A local version of the Projection Theorem

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Abstract

We prove the following theorem:
Suppose that $1 \leq m \leq n$ are integers and μ is a Borel measure on $\mathbf{R}^n$ such that for μ-a.e. x,

(1) The upper and lower m-densities of μ at x are positive and finite.

(2) If ν is a tangent measure of μ at x then for all $V \in \mathrm{G}(n, m)$ the orthogonal projection of the support of ν onto V is a convex set.

Then μ is m-rectifiable.

1 Introduction

In his pioneering papers [Bes28, Bes38, Bes39], Besicovitch investigated ways of differentiating between regular and irregular 1-sets in the plane. It is easy to see that for almost all straight lines (through the origin) the orthogonal projection of a rectifiable (or regular) 1-set is a set of positive length. In [Bes39] Besicovitch studied the projection properties of irregular 1-sets and showed that the orthogonal projections onto lines of a purely unrectifiable (or irregular) 1-set in the plane would have zero length for almost every line. This work was later extended to higher dimensions by Federer in [Fed47]: a good account of the Besicovitch-Federer Projection Theorem and variants on it may be found in [Fed69].

The Projection Theorem has extended considerably our understanding of the structure of sets in Euclidean space and clearly illustrates the profound dichotomy which exists between rectifiable and unrectifiable sets. It is however a qualitative result whose hypotheses require global information about the behaviour of a set. It has been observed by G. David and S. Semmes in [DS91, DS94] that one of the difficulties in trying to find quantitative characterisations of rectifiability is the lack of a local version of this Projection Theorem. The result presented in this paper is a first step towards this goal:

Theorem 1.1 *Suppose that $1 \leq m \leq n$ are integers and μ is a non-zero, almost finite, Borel measure on $\mathbf{R}^n$ such that for μ-a.e. x*

(1) $0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$.

(2) If ν is a tangent measure of μ at x then for all $V \in \mathrm{G}(n, m)$, $\mathrm{P}_V(\mathrm{Spt}\,\nu)$ is convex.

Then μ is m-rectifiable.

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Tangent measures were introduced in [Pre87] and are an extension of ideas in [Mar61] and [Mat75]. They provide a natural framework within which to describe and investigate the local behaviour of measures.

The main advantage of Theorem 1.1 is that it only requires information about the local projection properties of the measure in order to deduce global information about its structure.

There are several directions in which further investigation may prove productive. The main problem is to find what an optimal version of *(2)* should read. The present requirement in *(2)* that *all* projections of the tangent measures be convex appears unnecessarily restrictive and it could possibly be relaxed to only requiring that *almost all* projections are convex.

2 Preliminaries

2.1 Notation

We use $\mathbf{R}^n$ to denote n-dimensional Euclidean space with $\|.\|$ denoting the usual Euclidean norm and $\langle . , . \rangle$ the associated inner product. For $E \subset \mathbf{R}^n$ and $x \in \mathbf{R}^n$ we define

$$\mathrm{d}(x, E) := \inf\{\|y - x\| : y \in E\}$$

and for $r \geq 0$ we set

$$\mathrm{B}(E, r) := \{y \in \mathbf{R}^n : \mathrm{d}(y, E) \leq r\},$$
$$\mathrm{U}(E, r) := \{y \in \mathbf{R}^n : \mathrm{d}(y, E) < r\}.$$

Observe that $\mathrm{U}(E, 0) = \emptyset$ and $\mathrm{B}(E, 0)$ is just the usual topological closure of E in $\mathbf{R}^n$ (usually denoted by $\mathrm{clos}\,(E)$). We abbreviate $\mathrm{B}(\{x\}, r)$ by $\mathrm{B}(x, r)$ and similarly for $\mathrm{U}(x, r)$. If $r > 0$ then $\mathrm{B}(x, r)$ is a non-degenerate ball. If $V \subset \mathbf{R}^n$ let $\mathrm{int}_V(E)$ denote the interior of E with respect to the induced topology on V and let $\partial_V E$ denote the boundary of $E \cap V$ (when considered as embedded in V). Define $\mathrm{int}\,(E) := \mathrm{int}_{\mathbf{R}^n}(E)$ and $\partial E := \partial_{\mathbf{R}^n}(E)$. Let $\mathrm{conv}\,(E)$ denote the closed convex hull of E. For a set E, $\mathrm{card}\,(E)$ will denote the cardinality of E.

$\mathbf{N}$ will denote the natural numbers and $\mathbf{Z}$ the integers. For $x \in \mathbf{R}$, $\lceil x \rceil$ will denote the least integer greater than or equal to x and $\lfloor x \rfloor$ will denote the largest integer less than or equal to x.

Let

$$\mathrm{G}(n, m) := \{V \subset \mathbf{R}^n : V \text{ is an } m\text{-dimensional linear subspace of } \mathbf{R}^n\}.$$

For $V \in \mathrm{G}(n, m)$ let P_V denote orthogonal projection onto V thus $\mathrm{P}_V : \mathbf{R}^n \to \mathbf{R}^n$ and has range V. Define $V_i \to V$ to mean that $\|\mathrm{P}_{V_i} - \mathrm{P}_V\| \to 0$ in the usual operator norm. Let $\mathrm{P}_V^{\perp}$ be the orthogonal projection onto the $(n - m)-$dimensional subspace of $\mathbf{R}^n$ which is orthogonal to V.

For $x \in \mathbf{R}^n$, $h > 0$, $k \geq 1$ and $V \in \mathrm{G}(n, m)$ define $\mathrm{X}(x, h, k, V)$ by

$$\mathrm{X}(x, h, k, V) := \{y \in \mathbf{R}^n \ : \|x - y\| \leq k\,[h + \|\mathrm{P}_V(x - y)\|]\}\,.$$

This is an expanded cone around x with central axis V.

2.2 Measures and densities

Throughout this paper by saying that μ is a Borel measure over $\mathbf{R}^n$ we shall understand that μ is a Borel regular outer measure over $\mathbf{R}^n$ such that all Borel sets are μ-measurable. A measure μ is locally finite if for all $x \in \mathbf{R}^n$ there is an $r > 0$ such that $\mu\mathrm{U}(x, r) < \infty$. Recall that locally finite, Borel measures on $\mathbf{R}^n$ are Radon measures (see [Fed69, 2.2.5]). Observe that this implies that for all compact sets $K \subset \mathbf{R}^n$, $\mu(K) < \infty$. All measures we shall consider in this paper are Borel measures and consequently we shall often just write 'measure' for 'Borel measure'. A measure μ is almost finite if

$$\mu\,\{x \in \mathbf{R}^n : \text{For all } r > 0,\ \mu\mathrm{U}(x, r) = \infty\} = 0.$$

We define the support of a measure μ by

$$\operatorname{Spt}\mu := \mathbf{R}^n \setminus \{x : \text{There is an } r > 0 \text{ with } \mu \mathrm{U}(x,r) = 0\}.$$

Notice that $\operatorname{Spt}\mu$ is a closed set and $\mu(\mathbf{R}^n \setminus \operatorname{Spt}\mu) = 0$. For a set $E \subset \mathbf{R}^n$ we define the restriction of μ to E, $\mu\lfloor_E$, by

$$\mu\lfloor_E(A) := \mu(A \cap E) \text{ for } A \subset \mathbf{R}^n.$$

Observe that if E is a Borel set and μ is a Borel measure then $\mu\lfloor_E$ is also a Borel measure.

A function $f: \mathbf{R}^n \to X$, where X is a topological space, is Borel-measurable if for all open sets $U \subset X$ we find that $f^{-1}(U)$ is a Borel set in $\mathbf{R}^n$. Observe that if μ is a Borel measure on $\mathbf{R}^n$ then such an$f$ is μ-measurable.

For $x \in \mathbf{R}^n$, $A \subset \mathbf{R}^n$ and $r > 0$ define

$$A_{x,r} := \{x + ra : a \in A\}$$

and for a measure μ on $\mathbf{R}^n$ define a new measure $\mu_{x,r}$ by, for $E \subset \mathbf{R}^n$,

$$\mu_{x,r}(E) := \mu(\{x + re : e \in E\}).$$

Thus $\mu_{x,r}(E) = \mu(E_{x,r})$.

