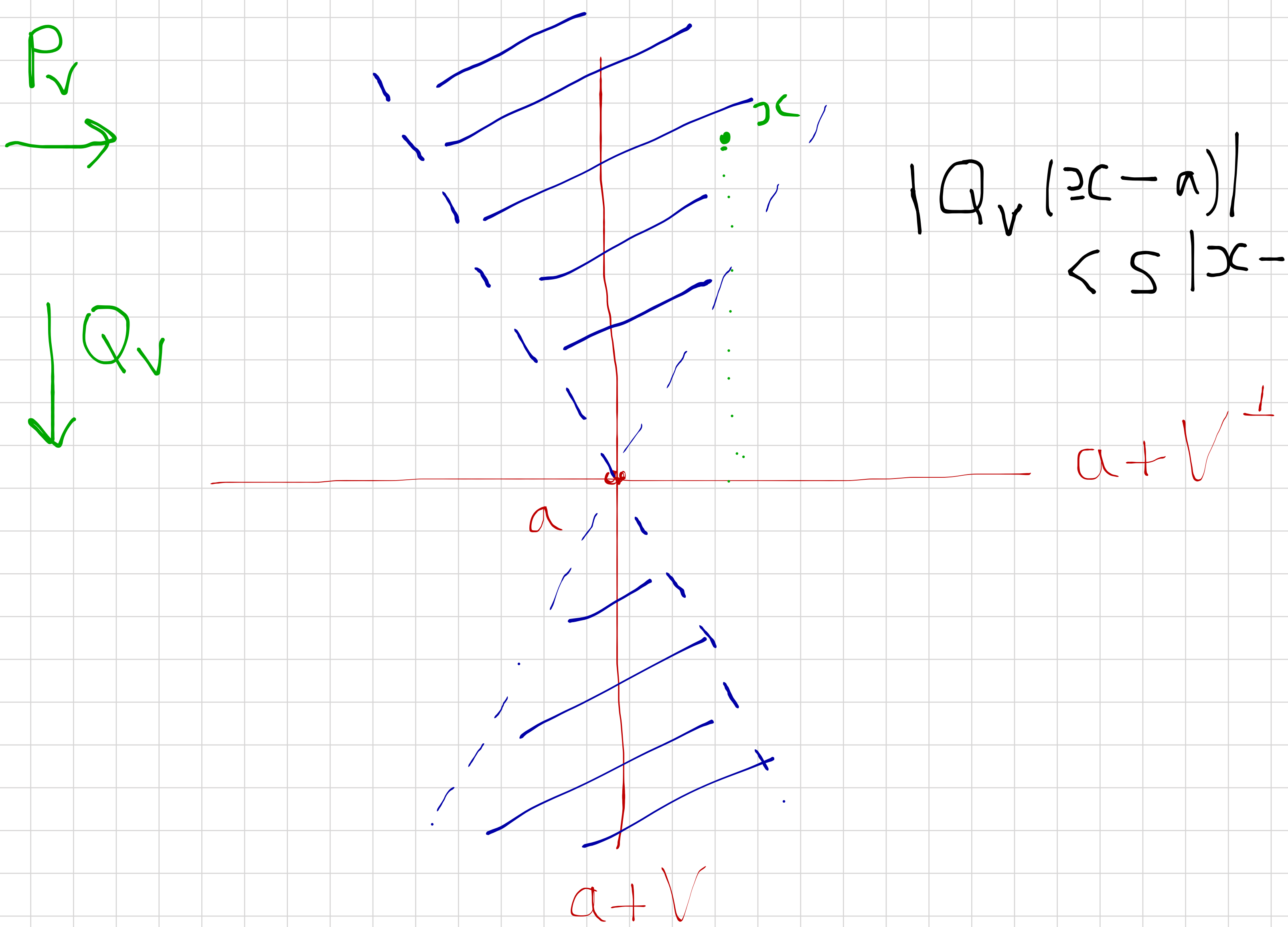
Let $V \in G(n, n-m)$ be an $(n-m)$-plane through the origin. Let $P_V : \mathbb{R}^n \to V$ denote orthogonal projection onto V and let $Q_V : \mathbb{R}^n \to V^\perp$ denote $P_{V^\perp}$ (so $P_V + Q_V$ is the identity map). For $a \in \mathbb{R}^n$, $0 < s < 1$ and $0 < r < \infty$, we set

$$X(a, V, s) = \{x \in \mathbb{R}^n : d(x-a, V) < s|x-a|\}$$
$$= \{x \in \mathbb{R}^n : |Q_V(x-a)| < s|x-a|\}$$

and

$$X(a, V, s, r) = X(a, V, s) \cap B(a, r).$$

$X(a, V, s)$



$$|Q_V(x-a)| < s\,|x-a|$$

A useful lemma

Lemma

Suppose $E \subseteq \mathbb{R}^n$, $V \in G(n, n-m)$, $0 < s < 1$ *and* $0 < r < \infty$. *If*

