An introduction to Geometric Measure Theory
Part 1: dimension
3 October 2016

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Proposed plan

- **Week 1** Introduce notions of dimension
- **Week 2** Develop theory of dimension
- **Week 3** Introduction to differentiation and rectifiability
- **Week 4** To be determined by audience interest

Today

1. Agree some notation and terminology
2. Agree a list of ‘test’ examples
3. What properties should dimension have?
4. Define some dimensions
5. Test the definitions against (some of) our examples

Our base assumptions

We shall usually work in $\mathbb{R}^n$ with the Euclidean metric. (And nearly always will specialise to the plane.)
Most of the ideas however make sense in complete separable metric spaces.

Part 1: Examples of sets

Simple sets

1. $\emptyset$
2. finite set: $\{a_1, \ldots, a_n\}$
3. $\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
4. traditional geometric shapes (line segment, triangle, square,...)

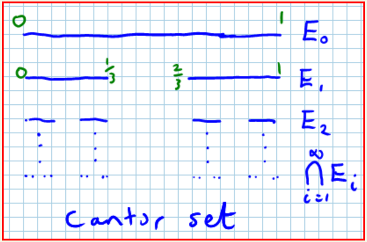
should have dimension zero.

dimension 1 ↑ 2 ↑

(filled)

Cantor sets

Definition

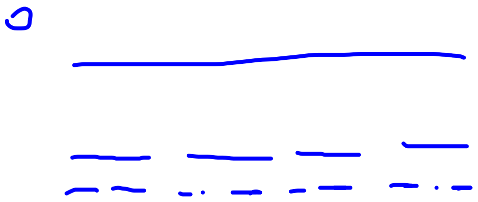


$\frac{1}{3}$-Cantor set $= \bigcap E_i$.

Some properties

- Uncountable, compact, totally disconnected (and perfect — no isolated points)
- Has zero length (whatever that means)

Cantor set



$$C = \bigcap_{k=1}^{\infty} E_k$$

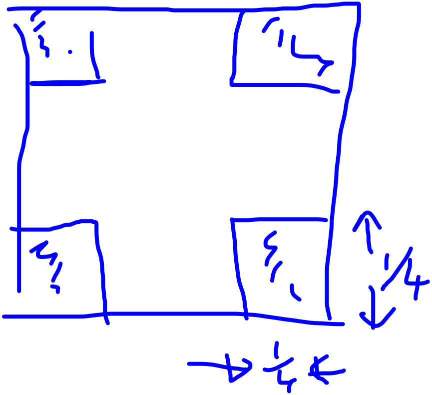
$$\mathcal{L}(C) \le 2^k \times 3^{-k} = \left(\frac{2}{3}\right)^k \to 0$$

$\mathcal{L}(C)$ ↑ length

2^k ↑ number of intervals

3^{-k} ↑ length (on level k)

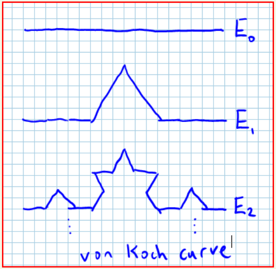
Cantor sets in the plane



$\left(\frac{1}{4} \times \frac{1}{4}\right)$-Cantor sets.

von Koch curve

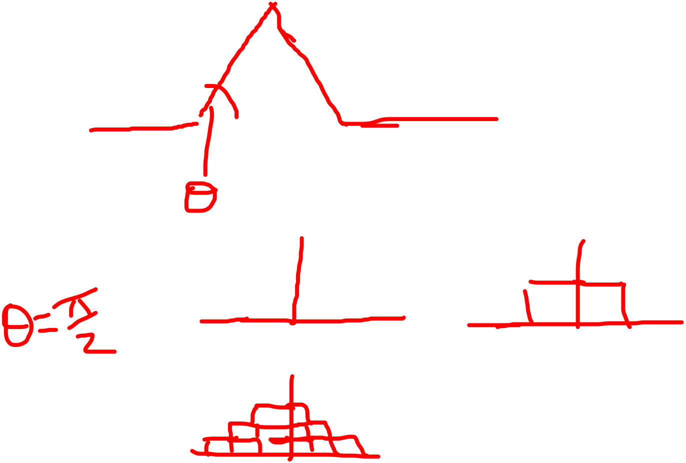
Definition



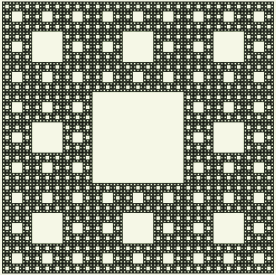
Some properties

- Has infinite length
- Has zero area

von Koch curve



Sierpinski gasket



Has area zero and is compact (and perfect). An example of a *universal set* — contains a homeomorphic image of every compact planar curve .