One measure which will appear on numerous occasions in this work is m-dimensional Hausdorff measure (where $0 \leq m \leq n$ and we are working in $\mathbf{R}^n$.) If $\alpha(m)$ denotes the Lebesgue measure of a unit ball in $\mathbf{R}^m$ then we define the m-dimensional Hausdorff measure of a set $E \subset \mathbf{R}^n$ by

$$\mathcal{H}^m(E) = \sup_{\delta > 0} \inf \left\{ \sum_{i=1}^{\infty} \alpha(m) \left(\frac{\operatorname{diam}(U_i)}{2} \right)^m : \{U_i\} \text{ is an open } \delta\text{-cover of } E \right\}.$$

For further discussion of Hausdorff measures and proofs that they are indeed Borel measures see either [Rog70] or [Fed69, 2.10].

It will also be helpful to define for integer m between 0 and n

$$\mathcal{G}(n,m) := \{c\mathcal{H}^m\lfloor_V : c > 0,\, V \in \mathrm{G}(n,m)\}$$

which may be thought of as the set of flat m-dimensional measures.

For a measure μ on $\mathbf{R}^n$, $x \in \mathbf{R}^n$ and $0 \leq m \leq n$ define the lower m-density of μ at x, $\underline{\mathrm{D}}_m(\mu, x)$, by

$$\underline{\mathrm{D}}_m(\mu, x) := \liminf_{r \to 0} \frac{\mu \mathrm{B}(x,r)}{\alpha(m) r^m}$$

and define the upper m-density of μ at x, $\overline{\mathrm{D}}_m(\mu, x)$, by

$$\overline{\mathrm{D}}_m(\mu, x) := \limsup_{r \to 0} \frac{\mu \mathrm{B}(x,r)}{\alpha(m) r^m}.$$

If these two limits are the same at a point x then we call their common value the m-density of μ at x and denote it by $\mathrm{D}_m(\mu, x)$.

The following lemma allows us to compare a measure μ with m-dimensional Hausdorff measure.

Lemma 2.1 *Suppose that μ is a locally finite, Borel regular measure on $\mathbf{R}^n$, $0 < \chi < \infty$ and $0 \leq m \leq n$.*

(1) If $E \subset \mathbf{R}^n$ is a Borel set such that for μ-a.e. x in E, $\overline{\mathrm{D}}_m(\mu, x) \leq \chi$ then $\mu(E) \leq 2^m \chi \mathcal{H}^m(\operatorname{Spt}\mu \cap E)$.

(2) If $E \subset \mathbf{R}^n$ is a Borel set such that for μ-a.e. x in E, $\overline{\mathrm{D}}_m(\mu, x) \geq \chi$ then $\mu(E) \geq \chi \mathcal{H}^m(\operatorname{Spt}\mu \cap E)$.

Proof: Both statements follow directly from [Fed69, 2.10.19(1)]. ∎

For a sequence of measures (μ_i) on $\mathbf{R}^n$ we say that μ_i converges to a (locally finite) measure μ (denoted by $\mu_i \to \mu$) if for all continuous functions $f : \mathbf{R}^n \to \mathbf{R}$ with compact support (that is, the set $\operatorname{clos}\{x : f(x) \neq 0\}$ is compact) we have

$$\int f \, d\mu_i \to \int f \, d\mu.$$

For a continuous function $f : \mathbf{R}^n \to \mathbf{R}$ we set $\operatorname{lip}(f)$ to be the least (possibly infinite) real number such that for all $x, y \in \mathbf{R}^n$

$$|f(x) - f(y)| \leq \operatorname{lip}(f)\|x - y\|.$$

If $D > 0$ and two Borel measures μ and ν are such that $(\mu + \nu)\mathrm{U}(0, D) < \infty$ then we define their distance apart on $\mathrm{U}(0, D)$ by

$$\mathrm{F}_D(\mu, \nu) := \sup\left\{\left|\int f \, d\mu - \int f \, d\nu\right| : f \geq 0, \operatorname{Spt}(f) \subset \mathrm{B}(0, D) \text{ and } \operatorname{lip}(f) \leq 1\right\}.$$

It can be shown, [Pre87, 1.11], that

$$\mu_i \to \mu$$

if and only if for all $D > 0$

$$\mathrm{F}_D(\mu_i, \mu) \to 0.$$

Four elementary observations to make about $\mathrm{F}_D(\mu, \nu)$ are

$$\left.\begin{array}{c} \text{if } \omega \text{ is also a measure then } \mathrm{F}_D(\mu,\nu) \leq \mathrm{F}_D(\mu,\omega) + \mathrm{F}_D(\omega,\nu), \\ \text{if } x \in \mathbf{R}^n \text{ then } D\mathrm{F}_1(\mu_{x,D}, \nu_{x,D}) \leq \mathrm{F}_{|x|+D}(\mu,\nu), \\ \text{if } D \leq E \text{ then } \mathrm{F}_D(\mu,\nu) \leq \mathrm{F}_E(\mu,\nu), \\ \mathrm{F}_D(\mu,\nu) = D\mathrm{F}_1(\mu_{0,D}, \nu_{0,D}). \end{array}\right\} \tag{2.1}$$

Let $\mathcal{M}(\mathbf{R}^n)$ denote the set of all locally finite, Borel measures on $\mathbf{R}^n$ (we shall usually write $\mathcal{M}$ for $\mathcal{M}(\mathbf{R}^n)$).

If for $\mu, \nu \in \mathcal{M}$ we define

$$\operatorname{dist}(\mu, \nu) := \sum_{i=1}^{\infty} 2^{-i} \min\{F_i(\mu, \nu), 1\}$$

then this is a metric on $\mathcal{M}$ and with this notion of distance $\mathcal{M}$ is both complete and separable (see [Pre87, 1.12(2)].) Also observe that for $\mu \in \mathcal{M}$, $\mathrm{F}_D(\mu, .)$ is an upper semicontinuous function with respect to the topology induced by dist on $\mathcal{M}$.

We shall have frequent recourse to the following lemma which is a consequence of Prohorov's Theorem.

Lemma 2.2 *If (μ_i) is a sequence of Borel measures on $\mathbf{R}^n$ such that for all $T > 0$*

$$\limsup_{i \to \infty} \mu_i \mathrm{B}(0, T) < \infty$$

then (μ_i) possesses a convergent subsequence.

Proof: See, for example, [Pre87, Lemma 1.12]. A version of this result for probability measures can be found in [Par67]. ∎

The following Lemma provides us with a basic technique for comparing two measures.

Lemma 2.3 *Suppose that μ and ν are in $\mathcal{M}(\mathbf{R}^n)$ and $D > 0$. If $\tau > 0$ and $E \subset \mathbf{R}^n$ are such that $\mathrm{B}(E, \tau) \subset \mathrm{B}(0, D)$ then*

$$\mu(E) \leq \nu \mathrm{B}(E, \tau) + \mathrm{F}_D(\mu,\, \nu)/\tau.$$

Proof: This is [Pre87, Proposition 1.10(3)]. ∎

2.3 Tangent Measures

In this section we shall give a brief outline of the theory of tangent measures. We shall, however, be using a slightly different definition of tangent measures to that given in [Pre87]. Under the density bounds which we shall assume it is easy to see that the two definitions are equivalent.

Suppose μ is a measure such that for some $0 \leq m \leq n$ we find that for μ-a.e. x

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty.$$

Then we may define the standardised tangent measures of μ at x, $\mathrm{Tan}_m(\mu, x)$, as follows:

$$\mathrm{Tan}_m(\mu, x) := \left\{ \nu \in \mathcal{M} : \nu = \lim_{k \to \infty} r_k^{-m} \mu_{x, r_k} \text{ for some sequence } r_k \searrow 0 \right\}.$$

$\mathrm{Tan}_m(\mu, x)$ has the following properties:

$$\text{For } \mu\text{-a.e. } x,\ \mathrm{Tan}_m(\mu, x) \neq \emptyset. \tag{2.2}$$

$$\mathrm{Tan}_m(\mu, x) \text{ is a closed set.} \tag{2.3}$$

If $\nu \in \mathrm{Tan}_m(\mu, x)$ then for all $\rho > 0$,

$$\alpha(m) \underline{\mathrm{D}}_m(\mu, x) \rho^m \leq \nu \mathrm{B}(0, \rho) \leq \alpha(m) \overline{\mathrm{D}}_m(\mu, x) \rho^m. \tag{2.4}$$

For μ-a.e. x if $\nu \in \mathrm{Tan}_m(\mu, x)$ and $\zeta \in \operatorname{Spt} \nu$ then

$$\nu_{\zeta, 1} \in \mathrm{Tan}_m(\mu, x). \tag{2.5}$$

Both (2.2) and (2.3) are elementary consequences of the density estimates on the measure μ together with Lemma 2.2. Equation (2.4) follows from the definition of convergence and it has been observed by P. Mörters that Equation (2.5) may be proved by making obvious modifications to the proof of shift invariance for tangent measures given in [Pre87, Lemma 2.12].

