Vector-valued multifractal measures*

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Abstract

We propose a framework for vector-valued measures which permits vector analogues of the singular measures which arise in multifractal theory. We obtain some of their properties and calculate the multifractal spectrum and L^p-dimensions of certain self-similar vector-valued measures.

1 Introduction

Fractals have been used to model a wide variety of objects which have an intricate structure at arbitrarily small scales and, more recently, multifractal measures have been introduced to describe phenomena that have widely varying densities at small scales, see [7, 10]. For example, the fractally homogenous model of turbulence, introduced by Mandelbrot [16, 17] to explain the phenomenon of intermittency, assumes a cascade of eddies of decreasing sizes with energy dissipation concentrated on an approximate fractal formed by the smallest scale eddies. Alternatively, the distribution of the rate of dissipation of kinetic energy in fully developed turbulence has been described as a multifractal measure of widely varying and singular density [16].

However, the behaviour of fluids, electric or magnetic fields and many other phenomena is more suited to a *vector* description than by a set or a density. Thus it is useful to have ways of representing fine scale 'fractal' features of a vectorial form, see Figure 1.

In seeking an extension of the multifractal formalism to the vector situation, it is natural to look to the well-established theory of vector-valued measures, see [4, Chapter IV, §10] or [5, Chapter 8.19]. For example, the instantaneous velocity of a moving continuum can be described by the vector valued momentum measure $\boldsymbol{\mu}$. Thus $\boldsymbol{\mu}(A)$ is the total (vector) momentum of the region A of the continuum. Vector addition of momenta corresponds to the additive property of measures: the vector identity

$$\boldsymbol{\mu}\left(\cup_{i=1}^{m} A_i\right) = \boldsymbol{\mu}(A_1) + \ldots + \boldsymbol{\mu}(A_m)$$

holds whenever $A_1, \ldots, A_m$ are disjoint regions.

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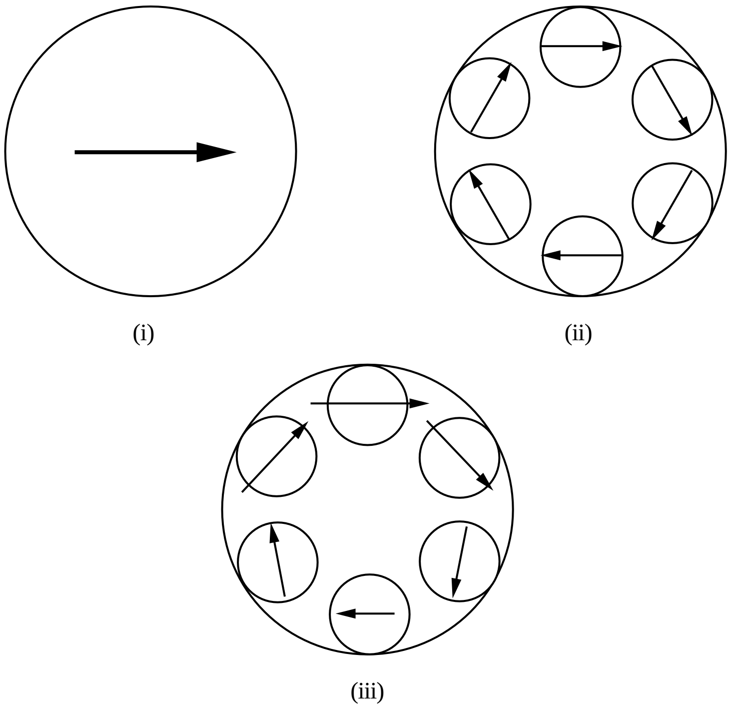

Figure 1: A simple model of a moving turbulent spot: in *(i)* we see the overall momentum of the spot, in *(ii)* we see the momenta of the subcomponents of the spot relative to its motion downstream and in *(iii)* we see the overall momenta of the subcomponents relative to a stationary observer. In a self-similar model we would iterate self-similar transformations which map the vector in *(i)* to those in *(iii)*.

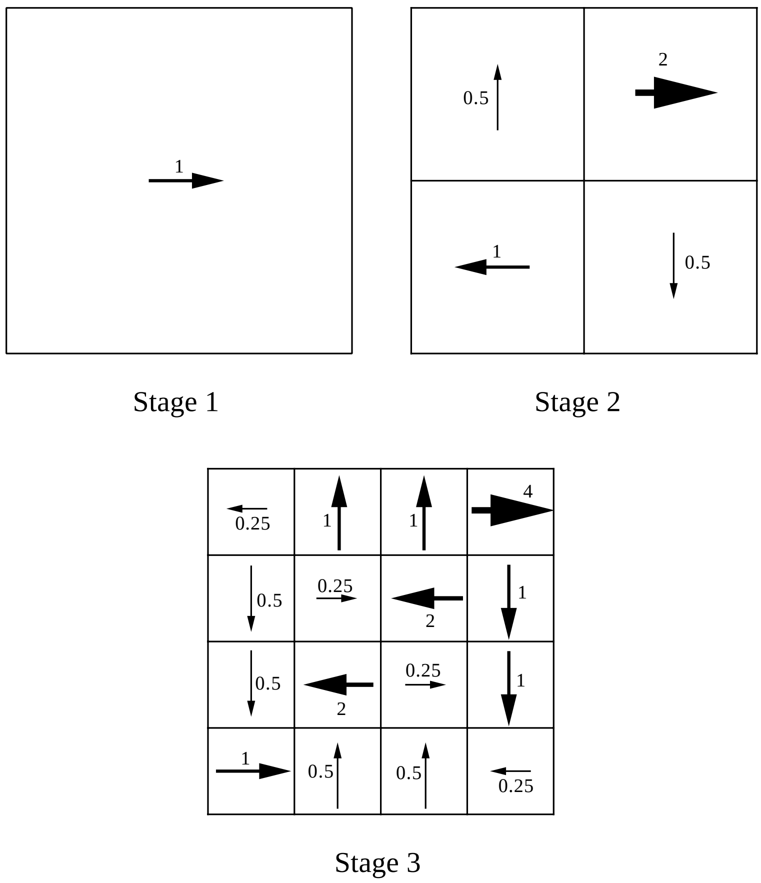

Figure 2: The first three steps in the construction of a highly singular vector measure supported by the unit square. The magnitude of vectors is indicated by their thickness.

The classical theory of vector-valued measures is applicable to measures that are (locally) absolutely integrable in the sense that $\int \|d\boldsymbol{\mu}\| < \infty$ or, equivalently, that for every bounded set B

$$\sup\left\{\sum_{i=1}^{\infty}\|\boldsymbol{\mu}(A_i)\| : \cup_{i=1}^{\infty}A_i \text{ is a disjoint decomposition of } B\right\} < \infty.$$

Vector measures that fail to satisfy such a condition are inherently unstable, for example integrals of functions cannot in general be defined with respect to such measures in a consistent manner. Whilst smoothly defined vector measures (that might represent momentum measures of non-turbulent flows) may be absolutely integrable, this will not be the case for the vector analogues of the highly singular measures, such as self-similar measures, that are considered in multifractal theory, see Figure 2 for an example of a singular self-similar construction supported by the unit square. Of course, in any physical situation, the question of infinitesimal decomposition does not arise, but from the theoretical point of view it is desirable to be able to work with vector analogues of self-similar and other multifractal measures. Therefore, a mathematical setting is required in which singular vector measures exist and can be manipulated in a consistent manner.

In this paper we establish a framework for vector-valued measures appropriate to the multifractal situation, and we examine properties of such measures. The crux of the matter is that, in order to enable us to work with sufficiently singular measures, we have to restrict integration to functions that are Lipschitz continuous. In practice, this does not seem to be a major restriction, and in some ways it is mathematically more natural to work with Lipschitz functions rather than continuous functions. (An alternative approach might be to work with vector-valued distributions, but this would lead to unnecessarily strong restrictions on the class of integrable functions.)

We go on to exhibit a class of 'self-similar vector measures' analogous to the self-similar fractals and self-similar multifractal measures that have been analysed extensively and used to model a wide variety of phenomenon. Iterated function schemes [8, Chapter 9] have become a standard tool for representing wide classes of fractals and multifractal measures, and we show how this idea may be extended to define vector measures.

Multifractal measures are often analysed by means of their 'multifractal spectrum' or 'singularity spectrum', which gives the dimension of the set of points at which the local density of the measure obeys a given power law. In Section 4 we examine vector-valued measures at scale r by integrating the measure against a suitable 'sampling function of width r',denoted by ϕ_r, which indicates the size of sets on which 'local activity' has a given concentration. In particular, we calculate a multifractal spectrum for certain self-similar vector multifractals. Such multifractals may be more appropriate than those obtained from conventional (scalar) multifractal models for representing situations where there is an underlying vector structure. In Section 5 we calculate the L^p-dimensions [22] of self-similar vector multifractals and relate these to their spectra.

2 Generalised vector-valued measures

We describe a natural setting for vector-valued measures with multifractal features, and obtain some results on the existence and uniqueness of such measures.

Let X be a locally compact metric space; usually X will be a closed subset of Euclidean space $\mathbf{R}^d$. Let $\mathrm{Lip}\,(X)$ denote the space of real-valued Lipschitz functions on X, with the Lipschitz constant $\mathrm{Lip}\,(f)$ of $f \in \mathrm{Lip}\,(X)$ given by

$$\mathrm{Lip}\,(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Fixing $x_0 \in X$ we define a norm on $\mathrm{Lip}\,(X)$ by

$$\|f\|_{\mathrm{Lip}} := |f(x_0)| + \mathrm{Lip}\,(f);$$

it is easily checked that $\mathrm{Lip}\,(X)$ is complete under this norm.

Let $(\mathbf{E}, \|.\|)$ be a Banach space (i.e. a complete normed space; in most examples $\mathbf{E}$ will be a Euclidean space with the usual norm). Define $\mathcal{M} \equiv \mathcal{L}(\mathrm{Lip}\,(X), \mathbf{E})$ as the space of bounded linear mappings from the space of functions $\mathrm{Lip}\,(X)$ to $\mathbf{E}$. We write $\int f \, d\boldsymbol{\mu}$ for the image of $f \in \mathrm{Lip}\,(X)$ under $\boldsymbol{\mu} \in \mathcal{M}$. Thus we think of $\mathcal{M}$ as a space of vector measures with values in $\mathbf{E}$, but with the proviso that they may only be used to integrate Lipschitz functions. For the purposes of this paper we refer to the members of $\mathcal{M}$ as *generalised vector* *measures*. The induced norm on $\mathcal{M}$ is given by $\|\boldsymbol{\mu}\| = \sup_{\|f\|_{\text{Lip}}=1} \left\| \int f\, d\boldsymbol{\mu} \right\|$ so that

$$\left\| \int f\, d\boldsymbol{\mu} \right\| \leq \|f\|_{\text{Lip}} \|\boldsymbol{\mu}\| \tag{1}$$

(with three different norms occurring here). Since both $\text{Lip}\,(X)$ and $\mathbf{E}$ are complete, so is $\mathcal{M}$.

This procedure should be compared with the usual definition of vector-valued measures as elements of $\mathcal{L}(C(X), \mathbf{E})$ where $C(X)$ is the space of continuous functions on X with the supremum norm [5, Chapter 8.19]. However, to include the highly singular multifractal vector measures that we are especially interested in, it is necessary to replace $C(X)$ by the smaller space $\text{Lip}\,(X)$. Nevertheless, it is often possible to extend $\boldsymbol{\mu} \in \mathcal{M}$ to act on certain functions that are not Lipschitz.