A weird set

$$E = \bigcup_{p,q \in \mathbb{Q},\, r \in \mathbb{Q}^+} S((p,q),r).$$

(Where $S((p,q),r)$ denotes the circle centre (p,q) with radius r.)

This set is dense in the plane and has area zero.
Can we talk about tangents for this set?

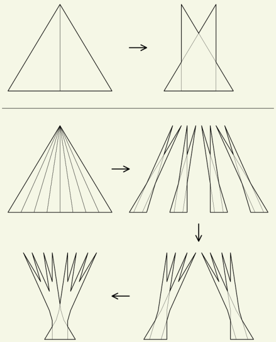
Besicovitch and Kakeya sets

1. What is the area of the smallest shape within which it is possible to rotate a unit needle so that it is facing the opposite way.
2. What is the area of the smallest shape that contains a line segment in every direction.

Besicovitch and Kakeya sets

1. For each $\epsilon > 0$, there is a plane set *E* with area at most ϵ inside which a unit segment may be moved continuously to lie in its original position but rotated through π. (Kakeya sets)
2. There is a plane set of area zero that contains a unit segment in every direction. (Besicovitch set)

Perron trees



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Part 2: Dimension and measures

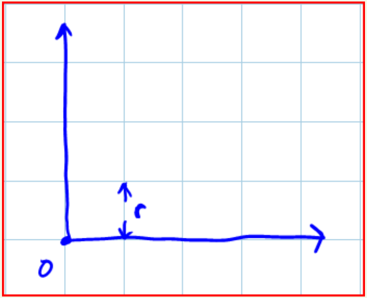
Properties expected of dimension

- points are zero-dimensional
- line segments are one-dimensional
- squares (and other such ‘nice’ shapes) are two-dimensional
- monotonicity: if $E \subseteq F$, then $\dim(E) \leq \dim(F)$.
- finite stability: $\dim(E \cup F) = \max\{\dim(E), \dim(F)\}$
- Also nice to have countable stability:

$$\dim\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup\{\dim(E_i) : i = 1, 2, \ldots\}.$$

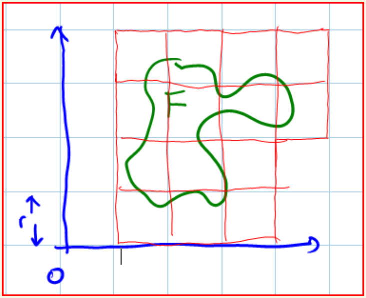
box dimension

- Suppose that F is a (non-empty) bounded set in $\mathbb{R}^2$.
- Fix a scale $r > 0$.
- Consider the (almost disjoint) squares of side r, parallel to the axes that cover the plane.



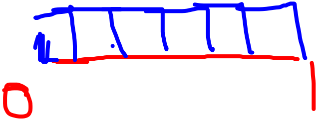
box dimension

- Let $N_r(F)$ be the number of squares from this grid that meet F.



Then $r \mapsto N_r(F)$ is a well-defined, monotonic decreasing function of r for $r > 0$.

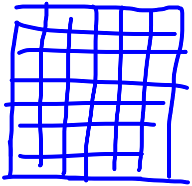
box dimension: line segment



$r \leq 1$

$$\frac{1}{r} \leq N_r([0,1]) \leq \frac{6}{r}$$

box dimension: square



$$N_r([0,1]^2) \sim \frac{1}{r^2}$$

box dimension: formal definition

Upper box dimension

$$\overline{\dim}_B(F) = \limsup_{r \searrow 0} \frac{\log N_r(F)}{-\log r}.$$

Lower box dimension

$$\underline{\dim}_B(F) = \liminf_{r \searrow 0} \frac{\log N_r(F)}{-\log r}.$$

If both limits exist and have a common value, then this is the box dimension, $\dim_B(F)$.

box dimension: Cantor set

box dimension: properties

- $\dim(E) \leq \dim(F)$ if $E \subseteq F$
- Gives right value to sets such as line segments, smooth curves.
- translations, rescalings, rotations and reflections do not change box dimension
- upper box dimension is finitely stable:

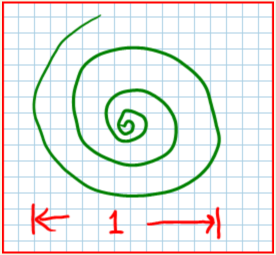
$$\overline{\dim}_B(E \cup F) = \max\{\overline{\dim}_B(E), \overline{\dim}_B(F)\}.$$

- lends itself to numerical estimation
- $\dim_B(F) = \dim_B(\operatorname{clos}(F))$

The last item is a problem: $\dim_B(\mathbb{Q} \cap [0, 1]) = 1$.

Estimating length: 1

Estimating length: spiral

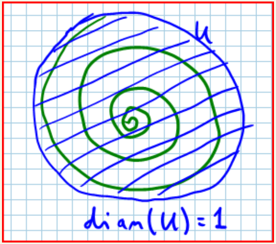


Spiral

Estimating length: 2

Estimating length: any cover

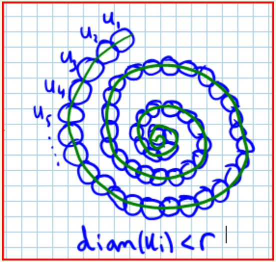
If we estimate length by summing diameters of any cover:



Estimating length: 3

Estimating length

We get a better estimate of length by summing diameters of covering sets whose diameter is at most r, where r is small:



definition: r-covers

r-cover

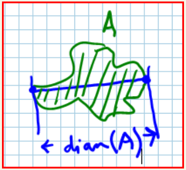
Let $r > 0$. A countable collection of sets $\{U_i : i \in \mathbb{N}\}$ in $\mathbb{R}^n$ is an r-cover of $E \subseteq \mathbb{R}^n$ if

1. $E \subseteq \bigcup_{i \in \mathbb{N}} U_i$
2. $\mathrm{diam}(U_i) \leq r$ for each $i \in \mathbb{N}$.

Interlude: some notation

For a set A and $s \geq 0$, we define

$$|A|^s = \begin{cases} 0 & \text{if A} = \emptyset, \\ 1 & \text{if } A \neq \emptyset \text{ and } s = 0, \\ \text{diam}(A)^s & \text{otherwise.} \end{cases}$$



Hausdorff measures

Definition (s-dimensional Hausdorff measure)

Suppose that F is a subset of $\mathbb{R}^n$ and $s \geq 0$. For any $r > 0$, we define

$$\mathcal{H}^s_r(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is an } r\text{-cover of } F \right\}.$$

The s-dimensional Hausdorff measure is then given by

$$\mathcal{H}^s(F) = \lim_{r \searrow 0} \mathcal{H}^s_r(F).$$

(It is possible to show that s-dimensional Hausdorff measure is in fact a measure — we shall do this later.[1].)

[1] small print: Borel regular outer measure

Simple examples

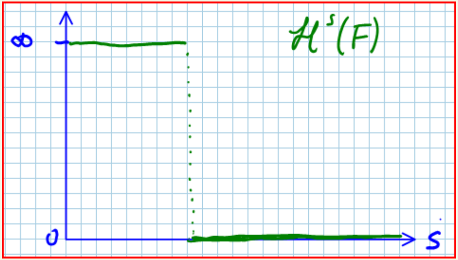
1. For any $s \geq 0$, $\mathcal{H}^s(\emptyset) = 0$.
2. For any point $x \in \mathbb{R}^n$,

$$\mathcal{H}^s(\{x\}) = \begin{cases} 1 & \text{if } s = 0, \\ 0 & \text{if } s > 0. \end{cases}$$

3. If A is a countable set, then $\mathcal{H}^s(A) = 0$ unless $s = 0$.

Properties of Hausdorff measure

1. If $s < t$ and $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$. (So for $t > n$, $\mathcal{H}^t(\mathbb{R}^n) = 0$.)



2. If s is a non-negative integer, then $\mathcal{H}^s$ is a constant multiple of the usual s-dimensional volume. (counting measure, length measure, area, volume etc.) — we shall not prove this.