If $0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$ then $\limsup_{r \to 0} \frac{\mu \mathrm{B}(x, 2r)}{\mu \mathrm{B}(x, r)} < \infty$ and so we may use Lemma 2.2 to deduce that for every sequence $r_i \searrow 0$, $(r_i^{-m} \mu_{x, r_i})$ possesses a convergent subsequence and hence $\mathrm{Tan}_m(\mu, x)$ is a compact set. As an immediate consequence of this together with (2.4) and (2.5) we deduce that:

Corollary 2.4 *Suppose that μ is a measure on $\mathbf{R}^n$, $0 \leq m \leq n$ and for μ-a.e. x*

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$$

then for μ-a.e. x if $\nu \in \mathrm{Tan}_m(\mu, x)$, $\zeta \in \operatorname{Spt} \nu$ and $\rho > 0$ then

$$0 < \alpha(m) \underline{\mathrm{D}}_m(\mu, x) \rho^m \leq \nu \mathrm{B}(\zeta, \rho) \leq \alpha(m) \overline{\mathrm{D}}_m(\mu, x) \rho^m < \infty.$$

Moreover there are tangent measures $\nu, \omega \in \mathrm{Tan}_m(\mu, x)$ such that

$$\underline{\mathrm{D}}_m(\nu, 0) = \underline{\mathrm{D}}_m(\mu, x) \text{ and } \overline{\mathrm{D}}_m(\omega, 0) = \overline{\mathrm{D}}_m(\nu, x).$$

2.4 Rectifiability

A set $E \subset \mathbf{R}^n$ is m-rectifiable for some m, an integer between 0 and n, if there is a countable set of Lipschitz maps $f_i : \mathbf{R}^m \to \mathbf{R}^n$ such that

$$\mathcal{H}^m \left(E \setminus \bigcup_i f_i \left(\mathbf{R}^m \right) \right) = 0.$$

A set E is purely m-unrectifiable if for all Lipschitz maps $f : \mathbf{R}^m \to \mathbf{R}^n$

$$\mathcal{H}^m \left(E \cap f \left(\mathbf{R}^m \right) \right) = 0.$$

A measure μ is m-rectifiable if there is an m-rectifiable Borel set E such that

$$\mu \left(\mathbf{R}^n \setminus E \right) = 0.$$

Observe that we have not required that μ be absolutely continuous with respect to $\mathcal{H}^m$. Thus, under our definition, $\mathcal{H}^{\log 2/\log 3}$ restricted to the usual 1/3-Cantor set is a 1-rectifiable measure. However the density estimates in the hypotheses of Theorem 1.1 ensure that any measure μ satisfying the hypotheses of the Theorem will also be absolutely continuous with respect to $\mathcal{H}^m$ measure.

A measure μ is purely m-unrectifiable if for all m-rectifiable Borel sets E

$$\mu \left(E \right) = 0.$$

One deep result about rectifiability which we shall use is the Besicovitch-Federer Projection Theorem:

Theorem 2.5 *Suppose that E is a purely m-unrectifiable, $\mathcal{H}^m$-measurable set with $\mathcal{H}^m(E) < \infty$ then for almost every $V \in \mathrm{G}(n,m)$*

$$\mathcal{H}^m \mathrm{P}_V(E) = 0.$$

Proof: See [Mat95, Theorem 18.1]. ∎

More recently Preiss has proved the following theorem concerning rectifiability:

Theorem 2.6 *Whenever μ is an almost finite Borel measure on $\mathbf{R}^n$, the following conditions are equivalent:*

(1) μ is m-rectifiable and is absolutely continuous with respect to $\mathcal{H}^m$.

(2) For μ-a.e. x, $0 < \mathrm{D}_m(\mu, x) < \infty$.

(3) For μ-a.e. x, $0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$ and $\mathrm{Tan}_m(\mu, x) \subset \mathcal{G}(n, m)$.

Proof: See [Pre87, Theorem 5.6]. ∎

We shall split the proof of Theorem 1.1 into two sections; the first section contains many of the preliminary Lemmas which are required for proving the Theorem and the second contains the actual proof.

3 Lemmas

Unless otherwise stated we shall always be working in $\mathbf{R}^n$ and $\mathcal{M} = \mathcal{M}(\mathbf{R}^n)$. Define for $0 < a \leq b < \infty$ and $0 \leq m \leq n$

$$\mathcal{M}^m(a,b) := \{0 \neq \nu \in \mathcal{M} : \text{ For all } \zeta \in \operatorname{Spt} \nu, \text{ for all } \rho > 0, \alpha(m) a \rho^m \leq \nu \mathrm{B}(\zeta, \rho) \leq \alpha(m) b \rho^m\}$$

and let

$$\mathcal{M}_C^m := \{\nu \in \mathcal{M} \,:\, \text{For all } V \in \mathrm{G}(n,m),\ \mathrm{P}_V(\mathrm{Spt}\,\nu) \text{ is convex }\}\,.$$

Finally define

$$\mathcal{M}_C^m(a,b) := \mathcal{M}_C^m \cap \mathcal{M}^m(a,b).$$

First let us observe that:

Lemma 3.1 *If $0 < a \le b < \infty$ and if $\nu \in \mathcal{M}_C^m(a,b)$ then for all $\zeta \in \mathbf{R}^n$ and all $\rho > 0$, $\rho^{-m}\nu_{\zeta,\rho} \in \mathcal{M}_C^m(a,b)$.*

Proof: This is obvious. ∎

The next lemma is a generalisation of Pythagoras' Theorem.

Lemma 3.2 *Suppose W is an $(\mathcal{H}^m, m)$-rectifiable set in $\mathbf{R}^n$. Define*

$$\Lambda(n,\,m) := \{\lambda \,:\, \lambda\colon\{1,\ldots,m\} \to \{1,\ldots,n\} \text{ and if } i<j \text{ then } \lambda(i) < \lambda(j)\}$$

and let $\{e_1,\ldots,e_n\}$ be the standard orthonormal basis in $\mathbf{R}^n$ and for $x = \sum_{i=1}^n x_i e_i$ in $\mathbf{R}^n$ and $\lambda \in \Lambda(n,\,m)$ define

$$P_\lambda(x) := \sum_{i=1}^m x_{\lambda(i)} e_{\lambda(i)}$$

and

$$V_\lambda := \{P_\lambda(x) \,:\, x \in \mathbf{R}^n\}\,.$$

Then if

$$a_\lambda := \int \mathrm{card}\,\left(P_\lambda^{-1}(y) \cap W\right)\, d\mathcal{H}^m \lfloor_{V_\lambda}(y)$$

we have that

$$\left[\sum_{\lambda\in\Lambda(n,m)} (a_\lambda)^2\right]^{1/2} \le \mathcal{H}^m(W) \le \sum_{\lambda\in\Lambda(n,m)} a_\lambda.$$

Proof: This is [Fed69, 3.2.27] . ∎

Using the same notation as the last Lemma it is easy to see that:

Lemma 3.3 *Suppose that E is an m-dimensional linear subspace of $\mathbf{R}^n$ and there is a $\sigma \in \Lambda(n,m)$ such that*

$$P_\sigma(E) = V_\sigma$$

and for all $\lambda \ne \sigma$ we have

$$\mathcal{H}^m(P_\lambda(E)) = 0.$$

Then $E = V_\sigma$.

Our next proposition is a variation on the covering lemmas more usually seen. The key point is that we know that the boundaries of the covering sets are not too large.