We define the support, $\text{Spt}\,\boldsymbol{\mu}$, of $\boldsymbol{\mu}$, to be the largest closed set $F \subset X$ such that if $f \in \text{Lip}\,(X)$ is zero on F then $\int f\, d\boldsymbol{\mu} = 0$. In general, $\boldsymbol{\mu}(A)$ has no meaning for $\boldsymbol{\mu} \in \mathcal{M}$ and $A \subset X$. However, if A and B are subsets of X that are separated by a positive distance and $\text{Spt}\,\boldsymbol{\mu} \subset A \cup B$ then we can define $\boldsymbol{\mu}(A) = \int f\, d\boldsymbol{\mu}$ for any $f \in \text{Lip}\,(X)$ with $f(x) = 1$ for $x \in A$ and $f(x) = 0$ for $x \in B$. In particular, we may take $\boldsymbol{\mu}(X) = \int 1\, d\boldsymbol{\mu}$.

The following construction, based on a hierarchy of subsets of $\mathbf{R}^d$ indexed by certain sequences, yields many examples of vector-valued multifractals. For $k = 0, 1, 2, \ldots$ let I_k be a family of k-term sequences $\{\mathbf{i} = (i_1, \ldots, i_k)\}$ of positive integers with the property that if $(i_1, \ldots, i_k) \in I_k$ then $(i_1, \ldots, i_j) \in I_j$ for every $j \leq k$. We write $I = \cup_{k=0}^{\infty} I_k$ for the set of all such finite sequences, and I_∞ for those infinite sequences of positive integers $(i_1, i_2, \ldots)$ such that $(i_1, \ldots, i_k) \in I_k$ for all k. (Frequently we will have $I_k = \{\mathbf{i} = (i_1, \ldots, i_k) : 1 \leq i_j \leq N\}$ for some integer $N \geq 2$, that is the set of all k-term sequences of integers between 1 and N, but this is not essential.) For each $\mathbf{i} \in I_k$ let $A_\mathbf{i}$ be a non-empty subset of $\mathbf{R}^d$, with these sets satisfying $A_\mathbf{i} \supset \cup_i A_{\mathbf{i},i}$ (where $\mathbf{i}, i$ denotes concatenation) with this union disjoint, and assume that $\text{diam}\,(A_{i_1,\ldots,i_k}) \to 0$ as $k \to \infty$ for every sequence $(i_1, i_2, \ldots) \in I_\infty$. Write $\mathcal{A}$ for the ring of sets generated by $\{A_\mathbf{i} : \mathbf{i} \in I_k\}$, that is the family of sets comprising finite unions of these. Typically, the sets $A_\mathbf{i}$ might be the hierarchy of sets used in defining a self-similar fractal; for example, we might have $I_k = \{\mathbf{i} = (i_1, \ldots, i_k) : 1 \leq i_j \leq 2\}$ and $\{A_\mathbf{i} : \mathbf{i} \in I_k\}$ the set of 2^k intervals of length 3^{-k} that occur at the k-th stage in the usual construction of the middle third Cantor set.

Assume that $\boldsymbol{\mu}_0(A_\mathbf{i}) \in \mathbf{E}$ is defined for $\mathbf{i} \in I$ in a formally consistent additive manner, i.e. for $\mathbf{i} \in I_k$

$$\boldsymbol{\mu}_0\,(A_\mathbf{i}) = \sum_{i:\, \mathbf{i},i \in I_{k+1}} \boldsymbol{\mu}_0\,(A_{\mathbf{i},i})\,. \tag{2}$$

Under certain circumstances this is sufficient to define a generalised vector measure $\boldsymbol{\mu} \in \mathcal{M}$. Condition (4) in the following proposition guarantees that $\boldsymbol{\mu}$ can be used to integrate Lipschitz functions. In many situations, for example when there is a constant c such that $\text{diam}\,(A_{\mathbf{i},i}) \geq c\, \text{diam}\,(A_\mathbf{i})$ for all $\mathbf{i}$ and i, this can be replaced by the more natural condition

$$\sum_{k=0}^{\infty} \sum_{\mathbf{i} \in I_k} \|\boldsymbol{\mu}_0(A_\mathbf{i})\|\, \text{diam}\,(A_\mathbf{i}) < \infty. \tag{3}$$

Proposition 2.1 *Suppose that*

$$\sum_{k=0}^{\infty} \sum_{\mathbf{i} \in I_k} \sum_{i:\, \mathbf{i},i \in I_{k+1}} \|\boldsymbol{\mu}_0(A_{\mathbf{i},i})\| \operatorname{diam}(A_{\mathbf{i}}) \equiv M < \infty. \tag{4}$$

Then for $x_{\mathbf{i}} \in A_{\mathbf{i}}$ *and* $f \in \operatorname{Lip}(X)$ *the limit*

$$\int f\, d\boldsymbol{\mu} \equiv \lim_{k\to\infty} \sum_{\mathbf{i} \in I_k} f(x_{\mathbf{i}}) \boldsymbol{\mu}_0(A_{\mathbf{i}}) \tag{5}$$

exists in $\mathbf{E}$ *and is independent of the points* $x_{\mathbf{i}} \in A_{\mathbf{i}}$ *chosen. Moreover, the linear mapping* $\boldsymbol{\mu} : f \to \int f\, d\boldsymbol{\mu}$ *defined in this way is in* $\mathcal{M}$.

Proof: Writing $S_k(f) = \sum_{\mathbf{i} \in I_k} f(x_{\mathbf{i}}) \boldsymbol{\mu}_0(A_{\mathbf{i}})$ we have, using (2), that

$$\begin{aligned} \|S_{k+1}(f) - S_k(f)\| &= \left\| \sum_{\mathbf{i} \in I_k} \sum_{i:\, \mathbf{i},i \in I_{k+1}} (f(x_{\mathbf{i},i}) - f(x_{\mathbf{i}})) \boldsymbol{\mu}_0(A_{\mathbf{i},i}) \right\| \\ &\le \sum_{\mathbf{i} \in I_k} \sum_{i:\, \mathbf{i},i \in I_{k+1}} \operatorname{diam}(A_{\mathbf{i}}) \operatorname{Lip}(f) \|\boldsymbol{\mu}_0(A_{\mathbf{i},i})\|. \end{aligned} \tag{6}$$

It follows using (4) that $S_k(f)$ is a Cauchy sequence in $\mathbf{E}$ and so converges to a limit (5) in $\mathbf{E}$. Now let $x'_{\mathbf{i}} \in A_{\mathbf{i}}$ be another set of representative points and write $S'_k(f) = \sum_{\mathbf{i} \in I_k} f(x'_{\mathbf{i}}) \boldsymbol{\mu}_0(A_{\mathbf{i}})$. Then

$$\begin{aligned} \|S'_k(f) - S_k(f)\| &= \left\| \sum_{\mathbf{i} \in I_k} (f(x'_{\mathbf{i}}) - f(x_{\mathbf{i}})) \boldsymbol{\mu}_0(A_{\mathbf{i}}) \right\| \\ &\le \sum_{\mathbf{i} \in I_k} \operatorname{diam}(A_{\mathbf{i}}) \operatorname{Lip}(f) \|\boldsymbol{\mu}_0(A_{\mathbf{i}})\| \\ &\le \sum_{\mathbf{i} \in I_{k-1}} \sum_{i:\, \mathbf{i},i \in I_k} \operatorname{diam}(A_{\mathbf{i}}) \operatorname{Lip}(f) \|\boldsymbol{\mu}_0(A_{\mathbf{i},i})\| \\ &\to 0 \end{aligned}$$

as $k \to \infty$. It follows that S_k and S'_k have a common limit, so (5) is well defined. Since $f \mapsto S_k(f)$ is linear and, as (6) gives that $\|S_k(f) - S_1(f)\| \le M \|f\|_{\mathrm{Lip}}$, it is uniformly bounded in k, (5) defines an element of $\mathcal{M}$ as required. ■

The support of $\boldsymbol{\mu}$ is contained in $\cap_{k=0}^{\infty} \cup_{\mathbf{i} \in I_k} \overline{A_{\mathbf{i}}}$ in the sense that $\int f\, d\boldsymbol{\mu} = 0$ for any $f \in \operatorname{Lip}(X)$ that vanishes on this set. Care is required in identifying $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_0$, since $\boldsymbol{\mu}(A_{\mathbf{i}}) = \int_{A_{\mathbf{i}}} f\, d\boldsymbol{\mu}$ is undefined unless $\mathbf{i} \in I_k$ and $A_{\mathbf{i}}$ is a positive distance from $A_{\mathbf{i}'}$ for all $\mathbf{i}' \in I_k$ with $\mathbf{i}' \ne \mathbf{i}$. However, in this case we may define $\boldsymbol{\mu}(A_{\mathbf{i}}) = \int_{A_{\mathbf{i}}} f\, d\boldsymbol{\mu}$ for any $f \in \operatorname{Lip}(X)$ with $f(x) = 1$ if $x \in A_{\mathbf{i}}$ and $f(x) = 0$ if $x \in A_{\mathbf{i}'}$ for $\mathbf{i}' \in I_k$ not equal to $\mathbf{i}$ and we recover $\boldsymbol{\mu}_0(A_{\mathbf{i}}) = \boldsymbol{\mu}(A_{\mathbf{i}})$.

By way of example, a simple but natural way of defining generalised vector measures on certain curves is to try to make the vector measure of an arc equal to the vector difference of the end points of the arc. Let C be a curve in X (i.e. a homeomorphic image of $[0, 1]$ contained in X). We define the curve dimension of C in an analogous way to the box-counting dimension of C but using coverings by subarcs of C. For all $\delta > 0$ we define $N(\delta)$ to be the least integer N such that there exists a dissection $x_1, \ldots, x_N$ of C with x_0 and x_N the endpoints of C such that the diameter of the subarc of C between x_{i-1} and x_i is at most δ for $i = 1, 2, \ldots, N$. Then we define the *curve dimension* of C as

$$\dim_{\mathrm{C}} C := \limsup_{\delta \to 0} \frac{\log N(\delta)}{-\log \delta}.$$

In particular, if $s > \dim_{\mathrm{C}} C$ then there is a constant $c > 0$ such that $N(\delta) \le c\delta^{-s}$ for $0 < \delta \le 1$.