Properties of Hausdorff measure

1. If $0 < r < \inf\{d(x,y) : x \in E,\, y \in F\}$, then

$$\mathcal{H}^s_r(E \cup F) = \mathcal{H}^s_r(E) + \mathcal{H}^s_r(F)$$

and so, in this case,

$$\mathcal{H}^s(E \cup F) = \mathcal{H}^s(E) + \mathcal{H}^s(F).$$

2. Let $F \subseteq \mathbb{R}^n$ and suppose that $f\colon F \to \mathbb{R}^m$ is a mapping such that for some fixed constants c and α

$$|f(x) - f(y)| \leq c|x - y|^\alpha \text{ whenever } x, y \in F.$$

Then for each s, $\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F)$.

Hausdorff dimension

Definition (Hausdorff dimension)

For a set F, we define the Hausdorff dimension of F by

$$\dim_H(F) = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

Observations

1. Hausdorff dimension is defined for any set F (unlike box dimension).
2. $\dim_H(\emptyset) = 0$
3. If A is a countable set, then $\dim_H(A) = 0$. In particular, $\mathbb{Q}$ has Hausdorff dimension 0.

Properties of Hausdorff dimension

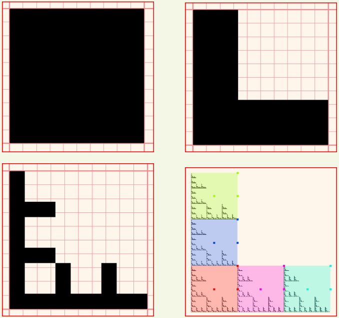
1. If $A \subseteq B$, then $\dim_H(A) \leq \dim_H(B)$.
2. If $F_1, F_2, F_3, \ldots$ is a (countable) sequence of sets, then

$$\dim_H\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup\{\dim_H(F_i) : 1 \leq i \leq \infty\}.$$

3. for bounded sets F, $\dim_H(F) \leq \underline{\dim}_B(F)$.

Another simple example

Let F be the compact set formed as the limit of the following construction.



So we split each square up into 3×3 subsquares and remove the 2×2 subsquares at the top right. What is $\dim_H(F)$?

Heuristic for finding what the dimension could be

Assume that F has positive and finite s-dimensional Hausdorff measure when $s = \dim_H(F)$ and then represent F as a finite disjoint union of scaled copies of F, F_i, say where F_i is a copy of F scaled by λ_i. Then

$$\mathcal{H}^s(F) = \mathcal{H}^s\left(\bigcup_i F_i\right) = \sum_i \mathcal{H}^s(F_i) = \sum_i \lambda_i^s \mathcal{H}^s(F).$$

Dividing through by $\mathcal{H}^s(F)$ then gives

$$1 = \sum_i \lambda_i^s$$

and solving this for s gives the expected Hausdorff dimension, $\log 5/\log 3$.

Covers provide an upper bound

Can use box dimension to find an upper bound, since

$$\dim_H(F) \leq \underline{\dim}_B(F).$$

Upper bound for our example.

Digression: minimal measure theory

(X, d), a metric space. (Usually complete and separable)

Definition: (outer) measures

A set function $\mu\colon \{A : A \subseteq X\} \to [0, \infty]$ is a measure if

1. $\mu(\emptyset) = 0$
2. if $A \subseteq B$, then $\mu(A) \leq \mu(B)$
3. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Definition: measurable set

$A \subseteq X$ is measurable if

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A) \text{ for each } E \subseteq X$$

Basic results

Theorem

If μ is a measure and $\mathcal{M}$ is the μ-measurable sets, then

1. $\mathcal{M}$ is a σ-algebra. [$\emptyset \in \mathcal{M}$, closed under complements and countable unions]
2. if $\mu(A) = 0$, then $A \in \mathcal{M}$
3. if $A_1, A_2, \ldots \in \mathcal{M}$ are pairwise disjoint, then $\mu\left(\bigcup A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$
4. if $A_1, A_2, \ldots \in \mathcal{M}$, then
   1. $\mu\left(\bigcup A_i\right) = \lim_{i \to \infty} \mu(A_i)$, provided $A_1 \subseteq A_2 \subseteq \cdots$
   2. $\mu\left(\bigcap A_i\right) = \lim_{i \to \infty} \mu(A_i)$, provided $A_1 \supseteq A_2 \supseteq \cdots$ and $\mu(A_1) < \infty$

Regular measures

μ is a regular measure if for each $A \subseteq X$, there is a μ-measurable set B with $A \subseteq B$ and $\mu(A) = \mu(B)$.