Proposition 3.4 *Suppose that $A \subset \mathbf{R}^m$ is a bounded set and that $\{\mathrm{B}(x, r(x)) \,:\, x \in A\}$ is a collection of non-degenerate balls in $\mathbf{R}^m$ such that $\sup_{x\in A} r(x) < \infty$. Then we may find a countable (possibly finite) set $D \subset A$ and an associated disjoint collection of Borel sets $\mathcal{C} := \{C_x \,:\, x \in D\}$ such that*

(1) for all $x \in D$, $\mathrm{B}(x, r(x)) \subset C_x \subset \mathrm{B}(x, 4r(x))$,

(2) $A \subset \cup\mathcal{C}$,

(3) for all $0 < \epsilon < [2/(3m)]^{m+1}$ and for all $x \in D$

$$\mathcal{H}^m\left[\mathrm{B}(\partial C_x, \epsilon r(x))\right] \leq c(m)\epsilon^{1/(m+1)}[r(x)]^m.$$

(The constant $c(m)$ depends only on m.)

Proof: Since $\sup_{x \in A} r(x) < \infty$ we may use [Fed69, 2.8.4] to find a countable set $D \subset A$ such that $\{\mathrm{B}(x, r(x)) : x \in D\}$ is a disjoint collection and yet $\{\mathrm{B}(x, 4r(x)) : x \in D\}$ covers A. Moreover as A is a bounded set and $\{\mathrm{B}(x, r(x)) : x \in D\}$ is a disjoint collection we conclude that for all $x \in D$ the set

$$\{y \in D : \mathrm{B}(x, 4r(x)) \subset \mathrm{B}(y, 4r(y))\}$$

is finite. Hence, as A is bounded, we can find an enumeration $x_1, x_2, \ldots$ of D such that the sequence $(r(x_k))$ is decreasing. Therefore, by a selection process, we may assume that if x and y are distinct elements of D then both $\mathrm{B}(x, 4r(x)) \setminus \mathrm{B}(y, 4r(y))$ and $\mathrm{B}(y, 4r(y)) \setminus \mathrm{B}(x, 4r(x))$ are non-empty. Define the collection $\mathcal{C}$ inductively as follows: For $k \geq 1$,

$$C_k := \mathrm{B}(x_k, 4r(x_k)) \setminus \left[\bigcup_{1 \leq i \leq k-1} C_i \cup \bigcup_{i \geq k+1} \mathrm{B}(x_i, r(x_i))\right].$$

Clearly the family $\mathcal{C} := \{C_k : k \geq 1\}$ is disjoint and each member of it is a Borel set. Claims (1) and (2) are similarly straightforward to verify.

For the third claim fix $j \geq 1$ and for all $i \geq 1$ let $r_i = r(x_i)$. We split the collection $\{x_i\}$ into three classes:

$$D_1 := \{x_i : i < j \text{ and } \mathrm{B}(x_i, 4r_i) \cap \mathrm{B}(x_j, 4r_j) \neq \emptyset\},$$

$$D_2 := \{x_i : i > j, r_i > \epsilon^{1/(m+1)} r_j \text{ and } \mathrm{B}(x_i, r_i) \cap \mathrm{B}(x_j, 4r_j) \neq \emptyset\}$$

and

$$D_3 := \{x_i : i > j, r_i \leq \epsilon^{1/(m+1)} r_j \text{ and } \mathrm{B}(x_i, r_i) \cap \mathrm{B}(x_j, 4r_j) \neq \emptyset\}$$

and observe that

$$\partial C_j \subset \operatorname{clos}\left[\partial \mathrm{B}(x_j, r_j) \cup \partial \mathrm{B}(x_j, 4r_j) \cup \bigcup_{x \in D_1} \partial \mathrm{B}(x, 4r(x)) \cup \bigcup_{x \in D_2 \cup D_3} \partial \mathrm{B}(x, r(x))\right].$$

Upon recalling that for distinct x and y in D, $\mathrm{B}(x, r(x)) \cap \mathrm{B}(y, r(y)) = \emptyset$ it is a simple application of the Besicovitch Covering Theorem [Fed69, 2.8.12] to deduce that there is a constant $c_1 = c_1(m)$ such that

$$\operatorname{card}(D_1) \leq c_1.$$

Furthermore it is clear that there is a constant $c_2 = c_2(m)$ such that for each $x \in D_1$

$$\mathcal{H}^m\left[\mathrm{B}(\partial \mathrm{B}(x, 4r(x)) \cap \mathrm{B}(x_j, 4r_j), \epsilon r_j)\right] \leq c_2 \epsilon r_j^m.$$

Hence

$$\mathcal{H}^m\left[\mathrm{B}(\cup_{x \in D_1} \partial(\mathrm{B}(x, 4r(x)) \cap \mathrm{B}(x_j, 4r_j)), \epsilon r_j)\right] \leq c_1 c_2 \epsilon r_j^m.$$

Since for all $x \in D_2$, $\epsilon^{1/(m+1)} r_j \leq r(x) \leq r_j$ it is easy to see that for all $x \in D_2$, $\mathrm{B}(x, r(x)) \subset \mathrm{B}(x_j, 6r_j)$. Consequently disjointness enables us to estimate that

$$\operatorname{card} D_2) \leq \frac{\alpha(m) 6^m r_j^m}{\alpha(m) \epsilon^{m/(m+1)} r_j^m} \quad = \quad 6^m \epsilon^{-m/(m+1)}.$$

and thus as there is a $c_3 = c_3(m)$ such that for all $x \in D_2$

$$\mathcal{H}^m\left[\mathrm{B}(\partial \mathrm{B}(x, r(x)), \epsilon r_j)\right] \leq c_3 \epsilon r_j^m$$

we deduce that

$$\mathcal{H}^m\left[\mathrm{B}\left(\bigcup_{x\in D_2} \partial \mathrm{B}(x, r(x)), \epsilon r_j\right)\right] \leq 6^m c_3 \epsilon^{1/(m+1)} r_j^m.$$

Finally let us consider the contribution to the boundary of C_j due to D_3. It is sufficient to observe that, as for all $x \in D$ there is no $y \in D$ (distinct from x) such that $\mathrm{B}(x, 4r(x)) \subset \mathrm{B}(y, 4r(y))$, for any $x \in D_3$

$$\mathrm{B}(x, r(x)) \subset \mathrm{B}(x_j, 4r_j + 2r(x)) \setminus \mathrm{B}(x_j, 4r_j - 5r(x))$$

and so

$$\mathrm{B}\left(\partial \mathrm{B}(x, r(x)), \epsilon^{1/(m+1)} r_j\right) \subset \mathrm{B}\left(x_j, (4 + 3\epsilon^{1/(m+1)})r_j\right) \setminus \mathrm{B}\left(x_j, (4 - 6\epsilon^{1/(m+1)})r_j\right).$$

Hence

$$\begin{aligned}
\mathcal{H}^m\left[\mathrm{B}\left(\bigcup_{x\in D_3} \partial \mathrm{B}(x, r(x)), \epsilon r_j\right)\right] &\leq \alpha(m)4^m r_j^m \left[(1 + 3\epsilon^{1/(m+1)}/2)^m - (1 - 3\epsilon^{1/(m+1)}/2)^m\right] \\
\text{which, as } \epsilon < [2/(3m)]^{m+1}, \text{ is} \quad &\leq 3 \times 4^m m^2 \alpha(m) \epsilon^{1/(m+1)} r_j^m \\
&= c_4 \epsilon^{1/(m+1)} r_j^m, \text{ say.}
\end{aligned}$$

Finally putting all these estimates together we deduce that there is a constant $c(m)$ so that

$$\mathcal{H}^m\left[\mathrm{B}\left(\partial C_j, \epsilon r_j\right)\right] \leq c(m)\epsilon^{1/(m+1)} r_j^m$$

as required. ∎

Lemma 3.5 *Suppose $0 < a \leq b < \infty$, $0 < \epsilon < (m+1)^{-1/2}/3$ and both R and $D > 0$. If $\mu \in \mathcal{M}$ and $y \in \operatorname{Spt}\mu \cap \mathrm{B}(0, D)$ are such that for $0 \leq r \leq R$*

$$\mu \mathrm{B}(y, r) \geq \alpha(m) a r^m$$

and if $\nu \in \mathcal{M}^m(a, b)$ is such that

$$\mathrm{F}_{D+R}(\mu_{0,r}/r^m, \nu) < \alpha(m) a \epsilon^{m+3}$$

then there is a $\zeta \in \operatorname{Spt}\nu$ such that

$$\|\zeta - y/r\| \leq \epsilon.$$

Proof: This is a consequence of Lemma 2.3 with $E = \mathrm{B}(y, m\epsilon/(m+1))$ and $\tau = \epsilon/(m+1)$. ∎

The next lemma is a straightforward consequence of the compactness of $\operatorname{Tan}_m(\mu, x)$ for 'nice' x.