Example 2.2 *Let C be a curve in $\mathbf{R}^d$ with $\dim_{\mathrm{C}} C < 2$. Then writing $\boldsymbol{\mu}_0(A) = \mathbf{b} - \mathbf{a} \in \mathbf{R}^d$ where A is the arc of C between $\mathbf{a}$ and $\mathbf{b}$, defines a generalised measure $\boldsymbol{\mu}$ on C taking values in $\mathbf{R}^d$.*

Proof: We apply Proposition 2.1. Let $\dim_{\mathrm{C}} C < s < 2$; then

$$N(\delta) \le c\delta^{-s} \tag{7}$$

for some constant $c > 0$ and for all $\delta \le 1$. Notice that C may be divided into $N(2^{-k})$ subarcs of diameters at most 2^{-k} for $k = 1, 2, \ldots$. We proceed to define inductively indexing sets $I_{k+1} \subset \prod_{j=1}^{k+1} \{1, \ldots, N(2^{-j})\}$ and a nested hierarchy $\mathcal{A}_{k+1}$ of disjoint subarcs of C of diameter at most $2^{-(k+1)}$. Set $I_1 = \{1, \ldots, N(2^{-1})\}$ and let $\mathcal{A}_1 = \{A_i : i \in I_1\}$ be a dissection of C into $N(2^{-1})$ disjoint subarcs of diameter at most 2^{-1}. Supppose $k \in \mathbf{N}$ and that $\mathcal{C}_{k+1} = \{C_i : 1 \le i \le N(2^{-(k+1)})\}$ is a dissection of C into $N(2^{-(k+1)})$ disjoint subarcs of diameter at most $2^{-(k+1)}$. Define

$$I_{k+1} = \left\{(i_1, \ldots, i_k, j) : (i_1, \ldots, i_k) \in I_k,\ A_{(i_1,\ldots,i_k)} \cap C_j \ne \emptyset\right\}$$

and set

$$\mathcal{A}_{k+1} = \left\{A_{(i_1,\ldots,i_k)} \cap C_j : (i_1, \ldots, i_k, j) \in I_{k+1}\right\}.$$

Using (7)

$$\operatorname{card} \mathcal{A}_k \le \sum_{j=0}^{k} N(2^{-j}) \le \frac{c}{2^s - 1} 2^{(k+1)s} \tag{8}$$

and so

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{\mathbf{i} \in I_k} \sum_{i:\, \mathbf{i},i \in I_{k+1}} \|\boldsymbol{\mu}_0(A_{\mathbf{i},i})\| \operatorname{diam}(A_{\mathbf{i}}) &\le \sum_{k=0}^{\infty} \sum_{\mathbf{i} \in I_k} \sum_{i:\, \mathbf{i},i \in I_{k+1}} 2^{-(k+1)} \times 2^{-k} \\ &\le \sum_{k=0}^{\infty} 2^{-(2k+1)} \operatorname{card} \mathcal{A}_k \\ &\le \frac{c 2^{s-1}}{2^s - 1} \sum_{k=0}^{\infty} 2^{k(s-2)} \text{ by (8)} \\ &< \infty \text{ since } s < 2. \end{aligned}$$

It now follows directly from Proposition 2.1 that $\boldsymbol{\mu}_0$ defines a generalised measure $\boldsymbol{\mu} \in \mathcal{M}$. ∎

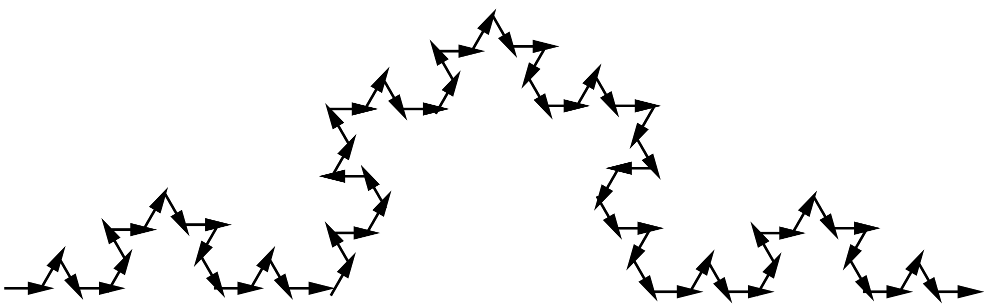

Figure 3: One stage in the construction of a generalised vector-valued measure supported by the von Koch curve.

For a specific example of this construction, let C be the von Koch curve, obtained from a unit segment by repeated replacement of the middle third of intervals by the other two sides of the equilateral triangle on the same base, see Figure 3. Then C has curve dimension $\log 4/\log 3$. Taking $I_k = \{\mathbf{i} = (i_1, \ldots, i_k) : 1 \leq i_j \leq 4\}$ and $\{A_\mathbf{i} : \mathbf{i} \in I_k\}$ as the natural hierarchy of scaled down similar copies of the von Koch curve, this procedure gives $\|\boldsymbol{\mu}_0(A_\mathbf{i})\| = 3^{-k}$ for $\mathbf{i} \in I_k$, so $\sum_{\mathbf{i}\in I_k} \|\boldsymbol{\mu}_0(A_\mathbf{i})\| = 4^k 3^{-k}$ which is unbounded in k. Attempting to extend $\boldsymbol{\mu}_0$ directly to subsets of C leads to a vector measure that fails to be absolutely integrable, and it is easy to define a bounded (non-Lipschitz) function f on C such that $\lim_{k\to\infty} \sum_{\mathbf{i}\in I_k} f(x_\mathbf{i})\boldsymbol{\mu}_0(A_\mathbf{i})$ takes different values (or does not exist at all) according to the $x_\mathbf{i} \in A_\mathbf{i}$ chosen. Nevertheless, in our framework, $\boldsymbol{\mu}_0$ leads to a vector-measure $\boldsymbol{\mu}$ on C that is enough to integrate Lipschitz functions. (Observe that the curves we have discussed here do not satisfy a strong separation condition and hence are not covered by the multifractal framework developed in the rest of this paper.)

We now show how certain generalised vector measures may be specified by their invariance under certain transformations. This is analogous to the way that conventional self-similar multifractal measures are invariant under iterated function schemes weighted by probabilities, see [11]. For each $\mathbf{e} \in \mathbf{E}$ write $\mathcal{M}_\mathbf{e}$ for the set of $\boldsymbol{\mu} \in \mathcal{M}$ such that $\boldsymbol{\mu}(X) = \int 1\, d\boldsymbol{\mu} = \mathbf{e}$. Then $\mathcal{M}_\mathbf{e}$ is a closed, and therefore a complete, subset of $\mathcal{M}$. Moreover, if $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathcal{M}_\mathbf{e}$ for some $\mathbf{e} \in \mathbf{E}$ then, since $\int f\, d(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$ is unaffected by adding a constant to f,

$$\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\| = \sup_{\operatorname{Lip} f \leq 1} \| \int f\, d(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\|. \tag{9}$$

(Thus we may regard $\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|$ as defining a distance on the subset $\mathcal{M}_\mathbf{e}$.)

Proposition 2.3 *For $i = 1, 2, \ldots, N$, let S_i be contractions on X and let $T_i \in \mathcal{L}(\mathbf{E}, \mathbf{E})$ satisfy $\sum_{i=1}^N T_i = I$ (the identity on $\mathbf{E}$). Suppose that*

$$\sum_{i=1}^N \|T_i\| \operatorname{Lip} S_i < 1. \tag{10}$$

Then for all $\mathbf{e} \in \mathbf{E}$ *there exists a unique* $\boldsymbol{\mu} \in \mathcal{M}_{\mathbf{e}}$ *such that*

$$\int f\, d\boldsymbol{\mu} = \sum_{i=1}^{N} T_i \left(\int [f \circ S_i]\, d\boldsymbol{\mu} \right) \tag{11}$$

for all $f \in \operatorname{Lip}(X)$.

Proof: The proof mirrors that given by Hutchinson for the non-vector case in [11]: Since $\sum_{i=1}^{N} T_i \mathbf{e} = \mathbf{e}$, the formula,

$$\int f\, d(\psi(\boldsymbol{\mu})) = \sum_{i=1}^{N} T_i \left(\int [f \circ S_i]\, d\boldsymbol{\mu} \right)$$

for $f \in \operatorname{Lip}(X)$ defines a mapping $\psi : \mathcal{M}_{\mathbf{e}} \to \mathcal{M}_{\mathbf{e}}$. If $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathcal{M}_{\mathbf{e}}$ then by (9)

$$\begin{aligned}
\|\psi\boldsymbol{\mu}_1 - \psi\boldsymbol{\mu}_2\| &= \sup_{\operatorname{Lip} f \le 1} \left\| \int f\, d(\psi\boldsymbol{\mu}_1 - \psi\boldsymbol{\mu}_2) \right\| \\
&= \sup_{\operatorname{Lip} f \le 1} \left\| \sum_{i=1}^{N} T_i \left(\int [f \circ S_i]\, d(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \right) \right\| \\
&\le \sum_{i=1}^{N} \|T_i\| \sup_{\operatorname{Lip} f \le 1} \left\| \int [f \circ S_i]\, d(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \right\| \\
&\le \sum_{i=1}^{N} \|T_i\| \sup_{\operatorname{Lip} g \le \operatorname{Lip} S_i} \left\| \int g\, d(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \right\| \\
&\le \sum_{i=1}^{N} \|T_i\| (\operatorname{Lip} S_i) \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|.
\end{aligned}$$

Thus ψ is a contraction on the complete metric space $\mathcal{M}_{\mathbf{e}}$ and so the result follows on using the contraction mapping theorem. ∎

Observe that if $\mathbf{e} = \mathbf{0}$ in the above proposition then $\boldsymbol{\mu}$ is the zero vector measure.

In the special case when $\mathbf{E}$ is $\mathbf{R}$ then $\sum_{i=1}^{N} T_i = 1$ and so (10) becomes the usual condition for the existence of self-similar signed measures; in particular if the T_i are all non-negative real numbers (necessarily summing to 1) then we obtain a self-similar measure supported by the non-empty compact set determined by the S_i. If $\mathbf{E}$ is $\mathbf{C}$, the complex numbers, then we obtain the self-similar complex-valued distributions discussed by Strichartz in [21].

Recall that by the standard theory of iterated function schemes, see [8, Section 9.1] or [11], since the S_i are contractions there is a unique non-empty compact set K satisfying

$$K = \bigcup_{i=1}^{N} S_i(K) \tag{12}$$

and that writing $K(\mathbf{i}) = K((i_1, \ldots, i_k)) = S_{i_1} \circ \ldots \circ S_{i_k}(K)$ where $\mathbf{i} = (i_1, \ldots, i_k)$ and $1 \le i_j \le N$, we have $K = \cup_{\mathbf{i} \in I_k} K(\mathbf{i})$ for each k.

In the event that the union in (12) is disjoint we may obtain a characterisation of the generalised measure of Proposition 2.3 using Proposition 2.1.

Corollary 2.4 *With the S_i and T_i as in Proposition 2.3 and K as in (12), with this union disjoint, let $\boldsymbol{\mu}_0(K((i_1,\ldots,i_k))) = T_{i_1}\circ\ldots\circ T_{i_k}(\mathbf{e})$. Then the generalised measure $\boldsymbol{\mu}\in\mathcal{M}_{\mathbf{e}}$ defined by* (5) *(with $A_{\mathbf{i}} = K(\mathbf{i})$) is just the invariant measure characterised by* (11)*. Moreover, $\boldsymbol{\mu}$ is supported by the set K.*

Proof: For $\mathbf{i} = (i_1,\ldots,i_k)$and $1\le i\le n$ we have

$$\begin{aligned}&\|\boldsymbol{\mu}_0(K(\mathbf{i},i))\|\mathrm{diam}\,(K(\mathbf{i}))\\ &\qquad\le\quad \|T_{i_1}\|\ldots\|T_{i_k}\|\|T_i\|\|\mathbf{e}\|(\mathrm{Lip}\,S_{i_1})\ldots(\mathrm{Lip}\,S_{i_k})\mathrm{diam}\,(K)\end{aligned}$$

and so

$$\sum_{\mathbf{i}\in I_k}\sum_{i=1}^N\|\boldsymbol{\mu}_0(K(\mathbf{i},i))\|\mathrm{diam}\,(K_{\mathbf{i}})\le\left(\sum_{i=1}^N\|T_i\|\mathrm{Lip}\,S_i\right)^k\left(\sum_{i=1}^N\|T_i\|\|\mathbf{e}\|\mathrm{diam}\,(K)\right).$$

Thus (4) holds, so there exists a generalised vector measure $\boldsymbol{\mu}$ satisfying (5). Then taking $x_{\mathbf{i}}\in K(\mathbf{i})$ we have

$$\begin{aligned}\sum_{i=1}^N T_i\left(\int[f\circ S_i]\,d\boldsymbol{\mu}\right) &= \lim_{k\to\infty}\sum_{\mathbf{i}\in I_k}\sum_{i=1}^N f(S_i(x_{\mathbf{i}}))T_i(\boldsymbol{\mu}_0(K(\mathbf{i})))\\ &= \lim_{k\to\infty}\sum_{\mathbf{i}\in I_k}\sum_{i=1}^N f(x_{i,\mathbf{i}})\boldsymbol{\mu}_0(K(i,\mathbf{i}))\\ &= \int f\,d\boldsymbol{\mu},\end{aligned}$$

where $x_{i,\mathbf{i}}\in K(i,\mathbf{i})$, using (5) again. Thus $\boldsymbol{\mu}$ is the unique generalised vector measure satisfying (11). It is now immediate that $\mathrm{Spt}\,\boldsymbol{\mu}\subset K$, since if $f(x)=0$ for $x\in K$ then $\int f\,d\boldsymbol{\mu}=0$ by (5). ∎

We will require the following adapted version of Fubini's Theorem for generalised vector measures. Conditions (13) and (14) are satisfied, for example, by functions f with bounded second derivatives. Under such conditions we can interchange the order of integration with respect to Lebesgue measure and with respect to our generalised measures. Some care is required however, since with generalised measures we lack the 'order property' of integration.