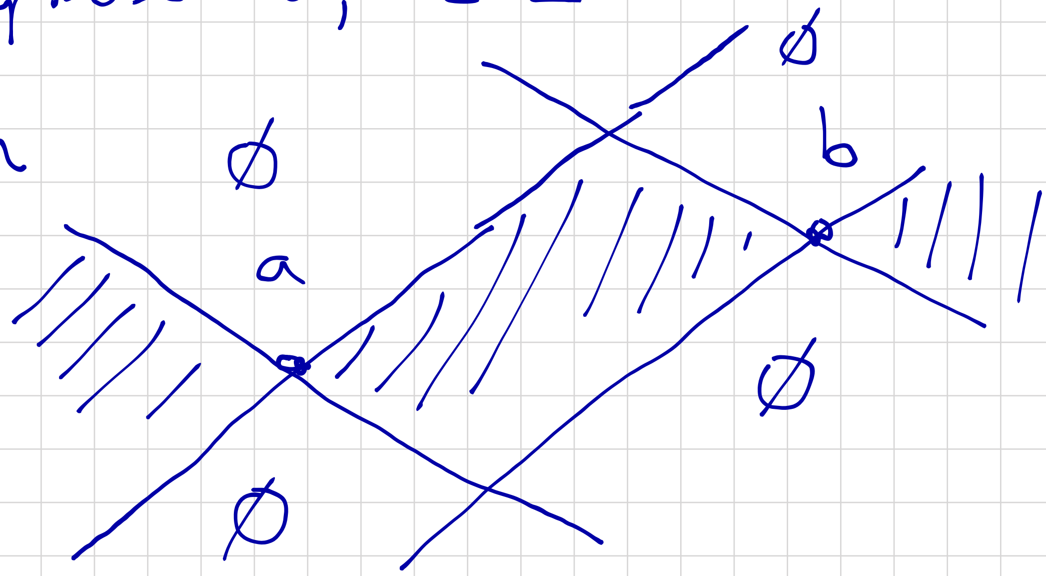
$$E \cap X(a, r, V, s) = \emptyset \text{ for each } a \in E,$$

then E *is m-rectifiable.*

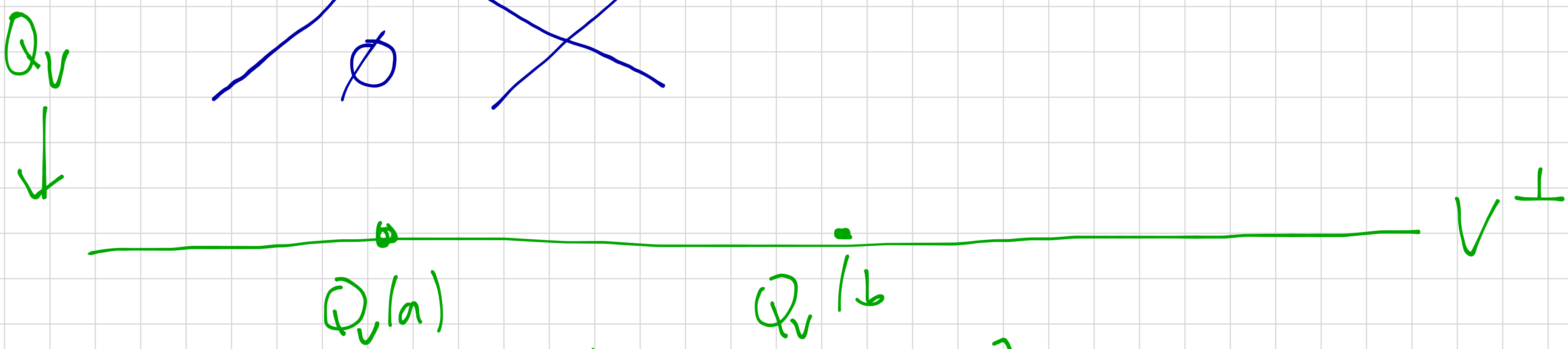
Why lemma B obvious! WLOG $r = \infty$.