Lemma 3.6 *If $\mu \in \mathcal{M}$ is such that for μ-a.e. x*

$$0 < \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) < \infty$$

then there is a Borel set B of positive μ-measure such that if $x_i \in B$ for all i and $x_i \to x \in B$ and if $\nu_i \in \operatorname{Tan}_m(\mu, x_i)$ is a sequence of measures converging to a measure ν then ν is in $\operatorname{Tan}_m(\mu, x)$.

Proof: We may find $0 < a \le b < \infty$ and a Borel set E of finite and positive μ-measure which is contained in the support of μ such that for all $x \in E$

$$a \le \underline{\mathrm{D}}_m(\mu, x) \le \overline{\mathrm{D}}_m(\mu, x) \le b.$$

This implies that for all $x \in E$, $\mathrm{Tan}_m(\mu, x)$ is a non-empty, compact set and $\mathrm{Tan}_m(\mu, x) \subset \mathcal{M}(a, b)$. For non-empty compact subsets M, $N \subset \mathcal{M}$ we may define their Hausdorff distance to be

$$\mathrm{H}(M, N) := \max\{d(M, N),\, d(N, M)\}$$

where

$$d(M, N) := \sup_{\mu \in M} \inf_{\nu \in N} \mathrm{dist}(\mu, \nu).$$

If $\mathcal{K}$ is defined to be the collection of non-empty compact subsets of $\mathcal{M}$ then $(\mathcal{K}, \mathrm{H})$ is a complete separable metric space (see [Mic51, Propositions 4.5(1), 4.1(3)]). Moreover it is a straightforward exercise to verify that $t\colon E \to \mathcal{K}$ defined by $t(x) := \mathrm{Tan}_m(\mu, x)$ is Borel-measurable (see [O'N94, 1.4.5]). Hence we may use Lusin's Theorem [Fed69, 2.3.5] to find a compact subset B contained in E which is of positive μ-measure and upon which t is continuous. Thus if $x_i \to x$ in B and if $\nu_i \in \mathrm{Tan}_m(\mu, x_i)$ converge to a measure ν then since $t(x_i) \to t(x)$ we conclude that $\nu \in \mathrm{Tan}_m(\mu, x)$ — this implies the lemma. ∎

The following (technical) lemma reduces the notion of convex projections to considerations of finite sets of points. Observe that the constant R in the following lemma is a uniform bound.

Lemma 3.7 *Suppose $\mu \in \mathcal{M}(\mathbf{R}^n)$, B is a compact set of positive μ-measure and $0 < a \le b < \infty$ are such that*

(1) for all $x \in B$, $a \le \underline{\mathrm{D}}_m(\mu, x) \le \overline{\mathrm{D}}_m(\mu, x) \le b$,

(2) if $(x_i) \subset B$ and $x_i \to x \in B$ and $\nu_i \in \mathrm{Tan}_m(\mu, x_i)$ converge to a measure ν then $\nu \in \mathrm{Tan}_m(\mu, x)$,

(3) for all $x \in B$, $\mathrm{Tan}_m(\mu, x) \subset \mathcal{M}^m_C(a, b)$.

(4) for all $x \in B$, if $\nu \in \mathrm{Tan}_m(\mu, x)$ and $\zeta \in \mathrm{Spt}\,\nu$ then $\nu_{\zeta,1} \in \mathrm{Tan}_m(\mu, x)$.

Then for all $\xi, \gamma \in (0, 1)$ and integer $M \ge 2$, there is an $R \ge 1$ so that for all $x \in B$, all $\nu \in \mathrm{Tan}_m(\mu, x)$, all $V \in \mathrm{G}(n, m)$ and all distinct points $\{\zeta^1, \ldots, \zeta^M\} \subset \mathrm{Spt}\,\nu$ which satisfy

$$\min_{i \ne j}\{||\mathrm{P}_V(\zeta^i - \zeta^j)||\} \ge \gamma \max_{i,j}\{||\zeta^i - \zeta^j||\}$$

we have that if $u \in \mathrm{conv}\,\{\mathrm{P}_V\zeta^1, \ldots, \mathrm{P}_V\zeta^M\}$ then there is a $\zeta \in \mathrm{Spt}\,\nu \cap \mathrm{B}(\zeta^1, R\max_{i,j}\{||\zeta^i - \zeta^j||\})$ with

$$\mathrm{P}_V\zeta \in \mathrm{B}(u, \xi \min_{i \ne j}\{||\mathrm{P}_V(\zeta^i - \zeta^j)||\}).$$

Proof: This is a technical but straightforward exercise in using compactness. For the unedifying details see [O'N94, 2.2.7]. ∎

Lemma 3.8 *If $0 < a \le b < \infty$, μ is a measure, $x \in \mathrm{Spt}\,\mu$, $s > 0$ and $\nu \in \mathcal{M}^m(a, b)$ are such that for some $R \ge 1$ and $0 < \epsilon < 1/m$*

$$\mathrm{F}_{R+3}\left(\nu, \frac{\mu_{x,s}}{s^m}\right) < \alpha(m) a \epsilon^{m+3}$$

then for all $z \in (x + s\mathrm{Spt}\,\nu) \cap \mathrm{B}(x, Rs)$ and all $t \in [\epsilon s, s]$ we have that

$$\alpha(m) a(1 - 3m\epsilon)t \le \mu\mathrm{B}(z, t) \le \alpha(m) b(1 + 3m\epsilon)t.$$

Proof: This is an application of Lemma 2.3 with $E = \mathrm{B}((z-x)/s, t/s)$ and $\tau = \epsilon t/s$. ∎

Lemma 3.9 *Suppose $0 \in E \subset \mathbf{R}^n$ is such that for all $V \in \mathrm{G}(n,m)$, $\mathrm{P}_V(E)$ is a convex set and for almost every $V \in \mathrm{G}(n,m)$, $\mathcal{H}^m[\mathrm{P}_V(E)] = 0$ then there is an $(m-1)$-dimensional subspace of $\mathbf{R}^n$ which contains E.*

Proof: If there were $m+1$ points , $\{0, e_1, \ldots, e_m\}$, such that the linear span, V, of $\{0, e_1, \ldots, e_m\}$ was m-dimensional then $\mathcal{H}^m(\mathrm{P}_V \operatorname{conv}\{0, e_1, \ldots, e_m\}) > 0$ and moreover for all $W \in \mathrm{G}(n,m)$ sufficiently close to V we would have

$$\mathcal{H}^m(\mathrm{P}_W(\operatorname{conv}\{0, e_1, \ldots, e_m\})) > 0$$

which is impossible by the hypotheses of the Lemma. Thus, for any m points, $\{e_1, \ldots, e_m\}$ in E, the linear span of $\{0, e_1, \ldots, e_m\}$ has dimension strictly less than m. This gives the Lemma.∎

As a consequence of this we have our first result concerning convex projections of purely unrectifiable measures:

Lemma 3.10 *If $0 < a \leq b < \infty$ and ν is a purely m-unrectifiable, locally finite measure on $\mathbf{R}^n$ with*

(1) for all $\zeta \in \operatorname{Spt}\nu$, $a \leq \underline{\mathrm{D}}_m(\nu, \zeta) \leq \overline{\mathrm{D}}_m(\nu, \zeta) \leq b$,

(2) for all $V \in \mathrm{G}(n,m)$, $\mathrm{P}_V(\operatorname{Spt}\nu)$ is a convex set.

Then $\nu \equiv 0$.