Proposition 2.5 *Suppose that $f\colon\mathbf{R}^d\times\mathbf{R}^d\to\mathbf{R}$ and $C>0$ are such that for all $x,x',y,y'\in\mathbf{R}^d$*

$$|f(x,y)-f(x',y')|\le C\left(\|x-x'\|+\|y-y'\|\right), \tag{13}$$

$$|f(x,y)-f(x,y')-f(x',y)+f(x',y')|\le C\|x-x'\|\|y-y'\| \tag{14}$$

and

$$\int|f(x,y)|\,d\mathcal{L}(y)<\infty \text{ for all } x. \tag{15}$$

Let $\boldsymbol{\mu}$ be a generalised vector measure on $\mathbf{R}^d$ with compact support which takes values in $\mathbf{R}^M$. Then

$$\iint f(x,y)\,d\boldsymbol{\mu}(x)d\mathcal{L}(y)=\iint f(x,y)\,d\mathcal{L}(y)d\boldsymbol{\mu}(x) \tag{16}$$

where $\mathcal{L}$ denotes Lebesgue measure on $\mathbf{R}^d$.

Proof: Since f satisfies (15) it is sufficient to prove the result when f has compact support in $\mathbf{R}^d \times \mathbf{R}^d$ so suppose $X \subset \mathbf{R}^d$ is a compact set chosen so that $\operatorname{Spt} \boldsymbol{\mu} \subset X$ and $\operatorname{Spt} f(x,.) \subset X$, for all $x \in \mathbf{R}^d$. Let $\mathcal{A}_i$ be a sequence of finite Borel partitions of X such that

$$\lim_{i \to \infty} \sup_{A \in \mathcal{A}_i} \operatorname{diam}(A) = 0 \tag{17}$$

and for each $A \in \mathcal{A}_i$ let $y_A \in A$. We have that if $h : \mathbf{R}^d \to \mathbf{R}^M$ is continuous then

$$\|\sum_{A \in \mathcal{A}_i} h(y_A)\mathcal{L}(A) - \int_X h(y)\, d\mathcal{L}(y)\| \to 0 \text{ as } i \to \infty.$$

For each $i \in \mathbf{N}$ we have

$$\sum_{A \in \mathcal{A}_i} \int f(x, y_A)\, d\boldsymbol{\mu}(x)\mathcal{L}(A) = \int \left(\sum_{A \in \mathcal{A}_i} f(x, y_A)\mathcal{L}(A) \right) d\boldsymbol{\mu}(x).$$

In order to pass to the limit (16) it is sufficient to show that $\int f(x,.)d\boldsymbol{\mu}(x)$ is continuous (for the left hand side) and

$$\left\| \int_X f(.,y)d\mathcal{L}(y) - \sum_{A \in \mathcal{A}_i} f(.,y_A)\mathcal{L}(A) \right\|_{\text{Lip}} \to 0 \text{ as } i \to \infty$$

(for the right hand side). Observe that, by (13) and (14)

$$\begin{aligned} \| \int f(x,y)\, d\boldsymbol{\mu}(x) - \int f(x,z)\, d\boldsymbol{\mu}(x) \| & \\ &= \| \int (f(x,y) - f(x,z))\, d\boldsymbol{\mu}(x) \| \\ &\leq \|\boldsymbol{\mu}\| \|f(.,y) - f(.,z)\|_{\text{Lip}} \\ &= \|\boldsymbol{\mu}\| \left(|f(x_0,y) - f(x_0,z)| + \operatorname{Lip}[f(.,y) - f(.,z)] \right) \\ &\leq 2C\|\boldsymbol{\mu}\|\|y - z\| \end{aligned}$$

and so $\int f(x,.)\, d\boldsymbol{\mu}(x)$ is continuous.

It is clear that

$$\left| \int_X f(x_0,y)d\mathcal{L}(y) - \sum_{A \in \mathcal{A}_i} f(x_0,y_A)\mathcal{L}(A) \right| \to 0 \text{ as } i \to \infty$$

and thus it only remains to verify that

$$\operatorname{Lip}\left(\int_X f(x,y)d\mathcal{L}(y) - \sum_{A \in \mathcal{A}_i} f(x,y_A)\mathcal{L}(A) \right) \to 0 \text{ as } i \to \infty. \tag{18}$$

Fix $x, x' \in \mathbf{R}^d$ and observe that

$$\left| \int_X f(x,y)d\mathcal{L}(y) - \sum_{A \in \mathcal{A}_i} f(x,y_A)\mathcal{L}(A) - \int_X f(x',y)d\mathcal{L}(y) + \sum_{A \in \mathcal{A}_i} f(x',y_A)\mathcal{L}(A) \right|$$

$$
\begin{aligned}
&\le \sum_{A\in\mathcal{A}_i}\int_A |f(x,y)-f(x,y_A)-f(x',y)+f(x',y_A)|\,d\mathcal{L}(y)\\
&\le C\|x-x'\|\sum_{A\in\mathcal{A}_i}\int_A \|y-y_A\|\,d\mathcal{L}(y)\\
&\le C\sup_{A\in\mathcal{A}_i}\operatorname{diam}(A)\mathcal{L}(X)\|x-x'\|.
\end{aligned}
$$

Thus (18) follows on using (17). ∎

3 Self-similar vector-valued measures in code space

As with most fractal and multifractal calculations for self-similar sets and measures we first calculate within a code space that codes points of the self-similar set, and then transfer these calculations to generalised vector-valued measures in $\mathbf{R}^d$ using geometrical arguments. In this section we consider generalised vector-valued measures defined on a code space I_∞ with values in $\mathbf{R}^M$, i.e. the measures are members of $\mathcal{L}(\mathrm{Lip}\,(I_\infty),\mathbf{R}^M)$.

Fix $N\in\mathbf{N}$ and let $I_\infty=\prod_{i=1}^{\infty}\{1,\ldots,N\}$. For $i\in\{1,\ldots,N\}$ fix $0<r_i<1$ and define a metric on I_∞ by

$$
\mathrm{d}(\mathbf{i},\mathbf{j})=\begin{cases} 1 & \text{if } i_1\ne j_1\\ r_{i_1}\ldots r_{i_k} & \text{if } \mathbf{i}|_k=\mathbf{j}|_k \text{ and } i_{k+1}\ne j_{k+1}\\ 0 & \text{if } \mathbf{i}=\mathbf{j}\end{cases} \tag{19}
$$

where $\mathbf{i}|_k$ is the sequence $\mathbf{i}$ curtailed after k terms. Under this metric I_∞ is complete, separable, compact and totally disconnected.

Define, for $i=1,\ldots,N$, the 'right shift' maps $\sigma_i : I_\infty\to I_\infty$ by

$$
\sigma_i(\mathbf{i})=i,\mathbf{i}
$$

(where $i,\mathbf{i}$ denotes concatenation.) If $\mathbf{i}$ and $\mathbf{j}\in I_\infty$ then

$$
\mathrm{d}(\sigma_i\mathbf{i},\sigma_i\mathbf{j})=r_i\mathrm{d}(\mathbf{i},\mathbf{j})
$$

and so the maps σ_i are contracting similitudes on the space I_∞. Let $T_i\in\mathcal{L}(\mathbf{R}^M,\mathbf{R}^M)$ $(i=1,\ldots,N)$ be similitudes with norms $\|T_i\|=t_i>0$; observe that we allow t_i to be greater than 1. We assume that

$$
\sum_{i=1}^{N} T_i = I \tag{20}
$$

where I is the identity map on $\mathbf{R}^M$, that

$$
\sum_{i=1}^{N} t_i r_i < 1 \tag{21}
$$

and that

$$
0<t_1\le\ldots\le t_N. \tag{22}
$$

For a non-zero vector $\mathbf{e} \in \mathbf{R}^M$, Proposition 2.3 gives that there is a unique generalised vector-valued measure $\boldsymbol{\nu} \in \mathcal{M}_{\mathbf{e}}(I_\infty)$ such that for all $f \in \mathrm{Lip}\,(I_\infty)$

$$\int f\, d\boldsymbol{\nu} = \sum_{i=1}^{N} T_i \left(\int [f \circ \sigma_i]\, d\boldsymbol{\nu} \right). \tag{23}$$

For $\mathbf{i} \in I_\infty$ and $k \in \mathbf{N}$ define the *k-th level cylinder* of $\mathbf{i}$ by

$$C(\mathbf{i}|_k) := \{\mathbf{j} \in I_\infty : \mathbf{j}|_k = \mathbf{i}|_k\}$$

and write

$$\mathrm{diam}\, C(\mathbf{i}|_k) = r(\mathbf{i}|_k) \equiv r_{i_1} \ldots r_{i_k}.$$

If $C(\mathbf{i}|_k) \neq C(\mathbf{j}|_k)$ then $\mathrm{dist}(C(\mathbf{i}|_k), C(\mathbf{j}|_k)) \geq r_{\min} r(\mathbf{i}|_{k-1})$ where $r_{\min} = \min_{i=1,\ldots,N}\{r_i\}$. Using (23) inductively with f as the characteristic function of cylinders (which is Lipschitz) we see that for all $\mathbf{i} \in I_\infty$ and $k \in \mathbf{N}$ the vector $\boldsymbol{\nu} C(\mathbf{i}|_k)$ is well defined and is given by

$$\boldsymbol{\nu} C(\mathbf{i}|_k) = T_{\mathbf{i}|_k}(\mathbf{e}) = T_{i_1} \circ \ldots \circ T_{i_k}(\mathbf{e}). \tag{24}$$

By analogy to the usual theory of multifractal measures, we are led to investigate the set of points of I_∞ where these vector measures satisfy a power law of exponent α; that is, where the 'local dimension' of the vector measure is α. Thus for all $\alpha \in \mathbf{R}$ we define

$$\begin{aligned} I(\alpha) &= \left\{ \mathbf{i} \in I_\infty : \lim_{k\to\infty} \frac{\log \|\boldsymbol{\nu} C(\mathbf{i}|_k)\|}{\log \mathrm{diam}\, C(\mathbf{i}|_k)} = \alpha \right\} \\ &= \left\{ \mathbf{i} \in I_\infty : \lim_{k\to\infty} \frac{\log t(\mathbf{i}|_k)}{\log r(\mathbf{i}|_k)} = \alpha \right\}. \end{aligned} \tag{25}$$

We are particularly interested in the size of these sets as measured by Hausdorff dimension. Recall that we define the Hausdorff dimension of a set E, a subset of a metric space X, by

$$\dim_{\mathrm{H}}(E) = \sup\{s : \mathbf{H}^s(E) = \infty\}$$

where, for $\delta > 0$,

$$\mathbf{H}^s_\delta(E) = \inf \left\{ \sum_{U \in \mathcal{U}} (\mathrm{diam}\, U)^s : \mathcal{U} \text{ is a } \delta\text{-cover of } E \right\}$$

and

$$\mathbf{H}^s(E) = \sup_{\delta > 0} \mathbf{H}^s_\delta(E).$$

For a fuller discussion of Hausdorff dimension see Falconer [8] or Mattila [18].