Suppose $a, b \in E$

Then



E lies in shaded region.



Q_V is s.t. $Q_V^{-1} : Q_V(E) \to \mathbb{R}^n$ exists

and diagram shows Q_V^{-1} is Lipschitz.

Lemma

Let $V \in G(n, n-m)$, $0 < s < 1$, $0 < \delta < \infty$ and $0 < \lambda < \infty$. If $A \subseteq \mathbb{R}^n$ is purely m-unrectifiable and

$$\mathcal{H}^m(A \cap X(x, r, V, s)) \leq \lambda r^m s^m \text{ for } x \in A,\ 0 < r < \delta,$$

then

$$\mathcal{H}^m(A \cap B(a, \delta/6)) \leq 2 \cdot 20^m \lambda \delta^m.$$

Corollary

If $V \in G(n, n-m)$, $\delta > 0$ and $A \subseteq \mathbb{R}^n$ is purely m-unrectifiable with $\mathcal{H}^m(A) < \infty$, then

$$\limsup_{s \searrow 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m(A \cap X(a, r, V, s)) > 0$$

for $\mathcal{H}^m$-almost every $a \in A$.

proof of horrible technical lemma.

WLOG $A \subseteq B(a, \frac{1}{6}\delta)$

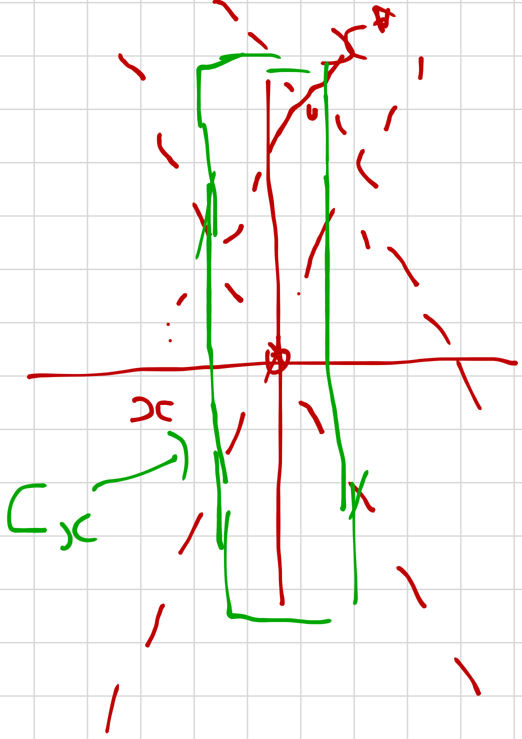
and $A \cap X(x, V, \frac{1}{4}s) \neq \emptyset$ for all $x \in A$

[since $\{x \in A : A \cap X(x, V, \frac{1}{4}s) = \emptyset\}$ has zero $\mathcal{H}^m$ measure because A is m purely unrectifiable.]

For $x \in A$, let

$$h(x) = \sup\{|y-x| : y \in A \cap X(x, V, \tfrac{1}{4}s)\}$$

Then $0 < h(x) \lesssim \frac{1}{3}\delta$ and so for each $x \in A$ can choose $x^* \in A \cap X(x, V, \frac{1}{4}s)$ with $|x^* - x| \geq \frac{3}{4} h(x)$.



Let $C_x = Q_V^{-1}[Q_V B(x, \frac{1}{4} s h(x))]$

Then

$$A \cap C_x \subseteq X(x, 2h(x), V, s) \cup X(x^*, 2h(x), V, s) \quad \text{— (1)}$$

[geometrically plausible – proved later]

Given (1), then hypotheses of lemma

$$\Rightarrow \quad \mathcal{H}^m(A \cap C_x) \leq 2\lambda\,(2h(x))^m s^m$$

Now use basic covering theorem to find countable $S \subseteq A$ with $\{Q_V B(x, \frac{1}{20} s h(x)) : x \in S\}$ disjoint and $Q_V A \subseteq \bigcup_S Q_V B(x, \frac{1}{4} s h(x))$

So $\quad A \subseteq \bigcup_{x \in S} C_x$

Hence
$$\mathcal{H}^m(A) \leq \sum_{x \in S} \mathcal{H}^m(A \cap C_x)$$
$$\leq 2\lambda 2^m \sum_{x \in S} (sh(x))^m$$
$$= 2^{m+1} \lambda 20^m 2^{-m} \sum_{x \in S} \mathcal{H}^m(V^\perp \cap B(Q_V x, \tfrac{1}{20} sh(x)))$$
$$\leq 2 \cdot 20^m \lambda \mathcal{H}^m(V^\perp \cap B(Q_V a, \delta/2))$$
$$= 2 \cdot 20^m \lambda \delta^m \quad \text{as claimed.}$$

only remains to prove (1).