Proof: From Lemma 2.1 we know that for all Borel sets E we have

$$2^m b\mathcal{H}^m(\operatorname{Spt}\nu \cap E) \geq \nu(E) \geq a\mathcal{H}^m(\operatorname{Spt}\nu \cap E)$$

and so we deduce from the unrectifiability of ν that if E is m-rectifiable then $\nu(E) = 0$. Thus $\operatorname{Spt}\nu$ is a purely $(\mathcal{H}^m, m)$-unrectifiable set. Hence Theorem 2.5 enables us to deduce that for almost every $V \in \mathrm{G}(n,m)$ and all $R \geq 0$

$$\mathcal{H}^m\left[\mathrm{P}_V(\operatorname{Spt}\nu \cap \mathrm{B}(0,R))\right] = 0.$$

Thus for almost every $V \in \mathrm{G}(n,m)$

$$\mathcal{H}^m\left[\mathrm{P}_V(\operatorname{Spt}\nu)\right] = 0$$

and so we can use Lemma 3.9 to deduce that there is an $(m-1)$-dimensional subspace, W say, which contains $\operatorname{Spt}\nu$. But then for any $\zeta \in \operatorname{Spt}\nu$ and $r > 0$

$$\begin{aligned} \nu[\mathrm{B}(\zeta, r)] &\leq 2^m b\mathcal{H}^m\left[\operatorname{Spt}\nu \cap \mathrm{B}(\zeta, r)\right] \\ &= 2^m b\mathcal{H}^m\left[W \cap \operatorname{Spt}\nu \cap \mathrm{B}(\zeta, r)\right] \quad = \quad 0 \end{aligned}$$

which implies that $\operatorname{Spt}\nu = \emptyset$ and so the Lemma follows. ∎

Lemma 3.11 *Suppose that $0 < a \leq \chi \leq b < \infty$, $x \in \mathbf{R}^n$ and $\mu \in \mathcal{M}$ are such that*

(1) $a \leq \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) \leq b$,

(2) if $\nu \in \operatorname{Tan}_m(\mu, x)$ and $\zeta \in \operatorname{Spt}\nu$ then $\nu_{\zeta,1} \in \operatorname{Tan}_m(\mu, x)$.

(3) $\chi \leq \inf\{\mathrm{D}_m(\omega, 0) : \omega \in \operatorname{Tan}_m(\mu, x) \cap \mathcal{G}(n,m)\} \leq b$.

Then for all $\nu \in \mathrm{Tan}_m(\mu, x)$ we have that for ν-a.e. ζ

$$\overline{\mathrm{D}}_m(\nu, \zeta) \geq \chi.$$

Proof: Suppose that the Lemma is false: Then there is a $\nu \in \mathrm{Tan}_m(\mu, x)$, a Borel set C of positive ν-measure and a $\theta \in (0, 1)$ such that for all $\zeta \in C$

$$\overline{\mathrm{D}}_m(\nu, \zeta) \leq \chi(1 - \theta), \tag{3.1}$$

$$\text{if } \omega \in \mathrm{Tan}_m(\nu, \zeta) \text{ and } \xi \in \mathrm{Spt}\, \omega \text{ then } \omega_{\xi,1} \in \mathrm{Tan}_m(\nu, \zeta). \tag{3.2}$$

Fix $\zeta \in C$ and consider $\omega \in \mathrm{Tan}_m(\mu, \zeta)$ then (3.1) and (3.2) enable us to conclude that for all $\xi \in \mathrm{Spt}\, \omega$

$$\overline{\mathrm{D}}_m(\omega, \xi) \leq \chi(1 - \theta).$$

However for any $\xi \in \mathrm{Spt}\, \omega$

$$\omega_{\xi,1} \in \mathrm{Tan}_m(\nu, \zeta) = \mathrm{Tan}_m(\nu_{\zeta,1}, 0) \subset \mathrm{Tan}_m(\mu, x)$$

and so

$$\mathrm{Tan}_m(\omega, \xi) \subset \mathrm{Tan}_m(\mu, x) \, (\subset \mathcal{M}_C^m(a, b)) \, .$$

Thus 3.11(1,2,3) together with Theorem 2.6 force us to conclude that ω is purely m-unrectifiable. But then Lemma 3.10 implies that $\omega = 0$. This is impossible and so our claim holds. ∎

It is easy to see that conditions (1),(2),(3) and (5) of the following proposition are not sufficient to guarantee its conclusion. For example we may consider the measure $\mathcal{H}^1 \lfloor_{\partial B(0,1)} + \mathcal{H}^1 \lfloor_C$ where $C \subset \mathrm{B}(0, 1)$ is any purely 1-unrectifiable 1-set with positive and finite upper and lower 1-densities.

Proposition 3.12 *Suppose that $V \in \mathrm{G}(n, m)$ and ν is a measure which satisfy the following:*

(1) For all $W \in \mathrm{G}(n, m)$, $\mathrm{P}_W(\mathrm{Spt}\, \nu)$ is a convex set,

(2) there is a $\chi > 0$ such that for ν-a.e.ζ, $\overline{\mathrm{D}}_m(\nu, \zeta) \geq \chi$,

(3) there are $0 < a \leq b < \infty$ such that for ν-a.e.ζ

$$a \leq \underline{\mathrm{D}}_m(\nu, \zeta) \leq \overline{\mathrm{D}}_m(\nu, \zeta) \leq b,$$

(4) for all $I \subset V$, $\nu(\mathrm{P}_V^{-1}(I)) = \chi \mathcal{H}^m(I)$,

(5) there is an $h \geq 0$ and $k \geq 0$ so that $\mathrm{Spt}\, \nu \subset \mathrm{X}(0, h, k, V)$.

Then ν is an m-rectifiable measure.

Proof: From the density hypothesis, 3.12(3), we may use Lemma 2.1 to deduce that for all Borel sets $E \subset \mathbf{R}^n$

$$\chi \mathcal{H}^m(E \cap \mathrm{Spt}\, \nu) \leq \nu(E) \leq 2^m b \mathcal{H}^m(E \cap \mathrm{Spt}\, \nu). \tag{3.3}$$

These density estimates enable us to conclude that a set A is (ν, m)-rectifiable if and only if $A \cap \mathrm{Spt}\, \nu$ is an $(\mathcal{H}^m, m)$-rectifiable set. Hence we may split $\mathrm{Spt}\, \nu$ into an $(\mathcal{H}^m, m)$-rectifiable Borel set, R, and a purely $(\mathcal{H}^m, m)$-unrectifiable Borel set, U, such that $\mathcal{H}^m(R \cap U) = \nu(R \cap U) = 0$ and $\mathcal{H}^m[(R \cup U) \setminus \mathrm{Spt}\, \nu] = 0$.

If $\mathcal{H}^m[\mathrm{P}_V U] = 0$ then, from 3.12(4), we conclude that $\nu(U) = 0$ and hence (3.3) implies that $\mathcal{H}^m(U) = 0$ and we are done.

So suppose instead that $\mathcal{H}^m[\mathrm{P}_V U] > 0$. We may suppose (by a suitable translation and relabeling of h and k) that 0 is a density point of $\mathrm{P}_V(U)$. Fix $0 < \xi < 1$ and recall that $\mathrm{Spt}\,\nu \subset \mathrm{X}(0, h, k, V)$. We can find an $r > 0$ such that for $0 < s \leq r$

$$\mathcal{H}^m(\mathrm{P}_V(U) \cap \mathrm{B}(0, s)) > (1 - \xi)\mathcal{H}^m(\mathrm{B}(0, s) \cap V).$$

So for such an s we find that

$$\nu[\mathrm{P}_V^{-1}(\mathrm{B}(0, s)) \cap R] \leq \chi\xi\mathcal{H}^m(\mathrm{B}(0, s) \cap V)$$

and so, by (3.3),

$$\mathcal{H}^m[\mathrm{P}_V^{-1}(\mathrm{B}(0, s)) \cap R] \leq \xi\mathcal{H}^m(\mathrm{B}(0, s) \cap V).$$

Fix $[1 - \xi^{1/m}]r < s < r$. Since $\mathrm{P}_V^{-1}(\mathrm{B}(0, s) \cap \mathrm{X}(a, h, k, V)$ is compact we may find a $\delta > 0$ such that if $W \in \mathrm{G}(n, m)$ has $\|\mathrm{P}_V - \mathrm{P}_W\| < \delta$ then

$$\mathrm{P}_W^{-1}(\mathrm{B}(0, s)) \cap \mathrm{X}(0, h, k, V) \subset \mathrm{P}_V^{-1}(\mathrm{B}(0, r)) \cap \mathrm{X}(0, h, k, V).$$

On observing that 3.12(1) and 3.12(4) imply that $\mathrm{P}_V \mathrm{Spt}\,\nu = V$ we deduce that we can find $0 < \delta' \leq \delta$ such that for all $W \in \mathrm{G}(n, m)$ satisfying $\|\mathrm{P}_V - \mathrm{P}_W\| < \delta'$ we have

$$\mathrm{P}_W[\mathrm{Spt}\,\nu \cap \mathrm{P}_V^{-1}(\mathrm{B}(0, r))] \cap \mathrm{B}(0, s) \supset \mathrm{B}(0, s) \cap W.$$