Let $\beta: \mathbf{R} \to \mathbf{R}$ be defined by

$$\sum_{i=1}^{N} t_i^p r_i^{\beta(p)} = 1. \tag{26}$$

(It is easy to see that for all p there is a unique number $\beta(p)$ satisfying this equation). This is analogous to the usual multifractal situation, except here the t_i need not sum to 1 and so $\beta(1)$ need not be zero.

Proposition 3.1 *The function β defined in (26) is real analytic. If*

1. $t_1 = t_N$ *then* $\beta(p) = \beta(0)$ *for all* p,
2. $t_i = r_i^{\beta(0)}$ *for all* i *then* $\beta(p) = (1-p)\beta(0)$,
3. $t_1 < t_N$ *and there is an* i *such that* $t_i \neq r_i^{\beta(0)}$ *then* $\beta''(p) > 0$ *for all* p *and thus* $\beta'(p)$ *is strictly increasing. Moreover, as* $p \to \infty$
   (a) *if* $t_1 > 1$ *then* $\beta(p) \to \infty$, $\beta(-p) \to -\infty$ *and* $\beta(p)$ *is strictly increasing,*
   (b) *if* $t_N < 1$ *then* $\beta(p) \to -\infty$, $\beta(-p) \to \infty$ *and* $\beta(p)$ *is strictly decreasing,*
   (c) *if* $t_1 = 1$ *then* $\beta(p) \to \infty$, $\beta(-p) \to 0$ *and* $\beta(p)$ *is positive and strictly increasing,*
   (d) *if* $t_N = 1$ *then* $\beta(p) \to 0$, $\beta(-p) \to \infty$ *and* $\beta(p)$ *is positive and strictly decreasing,*
   (e) *if* $t_1 < 1 < t_N$ *then* $\beta(p) \to \infty$, $\beta(-p) \to \infty$ *and* $\beta(p)$ *is positive and convex.*

Proof: This follows from a simple analysis of the expression (26) defining β together with implicit differentiation. ∎

In the event that (1) or (2) of Proposition 3.1 hold it becomes a trivial exercise to compute that $I(\alpha) \neq \emptyset$ for only one value of α and for this α, $I(\alpha) = I_\infty$ and thus its dimension is easily computed. Hence in what follows we shall implicitly assume that neither of cases (1) and (2) occur.

Let

$$\alpha_{\min} = \min_{1\le i\le N} \frac{\log t_i}{\log r_i}$$

and

$$\alpha_{\max} = \max_{1\le i\le N} \frac{\log t_i}{\log r_i}.$$

(Note that both $\alpha_{\min}$ and $\alpha_{\max}$ may be negative.)

Proposition 3.2 *The following hold:*

1. $\alpha_{\min} = \lim_{p\to\infty} -\beta'(p)$ *and* $\alpha_{\max} = \lim_{p\to-\infty} -\beta'(p)$
2. *For all* $\mathbf{i} \in I_\infty$
   $$\alpha_{\min} \le \liminf_{k\to\infty} \frac{\log t(\mathbf{i}|_k)}{\log r(\mathbf{i}|_k)} \le \limsup_{k\to\infty} \frac{\log t(\mathbf{i}|_k)}{\log r(\mathbf{i}|_k)} \le \alpha_{\max}$$
   and so $I(\alpha) = \emptyset$ *for* $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$.
3. $d_{\max} = \lim_{p\to-\infty} p\alpha_{\max} + \beta(p)$ *exists and satisfies*
   $$\sum_{i:t_i = r_i^{\alpha_{\max}}} r_i^{d_{\max}} = 1, \tag{27}$$

4. $d_{\min} = \lim_{p\to\infty} p\alpha_{\min} + \beta(p)$ *exists and satisfies*

$$\sum_{i:t_i = r_i^{\alpha_{\min}}} r_i^{d_{\min}} = 1. \tag{28}$$

Proof: The details may be found in Cawley and Mauldin [3, Section 1] — the fact that the t_i do not necessarily sum to one does not affect any of their calculations. ■

One immediate corollary of this proposition is that if $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$ then $I(\alpha) = \emptyset$ and thus has zero dimension.

The *Legendre Transform*, β^*, of β is defined by

$$\beta^*(\alpha) = \inf\{p\alpha + \beta(p) : p \in \mathbf{R}\} \tag{29}$$

for $\alpha \in (\alpha_{\min}, \alpha_{\max})$. We set

$$\beta^*(\alpha_{\max}) = d_{\min} \text{ and } \beta^*(\alpha_{\min}) = d_{\max}. \tag{30}$$

Since β' is strictly monotonic increasing on $\mathbf{R}$ (see Proposition 3.2) it is easy to see that for $\alpha_{\min} < \alpha < \alpha_{\max}$, $\beta^*(\alpha) = p\alpha + \beta(p)$ where p has been chosen so that $\alpha = -\beta'(p)$.

Our final result in this section is our main result about the code space: it characterises the size of the sets in I_∞ where the local dimension is α.

Theorem 3.3 *For all* $\alpha \in [\alpha_{\min}, \alpha_{\max}]$,

$$\dim_{\mathrm{H}} I(\alpha) = \beta^*(\alpha).$$

Moreover, if $\alpha \in (\alpha_{\min}, \alpha_{\max})$ *then*

$$\dim_{\mathrm{H}} I(\alpha) = p_\alpha \alpha + \beta(p_\alpha)$$

where p_α *is the unique* p *such that* $\alpha = -\beta'(p)$. *In particular* $\dim_{\mathrm{H}}(I_\alpha)$ *is independent of the value of* $\boldsymbol{\nu}(I_\infty)$ *(=*$\mathbf{e}$*).*

Proof: This is proved using methods identical to those used by Cawley and Mauldin in [3]. For $\alpha \in (\alpha_{\min}, \alpha_{\max})$ the upper bound on the dimension of $I(\alpha)$ follows from a standard covering argument and the lower bound follows on using the fact that $\sum t_i^{p_\alpha} r_i^{\beta(p_\alpha)} = 1$ to define a probability measure μ on I and then using the ergodic theorem to show that μ is concentrated on $I(\alpha)$. Standard techniques then allow the dimension of μ to be estimated. If $\alpha = \alpha_{\min}$ (or $\alpha_{\max}$) then one proceeds exactly as in [3, Theorem 2.14]. ■

Thus we have calculated the multifractal spectra for a class of vector measures defined on a code space. In the next section we shall attempt to push this result forward to self-similar vector measures in Euclidean space which satisfy a strong separation condition.

4 Multifractal spectrum of self-similar vector measures defined on Euclidean space.

Suppose that $S_i: \mathbf{R}^d \to \mathbf{R}^d$ for $i = 1, \dots N$ are similitudes with contraction ratios $0 < r_i < 1$ and let $T_i \in \mathcal{L}(\mathbf{R}^M, \mathbf{R}^M)$ be similitudes satisfying (20), (21) and (22). Let K be the unique, non-empty, compact set satisfying $K = \cup_i S_i(K)$. Further suppose that the S_i are such that if $i \neq j$ then $S_i(K) \cap S_j(K) = \emptyset$. Let d be the metric defined on I_∞ by (19) using the contraction ratios r_i. Define a projection map $\pi: I_\infty \to K$ by setting, for $\mathbf{i} \in I_\infty$,

$$\{\pi(\mathbf{i})\} = \cap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(K).$$

(π is well defined since for all k, $\operatorname{diam} S_{i_1} \circ \cdots \circ S_{i_k}(K) \leq r(\mathbf{i}|_k) \leq r_{\max}^k \to 0$.) The choice of metric for the code space I_∞ and the fact that K satisfies a strong separation property ensures that the following holds:

Proposition 4.1 *The map π is bi-Lipschitz and is such that for all $i \in \{1, \dots, N\}$ the following diagram commutes:*

$$\begin{array}{ccc} I_\infty & \xrightarrow{\sigma_i} & I_\infty \\ \downarrow \pi & & \downarrow \pi \\ K & \xrightarrow{S_i} & K \end{array}$$

Proof: It is clear from the definition of π, that the diagram commutes. Choose $\gamma > 0$ such that if $i \neq j$ then $\operatorname{dist}(S_i(K), S_j(K)) \geq \gamma$. Suppose that $\mathbf{i} \neq \mathbf{j} \in I_\infty$ and let k be the least integer such that $\mathbf{i}|_k \neq \mathbf{j}|_k$. Then $\operatorname{dist}(K(\mathbf{i}|_k), K(\mathbf{j}|_k)) \geq \gamma \operatorname{diam}(K) r_{i_1} \dots r_{i_{k-1}}$ where $K(\mathbf{i}|_k) = S_{i_1} \circ \dots \circ S_{i_k}(K)$. Hence

$$\gamma \operatorname{diam}(K) \mathrm{d}(\mathbf{i}, \mathbf{j}) \leq \|\pi(\mathbf{i}) - \pi(\mathbf{j})\| \leq \operatorname{diam}(K) \mathrm{d}(\mathbf{i}, \mathbf{j}) \tag{31}$$

as required. ∎

Consequently we have the following result on dimensions.

Lemma 4.2 *If $J \subset I_\infty$ then*

$$\dim_{\mathrm{H}} \big(\pi(J)\big) = \dim_{\mathrm{H}} J. \tag{32}$$

Proof: This follows since Hausdorff dimension is preserved by bi-Lipschitz maps. ∎

We can now transfer the results of Section 3 for code space to K.

Proposition 4.3 *For all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$,*

$$\dim_{\mathrm{H}} \pi\big(I(\alpha)\big) = \beta^*(\alpha)$$

and if $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$ then $\pi(I(\alpha)) = \emptyset$. Moreover, if $\alpha \in (\alpha_{\min}, \alpha_{\max})$ then

$$\dim_{\mathrm{H}} \pi\big(I(\alpha)\big) = p_\alpha \alpha + \beta(p_\alpha)$$

where p_α is the unique p such that $\alpha = -\beta'(p)$.

Proof: This is immediate from Lemma 4.2, Proposition 3.2 and Theorem 3.3. ∎

Let $\boldsymbol{\mu}$ be the unique vector-valued measure in $\mathcal{M}_{\mathbf{e}}(\mathbf{R}^d)$ given by Proposition 2.3 and recall that for all $f \in \mathrm{Lip}\,(\mathbf{R}^d)$

$$\int f \, d\boldsymbol{\mu} = \sum_{i=1}^{N} T_i \left(\int [f \circ S_i] \, d\boldsymbol{\mu} \right). \tag{33}$$

It is clear that there should be a relationship between the vector measure $\boldsymbol{\mu}$ defined on $\mathbf{R}^d$ and the vector measure $\boldsymbol{\nu}$ defined on I_∞ by (23); the next proposition formalises this.