Lemma

Suppose that μ is a regular measure. If $A_1 \subseteq A_2 \subseteq \cdots$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i).$$

Identifying measurable sets

Definition

The family of Borel sets in a metric space X is the smallest σ-algebra that contains the open subsets of X.

Definition

A measure μ is:

1. a *Borel measure* if the Borel sets are μ-measurable
2. *Borel regular* if it is a Borel measure and for each $A \subseteq X$, there is a Borel set B with $A \subseteq B$ and $\mu(A) = \mu(B)$.

Theorem

Let μ be a measure on X. Then μ is a Borel measure if, and only if,

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

whenever $\inf\{d(x, y) : x \in A,\ y \in B\} > 0$.

Theorem

$\mathcal{H}^s$ *is a Borel regular measure for each* $s \geq 0$.

Definition (support)

The support of a Borel measure μ is defined to be the smallest closed set F for which $\mu(X \setminus F) = 0$.
$\text{spt}(\mu) = X \setminus \bigcup\{U : U \text{ is open and } \mu(U) = 0\}$.

Definition (Mass distribution)

A Borel measure μ is a **mass distribution** on the set F if the support of μ is contained in F and $0 < \mu(F) < \infty$.

Mass distribution principle

Suppose that μ is a mass distribution on a set F and for some real number s, there are $c > 0$ and $r_0 > 0$ so that

$$\mu(U) \leq c|U|^s,$$

whenever U is a set with $|U| \leq r_0$. Then $\mathcal{H}^s(F) \geq \mu(F)/c > 0$ and so $\dim_H(F) \geq s$.

lower bound for our example

Plan

1. Use squares in construction of F and Proposition 1.7 to define a mass distribution on F.
2. Show hypothesis of Mass Distribution Principle is satisfied.
3. Profit! (Or, at the least, deduce a lower bound on Hausdorff dimension of F.)

lower bound for F

Packing measures

Definition (packing)

$\{(x_i, r_i) : i = 1, 2, \ldots, n\}$ is an r-packing of E if

1. $x_i \in E$ for each i
2. $0 < r_i \leq r$ for each i
3. $d(x_i, x_j) \geq \max\{r_i, r_j\}$ for $i \neq j$

Definition (packing measure)

Let $s \geq 0$ be given and let $E \subseteq X$. For $r > 0$, define

$$\mathcal{P}_r^s(E) = \sup\left\{\sum_{i=1}^{n} r_i^s : \{(x_i, r_i)\}_{i=1}^{n} \text{ is an } r\text{-packing of } E\right\}$$

and

$$\mathcal{P}_0^s(E) = \lim_{r \searrow 0} \mathcal{P}_r^s(E).$$

The s-packing measure of E is given by

$$\mathcal{P}^s(E) = \inf\left\{\sum_{i=1}^{\infty} \mathcal{P}_0^s(E_i) : E \subseteq \bigcup_{i=1}^{\infty} E_i\right\}.$$

This defines a Borel regular measure.

Definition (packing dimension)

For $F \subseteq \mathbb{R}^n$, the packing dimension of F is given by

$$\dim_P(F) = \sup\{s : \mathcal{P}^s(F) = +\infty\}.$$

Fortunately in $\mathbb{R}^n$...

Theorem

If $F \subseteq \mathbb{R}^n$, then the packing dimension of F is given by

$$\dim_P(F) = \inf\left\{\sup_i \overline{\dim}_B(F_i) : F \subseteq \bigcup_{i=1}^{\infty} F_i,\ F_i \textit{ bounded}\right\}.$$

Usage

- box commonly used in analysis of real-world phenomena
- Hausdorff dimension commonly used/investigated in pure mathematics
- packing dimension turns out to be important in the study of fine properties of probabilistic phenomena such as Brownian motion.

Some references

K. J. Falconer, *The geometry of fractal sets*, Cambridge University Press, 1985.

K. J. Falconer, *Fractal geometry*, Wiley 2nd Edition, 2003.

P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge University Press, 1995.

Problems

Homework, if interested.

1. Find an example of a compact set $E \subset \mathbb{R}$ for which

$$\underline{\dim}_B(E) < \overline{\dim}_B(E).$$

2. Determine the Hausdorff, packing and box dimensions of

$$\bigcup_{p,q\in\mathbb{Q},\, r\in\mathbb{Q}^+} S((p,q),r).$$

(Where $S((p,q),r)$ denotes the circle centre (p,q) with radius r.)