Hence, for such W,

$$\begin{aligned} \mathcal{H}^m[\mathrm{P}_W[U \cap \mathrm{P}_V^{-1}(\mathrm{B}(0, r))] \cap \mathrm{B}(0, s)] &\geq \mathcal{H}^m[\mathrm{B}(0, s) \cap W) - \mathcal{H}^m(\mathrm{P}_W(R \cap \mathrm{P}_V^{-1}\mathrm{B}(0, r))] \\ &\geq \alpha(m)s^m - \alpha(m)(\xi\chi/a)r^m \\ &> 0. \end{aligned}$$

However, from the Projection Theorem, we know that for almost every $W \in \mathrm{G}(n, m)$ we have $\mathcal{H}^m[\mathrm{P}_W U] = 0$. But this contradicts the above and thus $\mathcal{H}^m(U) = 0$ as required. ∎

The following lemma is a key part of the proof of Theorem 1.1: The proof of this theorem is directed towards showing that if there is a purely m-unrectifiable measure μ satisfying the hypotheses of the theorem then we can construct a tangent measure, ν, satisfying the hypotheses of the following lemma. However the construction of ν is such that the following condition must also hold: For all $A \subset V$ (defined below) we have

$$\nu\mathrm{P}_V^{-1}(A) = \chi\mathcal{H}^m(A).$$

This is in direct contradiction with the conclusion of the lemma.

Lemma 3.13 *Suppose $0 < a \leq \chi \leq b < \infty$ and $\nu \in \mathcal{M}_C^m(a, b)$ is an m-rectifiable measure such that*

(1) for ν-a.e. x, $\overline{\mathrm{D}}_m(\nu, x) \geq \chi$

(2) there is a $V \in \mathrm{G}(n, m)$ with $\mathrm{diam}\,(\mathrm{P}_V^{\perp}\mathrm{Spt}\,\nu) > 0$ and $V = \mathrm{P}_V(\mathrm{Spt}\,\nu)$.

Then there is a Borel set $B \subset V$ such that

$$\nu(\mathrm{P}_V^{-1}(B)) > \chi\mathcal{H}^m(B).$$

Proof: This follows from Lemma 3.3 and Lemma 3.2: For suppose we choose an orthonormal basis of $\mathbf{R}^n$, $\{e_1, \ldots, e_n\}$, such that V is the linear subspace spanned by $\{e_1, \ldots, e_m\}$ and σ is its associated map in $\Lambda(n, m)$. If for all $\lambda \neq \sigma$ we have that

$$\mathcal{H}^m[P_\lambda(\mathrm{Spt}\,\nu)] = 0$$

then applying Lemma 3.3 we conclude that $\mathrm{Spt}\,\nu \subset V$ and this contradicts the fact that $\mathrm{diam}[\mathrm{P}_V^{\perp}\mathrm{Spt}\,\nu] > 0$. Thus there is a $\lambda \in \Lambda(n,m)$ which is different from σ and with

$$\mathcal{H}^m[P_\lambda(\mathrm{Spt}\,\nu)] > 0.$$

Hence we can find a closed ball $B \subset V$ such that for some positive ξ

$$\mathcal{H}^m[P_\lambda((\mathrm{P}_V^{-1}B) \cap \mathrm{Spt}\,\nu)] > \xi.$$

By Lemma 3.2 we conclude that

$$\mathcal{H}^m(\mathrm{P}_V^{-1}(B) \cap \mathrm{Spt}\,\nu) \geq \left[(\mathcal{H}^m(B))^2 + \xi^2\right]^{1/2}$$

and so as for ν-a.e. x, $\overline{\mathrm{D}}_m(\nu, x) \geq \chi$ we deduce from Lemma 2.1 that

$$\nu(\mathrm{P}_V^{-1}(B)) \quad \geq \quad \chi\left[(\mathcal{H}^m(B))^2 + \xi^2\right]^{1/2} \quad > \quad \chi\mathcal{H}^m(B)$$

as required. ∎

Our final lemma in this section is a technical result introduced to reduce repetition later.

Lemma 3.14 *Fix $L \in [0,\infty]$, $a, b, q > 0$ and suppose that S_i, Θ_i and Ξ_i are sequences of positive real numbers with $\limsup S_i \geq L$, $S_i\Theta_i \to 0$ and $\Xi_i \to 0$. If $V_i \in \mathrm{G}(n,m)$, $\nu_i \in \mathcal{M}_C^m(l,u)$ and $Y_i \in \mathrm{Spt}\,\nu_i \cap \mathrm{P}_{V_i}^{-1}[\mathrm{B}(0, 1+\Xi_i)] \cap \mathrm{X}(0, a+\Xi_i, b, V_i)$ are such that*

$$V_i \to V \in \mathrm{G}(n,m),\ Y_i \to Y \text{ and } \nu_i \to \nu \in \mathcal{M}_C^m(l,u),$$

$$\text{for all } i,\ \|\mathrm{P}_{V_i}^{\perp}Y_i\| \geq q - \Xi_i$$

and for all $w \in \mathrm{B}(0, S_i) \cap V_i$

$$\mathrm{P}_{V_i}^{-1}[\mathrm{B}(w, \Theta_i S_i)] \cap \mathrm{Spt}\,\nu_i \cap \mathrm{X}(0, a+\Xi_i, b, V_i) \neq \emptyset.$$

Then (on interpreting $\mathrm{B}(0,\infty)$ as $\mathbf{R}^n$)

(1) $\mathrm{P}_V[\mathrm{Spt}\,\nu \cap \mathrm{X}(0,a,b,V)] \supset V \cap \mathrm{B}(0,L)$,

(2) $\|\mathrm{P}_V^{\perp}Y\| \geq q$,

(3) $Y \in \mathrm{Spt}\,\nu \cap \mathrm{P}_V^{-1}\mathrm{B}(0,1) \cap \mathrm{X}(0,a,b,V)$.

Proof: First observe that an immediate consequence of the density estimates on the ν_i is that if $y_i \in \mathrm{Spt}\,\nu_i \in \mathcal{M}_C^m(l,u)$ for all i and $y_i \to y$ then $y \in \mathrm{Spt}\,\nu$. Hence we may immediately conclude that $Y \in \mathrm{Spt}\,\nu$. Moreover it is clear that $\|\mathrm{P}_V^{\perp}Y\| \geq q$. If there was a $\theta > 0$ such that

$$\mathrm{B}(Y,\theta) \cap \mathrm{P}_V^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,a,b,V) = \emptyset$$

then we would find that for all i sufficiently large

$$\mathrm{B}(Y_i, \theta/2) \cap \mathrm{P}_{V_i}^{-1}(\mathrm{B}(0, 1+\Xi_i)) \cap \mathrm{X}(0, a+\Xi_i, b, V_i) = \emptyset$$

which is impossible and so claim (3) holds.

In order to verify claim (1) fix $\theta > 0$ and (interpreting $\mathrm{B}(0,\infty) = \mathbf{R}^n$) suppose that there is a $v \in \mathrm{int}[\mathrm{B}(0,L)] \cap L$ and an $r > 0$ such that

$$\mathrm{P}_V^{-1}[\mathrm{B}(v,r)] \cap \mathrm{Spt}\,\nu \cap \mathrm{X}(0, a+\theta, b, V) = \emptyset.$$

Let $v_i \in V_i$ be such that $v_i \to v$. We may find a $0 < \rho < r$ such that there are arbitrarily large i so that

$$\mathrm{P}_{V_i}^{-1}[\mathrm{B}(v_i, \rho)] \cap \mathrm{X}(0, a + \Xi_i, b, V_i)$$

is a subset of

$$\mathrm{P}_V^{-1}[\mathrm{B}(v, r)] \cap \mathrm{X}(0, a + \theta, b, V)$$

and

$$\mathrm{P}_{V_i}^{-1}[\mathrm{B}(v_i, \rho)] \cap \operatorname{Spt} \nu_i \cap \mathrm{X}(0, a + \Xi_i, b, V) \neq \emptyset$$

which in view of our earlier note enables us to deduce that

$$\mathrm{P}_V^{-1}[\mathrm{B}(v, r)] \cap \operatorname{Spt} \nu \cap \mathrm{X}(0, a + \theta, b, V) \neq \emptyset,$$

contradicting the choice of v and r. Hence, as $\operatorname{Spt} \nu$ is a closed set

$$V \cap \mathrm{B}(0, L) \subset \mathrm{P}_V[\operatorname{Spt} \nu \cap \mathrm{X}(0, a + \theta, b, V)]$$

and thus as θ was arbitrary

$$V \cap \mathrm{B}(0, L) \subset \mathrm{P}_V[\operatorname{Spt} \nu \cap \mathrm{X}(0, a, b, V)]$$

as required. ∎

4 Proof of Theorem

It suffices for us to prove the Theorem in the case that μ is a locally finite measure. We shall prove the Theorem by contradiction: Suppose that there is a purely m-unrectifiable, non-zero, locally finite, Borel measure μ which satisfies the hypotheses of Theorem 1.1.