Proposition 4.4 *For all $g \in \mathrm{Lip}\,(I_\infty)$ and f any Lipschitz extension of $g \circ \pi^{-1}$ to $\mathbf{R}^d$ we have*

$$\int f \, d\boldsymbol{\mu} = \int g \, d\boldsymbol{\nu}.$$

In particular, for any $\mathbf{i} \in I_\infty$ and $k \in \mathbf{N}$ we have

$$\boldsymbol{\mu} K(\mathbf{i}|_k) = \boldsymbol{\nu} C(\mathbf{i}|_k). \tag{34}$$

Proof: Fix such an f and g. We have that $g = f \circ \pi$ so that from (23) and the definition of π we find that

$$\begin{aligned} \int g \, d\boldsymbol{\nu} &= \int (f \circ \pi) \, d\boldsymbol{\nu} \\ &= \sum_{i=1}^{N} T_i \left[\int (f \circ \pi \circ \sigma_i) \; d\boldsymbol{\nu} \right] \\ &= \sum_{i=1}^{N} T_i \left[\int (f \circ S_i \circ \pi) \; d\boldsymbol{\nu} \right]. \end{aligned}$$

Hence

$$\int f \, d\pi_\# \boldsymbol{\nu} = \sum_{i=1}^{N} T_i \left[\int (f \circ S_i) \; d\pi_\# \boldsymbol{\nu} \right]$$

where, for $h \in \mathrm{Lip}\,(\mathbf{R}^d)$,

$$\int h \, d\pi_\# \boldsymbol{\nu} = \int (h \circ \pi) \, d\boldsymbol{\nu}.$$

But as $\boldsymbol{\mu}$ is the only vector measure in $\mathcal{M}_{\mathbf{e}}(\mathbf{R}^d)$ satisfying this equation, see (33), we conclude that

$$\pi_\# \boldsymbol{\nu} = \boldsymbol{\mu}$$

as required.

Finally, by choosing g to be the Lipschitz function that equals 1 on $C(\mathbf{i}|_k)$ and 0 elsewhere we obtain (34). ∎

Corollary 4.5 *For all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$,*

$$\dim_{\mathrm{H}} \big(K_{\mathrm{C}}(\alpha) \big) = \beta^*(\alpha)$$

where

$$K_{\mathrm{C}}(\alpha) := \left\{ x \in K : \lim_{k\to\infty} \frac{\log \boldsymbol{\mu} K(\mathbf{x}|_k)}{\log \operatorname{diam} K(\mathbf{x}|_k)} = \alpha \text{ where } \mathbf{x} = \pi^{-1}x \right\}.$$

Proof: It follows from (31) that there is a constant $P \in (0, \infty)$ such that for all $x \in K$ and $k \in \mathbf{N}$

$$P^{-1}\operatorname{diam} C(\mathbf{x}|_k) \leq \operatorname{diam} K(\mathbf{x}|_k) \leq P \operatorname{diam} C(\mathbf{x}|_k).$$

Propositions 4.3, 4.4, Theorem 3.3 and (25) now give the result. ∎

Thus we have determined the multifractal spectrum of the cylinder sets of K. However we are really interested in the dimension of the subsets of K at which the local dimension calculated over *balls* equals α, that is

$$\left\{ x \in K : \lim_{r\to 0} \frac{\log \boldsymbol{\mu}\mathrm{B}(x,r)}{\log r} = \alpha \right\}.$$

Unfortunately, in the framework of Section 2, $\boldsymbol{\mu}\mathrm{B}(x,r)$ is not, in general, a well-defined quantity and thus we need to seek an alternative notion of local dimension for vector measures which parallels that for ordinary measures. This leads us to introduce convolution kernels which, in a sense made precise below, approximate the identity function on $\mathrm{B}(0,1)$.

Let $\phi : \mathbf{R}^d \to [0,1]$ be a Lipschitz function of compact support such that

1. ϕ is radially symmetric and decreasing (i.e. $\phi(x) \leq \phi(y)$ if $\|x\| \geq \|y\|$ with equality if $\|x\| = \|y\|$),
2. the function $f(x,y) := \phi(x-y)$ satisfies Equations (13) and (14) for some $C > 0$ (this will be satisfied if ϕ has continuous second derivatives),
3. $\int \phi(x)\, d\mathcal{L}(x) = 1$ where $\mathcal{L}$ denotes d-dimensional Lebesgue measure.

We define $\phi_r : \mathbf{R}^d \to [0,1]$ for $r > 0$ as dilations of ϕ:

$$\phi_r(x) = \phi\left(\frac{x}{r}\right). \tag{35}$$

Note that

$$\int \phi_r(x)\, d\mathcal{L}(x) = r^d. \tag{36}$$

We now make a few observations concerning the functions ϕ_r and their relationship with $\boldsymbol{\mu}$.

For $0 < \rho \leq 1$ we write $K^\rho = \{x : \operatorname{dist}(x,K) \leq \rho\}$ for the ρ neighbourhood of K. We assume that

$$K^\rho(i) \cap K^\rho(j) = \emptyset \text{ for } i \neq j \tag{37}$$

where $K^\rho(i) := S_i(K^\rho)$. Thus $\operatorname{diam} K^\rho(i) = r_i \operatorname{diam} K^\rho = r_i(2\rho + \operatorname{diam} K)$. Now choose $0 < \delta < \rho$ such that if $0 < r < \delta$ then

$$\operatorname{Spt} \phi_r \subset \mathrm{B}(0,\rho). \tag{38}$$

It is immediate from (36) and the translational invariance of Lebesgue measure that for all $y \in K$ and all $0 < r < \delta$

$$r^{-d} \int_{K^\rho} \phi_r(x-y)\, d\mathcal{L}(x) = 1. \tag{39}$$

For brevity we write for $x \in \mathbf{R}^d$ and $r > 0$

$$\Phi_r \boldsymbol{\mu}(x) := \int \phi_r(x-y)\, d\boldsymbol{\mu}(y)$$

and if $f\colon \mathbf{R}^d \to \mathbf{R}^d$ then we let

$$\Phi_r[\boldsymbol{\mu} \circ f^{-1}](x) := \int \phi_r(x - f(y))\, d\boldsymbol{\mu}(y).$$

We think of $\Phi_r\boldsymbol{\mu}(x)$ as a 'sample' of $\boldsymbol{\mu}$ at scale r in a neighbourhood of x. Our next lemma will enable us to relate $r^{-d} \int_{K^\rho} \|\Phi_r\boldsymbol{\mu}(x)\|^p\, d\mathcal{L}(x)$ to the vector measures of the 'cylinders' which make up K. (We take 0^0 to be 0 in all that follows.)

Lemma 4.6 *For all $p \geq 0$ there is a $C_p \geq 1$ such that for all $\delta r_{\min} \leq r < \delta$*

$$C_p^{-1} \leq r^{-d} \int_{K^\rho} \|\Phi_r\boldsymbol{\mu}(x)\|^p\, d\mathcal{L}(x) \leq C_p. \tag{40}$$

Proof: By Proposition 2.5 and (39)

$$r^{-d} \int_{K^\rho} \int \phi_r(x-y)\, d\boldsymbol{\mu}(y) d\mathcal{L}(x) = \mathbf{e}.$$

So in particular, for all r, $\|\Phi_r\boldsymbol{\mu}(x)\| \neq 0$ on a subset of K^ρ of positive Lebesgue measure. Since $(x, r) \mapsto \|\Phi_r\boldsymbol{\mu}(x)\|$ is uniformly continuous on $K^\rho \times [\delta r_{\min}, r_{\min}]$ we deduce immediately (40). ∎

Note, using the scaling and translational invariance of Lebesgue measure, that if $\mathbf{i} \in I_k$ for some k then for any $s > 0$ and $p \geq 0$

$$\int_{K^\rho(\mathbf{i})} \|\Phi_{sr(\mathbf{i})}[\boldsymbol{\mu} \circ S_{\mathbf{i}}^{-1}](x)\|^p\, d\mathcal{L}(x) = [r(\mathbf{i})]^d \int_{K^\rho} \|\Phi_s\boldsymbol{\mu}(x)\|^p\, d\mathcal{L}(x). \tag{41}$$

Lemma 4.7 *Let δ be given by (38) and fix $p \geq 0$. There is a $C_p \geq 1$ such that if $k \in \mathbf{N}$, $\mathbf{i} \in I_k$ and $\delta r_{\min} \leq s < \delta$ then*

$$C_p^{-1} t(\mathbf{i})^p \leq (sr(\mathbf{i}))^{-d} \int_{K^\rho(\mathbf{i})} \|\Phi_{sr(\mathbf{i})}\boldsymbol{\mu}(x)\|^p\, d\mathcal{L}(x) \leq C_p t(\mathbf{i})^p \tag{42}$$

where

$$t(\mathbf{i}) = t_{i_1} \ldots t_{i_k}$$

and

$$r(\mathbf{i}) = r_{i_1} \ldots r_{i_k}$$

for $\mathbf{i} = (i_1, \ldots, i_k)$.

Proof: Observe from (33) that

$$\int_{K^\rho(\mathbf{i})} \|\Phi_{sr(\mathbf{i})}\boldsymbol{\mu}(x)\|^p \, d\mathcal{L}(x) = \int_{K^\rho(\mathbf{i})} \|\sum_{\mathbf{j}\in I_k} T_{\mathbf{j}} \left(\Phi_{sr(\mathbf{i})}[\boldsymbol{\mu}\circ S_{\mathbf{j}}^{-1}](x)\right) \|^p \, d\mathcal{L}(x)$$

where for $\mathbf{j} = (j_1, \ldots, j_k)$,we set $T_{\mathbf{j}} = T_{j_1} \circ \cdots \circ T_{j_k}$ and $S_{\mathbf{j}} = S_{j_1} \circ \cdots \circ S_{j_k}$. However $\operatorname{Spt}\phi_{sr(\mathbf{i})} \subset \mathrm{B}(0, \rho r(\mathbf{i}))$ and so, if $\mathbf{j} \neq \mathbf{i}$, then for $x \in K(\mathbf{i})$ and $y \in K$

$$\|x - S_{\mathbf{j}}y\| \geq \rho(r(\mathbf{j}) + r(\mathbf{i})) > \rho r(\mathbf{i})$$

which implies that $\phi_{sr(\mathbf{i})}(x - S_{\mathbf{j}}y) = 0$. Hence this together with (41) gives

$$\begin{aligned}
(sr(\mathbf{i}))^{-d} \int_{K^\rho(\mathbf{i})} & \|\Phi_{sr(\mathbf{i})}\boldsymbol{\mu}(x)\|^p \, d\mathcal{L}(x) \\
&= (sr(\mathbf{i}))^{-d} \int_{K^\rho(\mathbf{i})} \|T_{\mathbf{i}} \left(\Phi_{sr(\mathbf{i})}[\boldsymbol{\mu}\circ S_{\mathbf{i}}^{-1}](x)\right) \|^p \, d\mathcal{L}(x) \\
&= t(\mathbf{i})^p s^{-d} \int_{K^\rho} \|\Phi_s\boldsymbol{\mu}(x)\|^p \, d\mathcal{L}(x)
\end{aligned}$$

and the result now follows from Lemma 4.6. ■

The hope is that, for small r, $\int \phi_r(x-y)\, d\boldsymbol{\mu}(y)$ will mirror the behaviour of $\mu\mathrm{B}(x,r)$ for an ordinary measure μ. Unfortunately there are problems associated with investigating this quantity — primarily the fact that in the vector case it may be unexpectedly small or even zero. In an attempt to surmount this difficulty we shall instead investigate the quantity

$$r^{-d} \int_{\mathrm{B}(z,ar)} \|\Phi_r\boldsymbol{\mu}(x)\| \, d\mathcal{L}(x) \tag{43}$$