Claim: $A \cap C_x \subseteq X(x, 2h(x), V, s) \cap X(x^*, 2h(x), V, s)$ ← radius

proof: Let $z \in A \cap C_x$.

Then $Q_V z \in Q_V B(x, \frac{1}{4} s h(x))$

$\Leftrightarrow \quad |Q_V(z-x)| \leq \frac{1}{4} s h(x)$.

If $h(x) < |x-z|$, then

$|Q_V(z-x)| < \frac{1}{4} s |z-x| \Rightarrow z \in X(x, V, \frac{1}{4} s)$

$\Rightarrow |x-z| \leq h(x)$ ↯

So $|x-z| \leq h(x)$

and so $|x^* - z| \leq 2h(x)$.

If $z \notin X(x^*, 2h(x), V, s)$, then

$s|x^* - z| \leq |Q_V x^* - Q_V z| \leq |Q_V(x^* - x)| + |Q_V(x-z)|$

$< \frac{1}{4} s |x^* - x| + \frac{1}{4} s h(x)$

$\leq \frac{1}{2} s h(x) \qquad \Rightarrow |x^* - z| \leq \frac{1}{2} h(x)$

But $|x - x^*| \geq \frac{3}{4} h(x)$

and so $|x - z| > \frac{3}{4} h(x) - \frac{1}{2} h(x) = \frac{1}{4} h(x)$

$$\geq \frac{1}{5} |Q_V(x - z)|$$

i.e. $z \in X(x, 2h(x), V, 5)$

as required! □

Hard work but geometrically obvious.

proof of Corollary:

Let B be the set of points in A for which the corollary fails.

Let $\lambda > 0$ and set

$$A_i = \{a \in A : \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m(A \cap X(a,r,V,s)) < \lambda \text{ for } 0 < s < 1/i\}$$

Then $A_1 \subseteq A_2 \subseteq \cdots$ and $B \subseteq \bigcup_i A_i$

But (technical) lemma gives for $i \in \mathbb{N}$

$$\mathcal{H}^m(A_i \cap B(a, \delta/6)) \leq 2 \cdot 20^m \lambda \delta^m \text{ for } a \in \mathbb{R}^n$$

So

$$\mathcal{H}^m(B \cap B(a, \delta/6)) \leq 2 \cdot 20^m \lambda \delta^m$$

But λ is arbitrary, so $\mathcal{H}^m(B \cap B(a, \delta/6)) = 0$

$$\Rightarrow \mathcal{H}^m(B) = 0$$

∎

2nd corollary Fix $V \in G(n, n-m)$ and $0 < s < 1$. Suppose $A \subseteq \mathbb{R}^n$ is m purely unrectifiable and $\mathcal{H}^m(A) < \infty$. Then for $\mathcal{H}^m$-a.e. $a \in A$

$$\overline{D}^m(A \cap X(a, V, s), a) \geq (240)^{-m-1} s^m \quad (*)$$

proof Set where $(*)$ fails is contained in $\bigcup_i A_i$ where

$$A_i = \{a \in A : \mathcal{H}^m(A \cap X(a, r, V, s)) < \lambda s^m r^m \text{ for } 0 < r < 1/i\}$$

$$\left(\lambda = \tfrac{1}{3}(120)^{-m}\right)$$

if $\delta \in (0, 1/i)$, then (technical) lemma gives

$$\mathcal{H}^m(A_i \cap B(a, \tfrac{1}{6}\delta)) \leq 2 \cdot 20^m \lambda \delta^m \text{ for } a \in \mathbb{R}^n$$

So $\quad \overline{D}^m(A_i, a) \leq 2 \cdot 60^m \lambda < 2^{-m}$

$\Rightarrow \mathcal{H}^m(A_i) = 0$ by results on upper density!