(where a is a constant to be defined later.) One advantage of this approach is that we can utilise monotonicity: if $E \subset F$ then

$$r^{-d} \int_E \| \int \phi_r(x-y)\, d\boldsymbol{\mu}(y)\| \, d\mathcal{L}(x) \leq r^{-d} \int_F \| \int \phi_r(x-y)\, d\boldsymbol{\mu}(y)\| \, d\mathcal{L}(x).$$

For an ordinary measure μ which satisfies a doubling condition (that is, $\limsup_{r\to 0} \mu\mathrm{B}(x,2r)/\mu\mathrm{B}(x,r) < \infty$ for all $x \in \operatorname{Spt}\mu$ — this is certainly true of self-similar measures) the method of averaging $\|\Phi_r\boldsymbol{\mu}(x)\|$ over $\mathrm{B}(z,ar)$ is essentially equivalent to just investigating $\mu\mathrm{B}(z,r)$. In addition the value that a takes is unimportant. Thus we may choose $a > 1$ such that if $\delta r_{\min} \leq r < \delta$ and $z \in K$ then

$$\mathrm{B}(z, ar) \supset K^\rho. \tag{44}$$

Our next lemma uses the strong separation property of K to enable us to compare $\mathrm{B}(z,as)$ with cylinder sets of K in a uniform manner. Recall that if $\mathbf{i} \in I_\infty$ and $k \in \mathbf{N}$ then

$$K^\rho(\mathbf{i}|_k) = S_{\mathbf{i}|_k} K^\rho.$$

Lemma 4.8 *There is an $l \in \mathbf{N}$ such that if $z \in K$ and $\mathbf{i} = \pi^{-1}(z)$ and $0 < s < \delta r(\mathbf{i}|_l)$ then*

$$K^\rho(\mathbf{i}|_k) \subset \mathrm{B}(z, as) \subset K^\rho(\mathbf{i}|_{k-l}) \tag{45}$$

where k is chosen so that

$$\delta r(\mathbf{i}|_k) \leq s < \delta r(\mathbf{i}|_{k-1}).$$

Proof: Since for any $y \in K$

$$\mathrm{B}(y, as/r(\mathbf{i}|_k)) \supset K^\rho$$

it is clear that

$$S_{\mathbf{i}|_k}\mathrm{B}(y, as/r(\mathbf{i}|_k)) \supset K^\rho(\mathbf{i}|_k)$$

and thus if y is chosen so that $S_{\mathbf{i}|_k} y = z$ then

$$\mathrm{B}(z, as) \supset K^\rho(\mathbf{i}|_k)$$

as required. For the other inclusion choose l to be an integer such that

$$l \geq 1 + \frac{\log \rho/(a\delta)}{\log r_{\max}}. \tag{46}$$

and observe that for $z \in K(\mathbf{i}|_k)$ we have using (46)

$$\begin{aligned}
\mathrm{dist}(z, \mathbf{R}^d \setminus K^\rho(\mathbf{i}|_{k-l})) &\geq \rho r(\mathbf{i}|_{k-l}) \\
&\geq \rho (r_{\max})^{1-l} r(\mathbf{i}|_{k-1}) \\
&\geq a\delta r(\mathbf{i}|_{k-1}) \\
&> as
\end{aligned}$$

and so the lemma follows. ■

This lemma enables us to compare $s^{-d}\int_{\mathrm{B}(z,as)} \|\Phi_s \boldsymbol{\mu}(x)\|\, d\mathcal{L}(x)$ with the size of appropriate level components of K as follows.

Corollary 4.9 *There are* $0 < c_1 \leq c_2 < \infty$ *and* $s_0 >$ *such that for all* $z = \pi(\mathbf{i}) \in K$ *and* $0 < s < s_0$ *we have*

$$c_1 t(\mathbf{i}|_k) \leq s^{-d} \int_{\mathrm{B}(z,as)} \|\Phi_s \boldsymbol{\mu}(x)\|\, d\mathcal{L}(x) \leq c_2 t(\mathbf{i}|_k) \tag{47}$$

where k *is such that* $\delta r(\mathbf{i}|_k) \leq s < \delta r(\mathbf{i}|_{k-1})$.

Proof: Let l be as in Lemma 4.8 and set $s_0 = \delta r_{\min}^{l+1}$. Then, for $0 < s < s_0$ we find that from (45)

$$s^{-d} \int_{\mathrm{B}(z,as)} \|\Phi_s \boldsymbol{\mu}(x)\|\, d\mathcal{L}(x) \geq s^{-d} \int_{K^\rho(\mathbf{i}|_k)} \|\Phi_s \boldsymbol{\mu}(x)\|\, d\mathcal{L}(x)$$

and the lower bound then follows immediately from Lemma 4.7. Similarly

$$s^{-d} \int_{\mathrm{B}(z,as)} \|\Phi_s \boldsymbol{\mu}(x)\|\, d\mathcal{L}(x) \leq s^{-d} \int_{K^\rho(\mathbf{i}|_{k-l})} \|\Phi_s \boldsymbol{\mu}(x)\|\, d\mathcal{L}(x)$$

which, again by Lemma 4.7,

$$\begin{aligned}
&\leq C_1 t(\mathbf{i}|_{k-l}) \\
&\leq C_1 (t_{\min})^{-l} t(\mathbf{i}|_k)
\end{aligned}$$

as required. ■

We can now relate our space averages to cylinder averages.

Proposition 4.10 *For all $z = \pi(\mathbf{i}) \in K$ and for all $\alpha \in \mathbf{R}$ we have*

$$\lim_{s\to 0} \frac{\log s^{-d} \int_{\mathrm{B}(z,as)} \|\Phi_s \boldsymbol{\mu}(x)\| \, d\mathcal{L}(x)}{\log s} = \alpha$$

if and only if

$$\lim_{k\to\infty} \frac{\log t(\mathbf{i}|_k)}{\log r(\mathbf{i}|_k)} = \alpha.$$

Proof: This is immediate from Corollary 4.9. ∎

Hence we can exhibit the multifractal spectrum in spatial form.

Corollary 4.11 *For $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ we have*

$$\dim_{\mathrm{H}} \left\{ z \in K : \lim_{s\to 0} \frac{\log s^{-d} \int_{\mathrm{B}(z,as)} \|\Phi_s \boldsymbol{\mu}(x)\| \, d\mathcal{L}(x)}{\log s} = \alpha \right\} = \beta^*(\alpha) \qquad (48)$$

where β^* is given by (29). Moreover, for $\alpha \in (\alpha_{\min}, \alpha_{\max})$,

$$\beta^*(\alpha) = p_\alpha \alpha + \beta(p_\alpha)$$

where p_α is the unique $p \in \mathbf{R}$ such that $\alpha = -\beta'(p)$ and

$$\beta^*(\alpha_{\min}) = d_{\min} \text{ and } \beta^*(\alpha_{\max}) = d_{\max}$$

(recall that $d_{\min}$ and $d_{\max}$ were defined in (27) and (28)). Finally, if $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$ then

$$\left\{ z \in K : \lim_{s\to 0} \frac{\log s^{-d} \int_{\mathrm{B}(z,as)} \|\Phi_s \boldsymbol{\mu}(x)\| \, d\mathcal{L}(x)}{\log s} = \alpha \right\} = \emptyset.$$

Proof: This follows immediately from Proposition 4.3 together with Proposition 4.10. ∎

Thus we have found the multifractal spectrum of $\boldsymbol{\mu}$, the self-similar vector-valued measure with the β function defined by (26) and, moreover, the spectrum is independent of the choice of the 'sampling function', ϕ. Let us illustrate the theory we have developed by means of the following simple example.

Example 4.12 *Consider the vector measure in the plane defined by Figure 4 where $\mathbf{E} = \mathbf{R}^2$, and $\mathbf{e}$ is a unit vector pointing along the x-axis and $T_1, \ldots, T_4$ are given as follows: $T_1(x) = \frac{1}{2}R(x)$, $T_2(x) = 2x$, $T_3(x) = -\frac{1}{2}R(x)$ and $T_4(x) = -x$ where R denotes a rotation of $\pi/2$. Let $r = 1/8$ and choose $S_1, \ldots, S_4$ so that they are similitudes which scale the unit square, Q, by a factor $1/8$ and then translate it to one of its four corners as illustrated in Figure 4. Then we may apply Proposition 2.3 to deduce the existence of a unique, self-similar vector measure supported by the cantor dust generated by these similitudes and satisfying (11).*

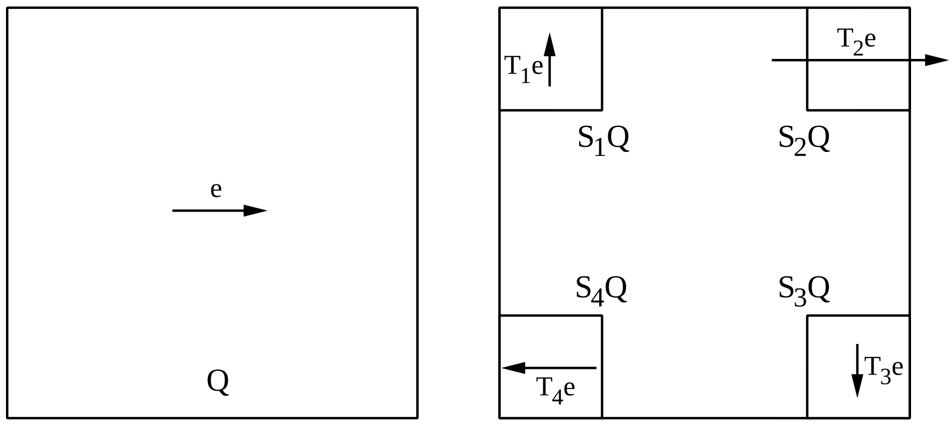

Figure 4: The generating transformations and similitudes for a self-similar vector-valued measure supported by a cantor dust.

It is a straightforward calculation to see that for this vector measure

$$\alpha_{\min} = -1/3, \quad \alpha_{\max} = 2/3,$$

$$d_{\min} = 0, \quad d_{\max} = 1/3$$

and

$$\beta(p) = \frac{\log[1 + 2^p + 2(0.25)^p]}{3 \log 2}.$$

Consequently the multifractal spectrum of this vector measure is as given in Figure 5. Observe that $\beta(0) \neq 1$ and that the set of points with negative local dimension has positive dimension; this is explained by considering how often points with a large number of 2's in their code space address appear.

5 The L^p-dimensions of self-similar generalised vector measures

The paper of Halsey et al [10] studied the multifractal spectrum of a measure μ by investigating the *L^p-dimensions* of the measure (this terminology was introduced by Strichartz in [22]) and relating them to the multifractal spectrum. In this section we find further parallels for the theory of generalised vector-valued measures involving L^p-dimensions. We continue to study the self-similar vector measure $\boldsymbol{\mu}$ discussed in the last section.

We define, for $p \geq 0$ and $r > 0$,

$$F_p(r) = r^{-d} \int \|\Phi_r \boldsymbol{\mu}(x)\|^p \, d\mathcal{L}(x) \tag{49}$$

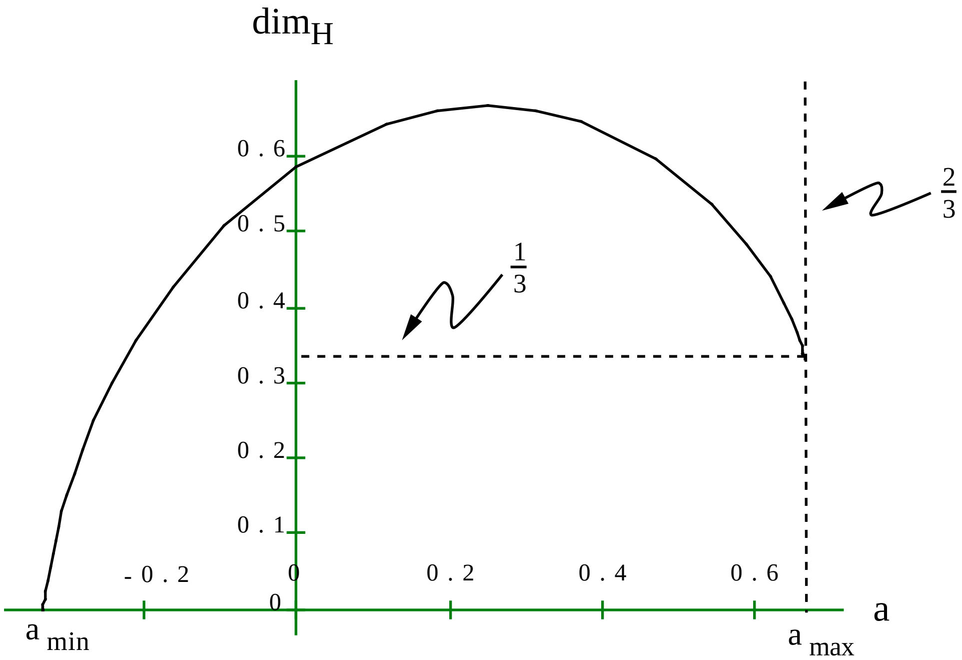

Figure 5: The multifractal spectrum of the vector measure described in Example 4.12

(where $||\mathbf{x}||^0$ is 1 if $\mathbf{x} \neq 0$ and 0 otherwise) and observe that F_p is a continuous function of r. We now define the *L^p-dimensions* of $\boldsymbol{\mu}$ to be

$$\tau(p) = \lim_{r \to 0} \frac{\log F_p(r)}{\log r}. \tag{50}$$

This is the natural vector analogue of the L^p-dimension function studied by Lau and Ngai in [15]. Theorem 5.1 shows that this limit does indeed exist for self-similar measures, is independent of the choice for ϕ and that, for $p \geq 0$, $\tau(p)$ is just the function $-\beta(p)$ given by (26).

Theorem 5.1 *Let $R = \{-\log r_1, \ldots, -\log r_N\}$ and suppose that $p \geq 0$.*

1. *If the group generated by R is dense in $\mathbf{R}$ then*

$$\lim_{r \to 0} r^{\beta(p)} F_p(r)$$

exists and is positive and finite.

2. *If the group generated by R is $\sigma\mathbf{Z}$ for some $\sigma > 0$ then there is a σ-periodic function h bounded away from 0 and ∞ such that*

$$\lim_{r \to 0} \frac{r^{\beta(p)} F_p(r)}{h(-\log r)} = 1.$$

Thus, in either case, $\tau(p)$ given in (50) exists and, moreover,

$$\beta(p) = -\tau(p).$$

Proof: Fix $p \geq 0$ and for $r > 0$ let $F(r) = F_p(r)$. Fix $i \in \{1, \ldots, n\}$ and suppose $0 < r < \delta r_i$. We examine the scaling properties of F. Observe that

$$\operatorname{Spt} \phi_r \subset \mathrm{B}(0, \rho r_i) \tag{51}$$

and if $y \in K(i)$ and $z \notin K^\rho(i)$ then

$$\phi_r(z - y) = 0. \tag{52}$$

Hence if $0 < r < \delta r_{\min}$ then

$$\begin{aligned}
r^d F(r) &= \int \| \int \phi_r(x-y)\, d\boldsymbol{\mu}(y) \|^p \, d\mathcal{L}(x) \\
&= \int_{K^\rho} \left\| \sum_{i=1}^N T_i \left(\int \phi_r(x - S_i y)\, d\boldsymbol{\mu}(y) \right) \right\|^p d\mathcal{L}(x), \quad \text{by (33)} \\
&= \sum_{i=1}^N \int_{K^\rho(i)} \left\| T_i \left(\int \phi_r(x - S_i y)\, d\boldsymbol{\mu}(y) \right) \right\|^p d\mathcal{L}(x), \quad \text{by (52)} \\
&= \sum_{i=1}^N t_i^p \int_{K^\rho(i)} \left\| \int \phi_r(x - S_i y)\, d\boldsymbol{\mu}(y) \right\|^p d\mathcal{L}(x) \\
&= \sum_{i=1}^N t_i^p r_i^d \int_{K^\rho} \left\| \int \phi_{r/r_i}(x - y)\, d\boldsymbol{\mu}(y) \right\|^p d\mathcal{L}(x), \quad \text{by (35)} \\
&= \sum_{i=1}^N t_i^p r^d F(r/r_i).
\end{aligned}$$

Thus, if $0 < r < \delta r_{\min}$ then

$$F(r) = \sum_{i=1}^N t_i^p F(r/r_i). \tag{53}$$

We now apply the renewal theory method, see Lalley [12, 13, 14], to study $F(r)$ as $r \to 0$. We define new functions $F_0 \colon (0, \infty) \to \mathbf{R}$ and $H \colon (0, \infty) \to \mathbf{R}$ by

$$F_0(r) = \begin{cases} F(r) & \text{if } 0 < r < \delta, \\ 0 & \text{if } r \geq \delta \end{cases} \tag{54}$$

and

$$H(r) = \begin{cases} 0 & \text{if } 0 < r < \delta r_{\min}, \\ F(r) - \sum_{i: \delta r_i > r} t_i^p F(r/r_i) & \text{if } \delta r_{\min} \leq r < \delta, \\ 0 & \text{if } r \geq \delta. \end{cases} \tag{55}$$

Thus for $r > 0$,

$$F_0(r) = \sum_{i=1}^N t_i^p F_0(r/r_i) + H(r). \tag{56}$$

By using Lemma 4.7 there are constants $0 < c_- \leq c_+ < \infty$ such that if $\delta r_{\min}^2 \leq r < \delta r_{\min}$ then

$$c_- \leq F_0(r) \leq c_+. \tag{57}$$

Define $f,\, g : \mathbf{R} \to \mathbf{R}$ by

$$f(t) = \exp(-st)F_0(\exp(-t)).$$

and

$$g(t) = \exp(-st)H(\exp(-t)).$$

Then (56) becomes

$$f(t) = \sum_{i=1}^{N} t_i^p r_i^s f(t + \log r_i) + g(t) \tag{58}$$

for $t \in \mathbf{R}$, where

$$g(t) = \begin{cases} 0 & \text{if } t > \log(\delta r_{\min})^{-1}, \\ e^{-st}\left(F(e^{-t}) - \sum_{i:\delta r_i > r} t_i^p F(r_i^{-1}e^{-t})\right) & \text{if } t \in [\log\delta, \log(\delta r_{\min})^{-1}], \\ 0 & \text{if } t \le -\log\delta^{-1}. \end{cases} \tag{59}$$

Hence g is a bounded, piecewise continuous function with compact support and if we choose $s \in \mathbf{R}$ such that $\sum t_i^p r_i^s = 1$ (that is, $s = \beta(p)$ by (26)) then we find from (57) and (58) that there are $0 < C_- \le C_+ < \infty$ such that for any $t > -\log\delta r_{\min}^2$

$$C_- \le f(t) \le C_+.$$

In this case, (58) is just the renewal equation, so we may apply the renewal theorem to deduce that one of the following occurs:

1. If $R = \{-\log r_1, \ldots, -\log r_N\}$ is non-arithmetic, that is the group generated by R under addition is dense in $\mathbf{R}$, then

$$\lim_{t\to\infty} f(t) = C$$

for some $C \in [C_-, C_+]$.

2. If R is σ-arithmetic, that is the group generated by R is $\sigma\mathbf{Z}$ for some $\sigma > 0$, then $f(t)$ is asymptotic (as $t \to \infty$) to a positive σ-periodic function h which is bounded above and below by C_- and C_+ respectively.

Consequently, in either case, by substituting back we find that either *(1)* or *(2)* of the theorem hold and thus $\tau(p)$ exists and equals $-\beta(p)$ as claimed. ∎

6 Discussion

We have set up a framework for vector-valued measures of a highly singular nature, which may be more appropriate for the study of certain physical phenomena than real-valued measures. We have calculated the multifractal spectrum and L^p-dimensions of self-similar vector measures. These quantities reflect the local intensity of $\boldsymbol{\mu}$ which is examined at scale r by integrating against a 'kernel of width r', ϕ_r. Whilst the multifractal spectrum and L^p-dimensions we obtain bear some resemblance to the real-valued measure analogues, a much wider variety of spectra are achievable in the self-similar case, permitting better matching to physical situations. In particular, the constraint that $\beta(0) = 1$ is no longer required.

There should be no difficulty in principle, in defining and calculating the spectra and dimensions of the vector-valued equivalents of all the (real-valued) multifractals that have been analysed. In particular, vector-valued analogues could be defined and analysed for graph-directed multifractal measures (see Edgar and Mauldin [6] for the real-valued case), for random self-similar measures (see Falconer [9], Olsen [19], Arbeiter and Patzschke [1]) and for self-conformal measures (see Bessis et al. [2]) where a vector-valued version of the thermodynamic formalism would be required. The theory could also be placed on the very general measure-theoretic foundation developed by Olsen in [20].

The model we have presented here could be refined in various ways. One would expect to be able to weaken the separation condition (37) to the open set condition (that $O \supset \cup_{i=1}^N S_i(O)$ with this union disjoint, for some non-empty bounded open set O); the approach of Arbeiter and Patzschke [1] may be applicable here. With some careful error analysis, the results in this paper would hold when the sampling kernels ϕ_r have non-compact support, provided they decrease sufficiently rapidly at infinity. In particular, the Gaussian kernel could be used.

It might seem more natural to consider the multifractal spectra with (43) replaced by $\Phi_r\boldsymbol{\mu}(x)$ so that we would examine

$$\dim_{\mathrm{H}} \left\{ x \in K : \lim_{s\to 0} \frac{\log \|\Phi_s \boldsymbol{\mu}(x)\|}{\log s} = \alpha \right\}$$

instead of (48). However, in the vector-valued case, $\Phi_s\boldsymbol{\mu}(x)$ can be rather erratic and, due to cancellation of vectors, may be very small or even vanish at certain x. Mathematically this would mean that the multifractal properties of $\boldsymbol{\mu}$ might differ from those occuring in the code space. It is possible to give rather complicated conditions that ensure that $\Phi_s\boldsymbol{\mu}(x) \neq 0$ for all $x \in K$, but it seems more appropriate to work with local averages.

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