s-densities

Definition

Suppose that A is an $\mathcal{H}^s$-measurable set with finite $\mathcal{H}^s$-measure. We define the upper and lower s-densities of A at a point x by

$$\overline{D}^s(A,x) = \limsup_{r \searrow 0} \frac{\mathcal{H}^s(A \cap B(x,r))}{(2r)^s}$$

and

$$\underline{D}^s(A,x) = \liminf_{r \searrow 0} \frac{\mathcal{H}^s(A \cap B(x,r))}{(2r)^s}.$$

Theorem

If A is an $\mathcal{H}^s$-measurable set with finite $\mathcal{H}^s$-measure, then

$$2^{-s} \leq \overline{D}^s(A, x) \leq 1$$

for $\mathcal{H}^s$-almost every x in A.
(In fact, if $s = m \in \mathbb{N}$, such a set A is m-rectifiable if, and only if, $\overline{D}^m(A, x) = \underline{D}^m(A, x) = 1$ for $\mathcal{H}^m$-almost every x.)

proof left hand $\leq$

$\{x \in A : \overline{D}^s(A, x) < 2^{-s}\} = \bigcup_{k=1}^{\infty} B_k$ where

$B_k = \{x \in A : \mathcal{H}^s(A \cap B(x, r)) < \frac{k}{k+1} r^s$ for $0 < r < 1/k\}$

Hence it is enough to show $\mathcal{H}^s(B_k) = 0$ for all k

Fix $k \in \mathbb{N}$ and let $t = k/(k+1) < 1$.

Let $\varepsilon > 0$.

Can cover B_k by sets $E_1, E_2, \ldots$ such that

- $0 < \operatorname{diam}(E_i) < 1/k$
- $B_k \cap E_i \neq \emptyset$

(for all i)

- $\sum \operatorname{diam}(E_i)^s < \mathcal{H}^s(B_k) + \varepsilon$

For each i, choose $x_i \in B_k \cap E_i$ and let $r_i = \operatorname{diam}(E_i)$

Then $B_k \cap E_i \subseteq A \cap B(x_i, r_i)$ for all i.

Hence

$$\mathcal{H}^s(B_k) \leq \sum_i \mathcal{H}^s(B_k \cap E_i)$$

$$\leq \sum_i{}' \mathcal{H}^s(A \cap B(x_i, r_i))$$

$$\leq t \sum r_i^s = t \sum{}' \operatorname{diam}(E_i)^s$$

$$\leq t\left(\mathcal{H}^s(B_k) + \varepsilon\right)$$

But $\varepsilon > 0$ was arbitrary,

so $\mathcal{H}^s(B_k) \leq t\,\mathcal{H}^s(B_k)$

and $B_k \subseteq A$ so $\mathcal{H}^s(B_k) < \infty$

$$\Rightarrow \mathcal{H}^s(B_k) = 0$$

$\square$

proof of right hand ≤

WLOG can assume A is Borel

and so, since, $\mathcal{H}^s(A) < \infty$, $\mathcal{H}^s \llcorner_A$ is Radon.

Let $t > 0$ and
define $B = \{x \in A : \overline{D}^s(A, x) > t\}$

It is enough to show that $\mathcal{H}^s(B) = 0$.

Fix $\varepsilon > 0$ and $\delta > 0$.

We can find an open set U with $B \subset U$
and $\mathcal{H}^s(A \cap U) < \mathcal{H}^s(B) + \varepsilon$

For each $x \in B$ can find $0 < r < \delta/2$ st $B(x, r) \subset U$
and $\mathcal{H}^s(A \cap B(x, r)) > t(2r)^s$

So Vitali Covering Thm $\Rightarrow \exists$ disjoint balls $B_1, B_2, \ldots$

$\left(\mathcal{H}^s(B \setminus \bigcup B_i) = 0 \right)$

Hence

$$\mathcal{H}^s(B) + \varepsilon > \mathcal{H}^s(A \cap U)$$
$$\geq \sum_i \mathcal{H}^s(A \cap B_i)$$
$$> t \sum \operatorname{diam}(B_i)^s$$
$$\geq t\, \mathcal{H}^s_\delta(B \cap \bigcup B_i)$$
$$= t\, \mathcal{H}^s_\delta(B)$$

since $\mathcal{H}^s_\delta$ is subadditive

But $\varepsilon > 0$ was arbitrary and can let $\delta \downarrow 0$

$$\Rightarrow \mathcal{H}^s(B) \geq t\, \mathcal{H}^s(B)$$

$$\Rightarrow \mathcal{H}^s(B) = 0 \quad \text{since } \mathcal{H}^s(A) < \infty \qquad \square$$

Besicovitch $\frac{1}{2}$-conjecture

Theorem

For $m \leq n$, there is a constant $0 < c(n,m) < 1$ such that if $A \subseteq \mathbb{R}^n$ is an $\mathcal{H}^m$-measurable set with finite $\mathcal{H}^m$-measure and for which for $\mathcal{H}^m$-almost every $x \in A$,

$$\underline{D}^m(A,x) \leq c(n,m),$$

then A is purely m-unrectifiable.

It is known that $c(n,1) < 3/4$ and it is conjectured that $c(n,1) = 1/2$ for all n.

Besicovitch-Federer projection theorem

Theorem

Let A be an $\mathcal{H}^m$-measureable subset of $\mathbb{R}^n$ with $\mathcal{H}^m(A) < \infty$.

1. *A is m-rectifiable if, and only if, $\mathcal{H}^m(P_B(B)) > 0$ for $\gamma_{n,m}$-almost every $V \in G(n,m)$ whenever B is an $\mathcal{H}^m$-measurable subset of B with $\mathcal{H}^m(B) > 0$.*
2. *A is purely m-unrectifiable if, and only if, $\mathcal{H}^m(P_V(A)) = 0$ for $\gamma_{n,m}$-almost every $V \in G(n,m)$.*

Jacobians

Definition

Suppose that $L\colon \mathbb{R}^m \to \mathbb{R}^n$ is linear.

1. If $m \leq n$, define $\|L\| = \sqrt{\det(L^* \circ L)}$.
2. If $n \leq m$, define $\|L\| = \sqrt{\det(L \circ L^*)}$.

(Here L^* denotes the adjoint of L.)

Definition (Jacobian)

If $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz, then we define the Jacobian of f by

$$Jf(x) = \|Df(x)\|$$

for $\mathcal{H}^m$-almost every x, where $Df(x)$ denotes the derivative of f at x.

Basic area formula

Theorem

If $f\colon \mathbb{R}^m \to \mathbb{R}^n$ *is Lipschitz and* $m \le n$*, then*

1. *if* $A \subseteq \mathbb{R}^m$ *is an* $\mathcal{H}^m$*-measurable set, then*

$$\int_A Jf(x)\, d\mathcal{H}^m(x) = \int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^m(y),$$

2. *if* u *is an* $\mathcal{H}^m$*-integrable function, then*

$$\int_{\mathbb{R}^m} u(x) Jf(x)\, d\mathcal{H}^m(x) = \int_{\mathbb{R}^n} \sum_{x \in f^{-1}(y)} u(x)\, d\mathcal{H}^m(y).$$

Full area formula

Theorem

If $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz and $m \leq n$, then

$$\int_A (g \circ f)(x) Jf(x)\, d\mathcal{H}^m(x) = \int_{\mathbb{R}^n} g(y)\mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^m(y),$$

whenever A is an $\mathcal{H}^m$-measurable set, $g\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, \infty\}$ and either

1. *g is $\mathcal{H}^m$-measurable, or*
2. *$\mathcal{H}^0(A \cap f^{-1}(y)) < \infty$ for $\mathcal{H}^m$-almost every y, or*
3. *$\mathbf{1}_A(g \circ f)J_m f$ is $\mathcal{H}^m$-measurable.*

Applications

Length of a curve

Suppose that $f\colon \mathbb{R} \to \mathbb{R}^n$ is Lipschitz and one-one. Then for $C = f([a, b])$

$$\mathcal{H}^1(C) = \int_a^b |\nabla f|\, dt.$$

Surface area of a graph

Assume that $g\colon \mathbb{R}^m \to \mathbb{R}$ is Lipschitz and for an open set $U \subseteq \mathbb{R}^m$, let $G = \{(u, g(u)) : u \in U\} \subseteq \mathbb{R}^{m+1}$. Then

$$\mathcal{H}^m(G) = \int_U \sqrt{1 + |(Dg)(u)|^2}\, d\mathcal{H}^m(u).$$

Basic co-area formula

Theorem

If $f\colon \mathbb{R}^m \to \mathbb{R}^n$ *is Lipschitz and* $m > n$*, then*

$$\int_A Jf(x)\, d\mathcal{H}^m(x) = \alpha_{m,n} \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(A \cap f^{-1}(y))\, d\mathcal{H}^n(y)$$

for every $\mathcal{H}^m$*-measurable set* A*.*