Our first task is to find a set in $\mathbf{R}^n$ within which we shall work.

4.1 Decomposition of the support of μ

From Section 2.3 we know that for μ-a.e. x both

$$\operatorname{Tan}_m(\mu, x) \neq \emptyset$$

and if $\nu \in \operatorname{Tan}_m(\mu, x)$ and $\zeta \in \operatorname{Spt} \nu$ then $\nu_{\zeta,1} \in \operatorname{Tan}_m(\mu, x)$. From the hypotheses of Theorem 1.1 we know that for μ-a.e. x

$$\operatorname{Tan}_m(\mu, x) \subset \mathcal{M}_C^m.$$

Thus we can find a Borel set $B \subset \operatorname{Spt} \mu$ of positive and finite μ-measure such that for all $x \in B$

$$\emptyset \neq \operatorname{Tan}_m(\mu, x) \subset \mathcal{M}_C^m$$

and

$$\text{if } \nu \in \operatorname{Tan}_m(\mu, x) \text{ and } \zeta \in \operatorname{Spt} \nu \text{ then } \nu_{\zeta,1} \in \operatorname{Tan}_m(\mu, x).$$

By decomposing B into a set of measure zero and a countable number of (Borel) sets of the form $\{x \in B : p \leq \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) \leq q\}$ (where p and q are positive rationals) we may find a Borel set $B^{(0)} \subset B$ of positive μ-measure and $0 < l \leq u < \infty$ such that for all $x \in B^{(0)}$

$$2l \leq \underline{\mathrm{D}}_m(\mu, x) \leq \overline{\mathrm{D}}_m(\mu, x) \leq u/2.$$

By Lemma 2.1, for all Borel sets E we have

$$2l\mathcal{H}^m(E \cap B^{(0)}) \leq \mu(E \cap B^{(0)}) \leq 2^{m+1}u\mathcal{H}^m(E \cap B^{(0)})$$

and so if E is $\mathcal{H}^m$-rectifiable then $\mu(E \cap B^{(0)}) = 0$ which implies that $\mathcal{H}^m(E \cap B^{(0)}) = 0$. Thus $B^{(0)}$ is purely $(\mathcal{H}^m, m)$-unrectifiable and of positive and finite $\mathcal{H}^m$-measure. By the Projection Theorem we may conclude that for almost every $V \in \mathrm{G}(n,m)$

$$\mathcal{H}^m \left[\mathrm{P}_V B^{(0)} \right] = 0.$$

By applying Lemma 3.6 we can find a compact subset $B^{(1)}$ of $B^{(0)}$ of positive μ measure with the following property:

$$\left. \begin{array}{l} \text{If } x_i \in B^{(1)} \text{ and } \nu_i \in \mathrm{Tan}_m(\mu, x_i) \text{ are such that} \\ x_i \to x \in B^{(1)} \text{ and } \nu_i \to \nu \text{ then } \nu \in \mathrm{Tan}_m(\mu, x). \end{array} \right\} \tag{4.1}$$

Let $K := (5m)^{2m} u/l$ let $\xi := 3/82$ and $\gamma := 1/(100K)$ and define M to be the maximum number of balls of radius $5/4$ and with centres in the boundary of $\mathrm{B}(0,4)$ in $\mathbf{R}^m$ which may be packed disjointly. Then, by Lemma 3.7, there is an $R \geq 1$ so that for all $x \in B^{(1)}$, all $\nu \in \mathrm{Tan}_m(\mu, x)$, all $V \in \mathrm{G}(n,m)$ and all distinct points $\{\zeta^1, \ldots, \zeta^M\} \subset \mathrm{Spt}\,\nu$ which satisfy

$$\min_{i \neq j} \{ \|\mathrm{P}_V(\zeta^i - \zeta^j)\| \} \geq \gamma \max_{i,j} \{ \|\zeta^i - \zeta^j\| \}$$

we have that if $u \in \mathrm{conv}\,\{\mathrm{P}_V \zeta^1, \ldots, \mathrm{P}_V \zeta^M\}$ then there is a $\zeta \in \mathrm{Spt}\,\nu \cap \mathrm{B}(\zeta^1, R \max_{i,j} \{\|\zeta^i - \zeta^j\|\})$ with

$$\mathrm{P}_V \zeta \in \mathrm{B}(u, \xi \min_{i \neq j} \{ \|\mathrm{P}_V(\zeta^i - \zeta^j)\| \}). \tag{4.2}$$

We have now completed our initial decomposition of the support of μ. We now proceed to investigate the properties of $B^{(1)}$.

Fix $0 < \epsilon < [100KmR]^{-6m}$ and define $\alpha := 21KR\epsilon^{1/(3m)}$ and $\delta := (2/R)\epsilon^{1/(3m)}$.

4.2 Properties of $B^{(1)}$ dependent upon ϵ

If ν is a standardised tangent measure of μ at $x \in B^{(1)}$ then ν is not the zero measure and so Lemma 3.10 implies that ν is not purely m-unrectifiable. Thus, from Theorem 2.6, we deduce that for all $x \in B^{(1)}$, $\mathrm{Tan}_m(\mu, x) \cap \mathcal{G}(n,m) \neq \emptyset$ and so we may define

$$\chi := \inf \Big\{ \lambda : \text{There is a Borel set } C \subset B^{(1)} \text{ of positive } \mu\text{-measure so that}$$

$$\forall x \in C, \exists \nu \in \mathrm{Tan}_m(\mu, x) \cap \mathcal{G}(n,m) \text{ with } \mathrm{D}_m(\nu, 0) \leq \lambda \Big\}.$$

Thus

$$\mu \left\{ x \in B^{(1)} : \text{There is a } \nu \in \mathrm{Tan}_m(\mu, x) \cap \mathcal{G}(n,m) \text{ with } \mathrm{D}_m(\nu, 0) < \chi \right\} = 0$$

and so we may find a compact subset $B^{(2)}$ of $B^{(1)}$ which is of positive μ-measure such that for all $x \in B^{(2)}$, there is an $\omega \in \mathrm{Tan}_m(\mu, x) \cap \mathcal{G}(n,m)$ with

$$\chi \leq \mathrm{D}_m(\omega, 0) \leq \chi(1+\epsilon) \tag{4.3}$$

and if $\nu \in \mathrm{Tan}_m(\mu, x) \cap \mathcal{G}(n,m)$ then $\mathrm{D}_m(\nu, 0) \geq \chi$. Observe also that from Lemma 3.11 we have that if $x \in B^{(2)}$ and $\nu \in \mathrm{Tan}_m(\mu, x)$ then for ν-a.e. ζ

$$\overline{\mathrm{D}}_m(\nu, \zeta) \geq \chi. \tag{4.4}$$

We can find a Borel subset $B^{(3)}$ of $B^{(2)}$ of positive μ-measure and $0 < r' \leq 1$ such that for all $x \in B^{(3)}$ and all $0 \leq r \leq r'$

$$\alpha(m) l r^m \leq \mu \mathrm{B}(x, r) \leq \alpha(m) u r^m.$$

We can find a Borel subset $B^{(4)}$ of $B^{(3)}$ of positive μ-measure and $0 < r'' \le r'$ so that for all $x \in B^{(4)}$ and all $0 < r \le r''$ there is a $\nu \in \mathrm{Tan}_m(\mu,\, x)\,(\subset \mathcal{M}_C^m(l, u))$ so that

$$\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{x,r}}{r^m}, \nu\right) < \alpha(m) l \epsilon^{m(m+3)}. \tag{4.5}$$

Let $B^{(5)}$ be a compact subset of $B^{(4)}$ of positive μ-measure and recall that as $B^{(5)}$ is a subset of $B^{(0)}$ we have that for almost every $V \in \mathrm{G}(n, m)$, $\mathcal{H}^m[\mathrm{P}_V B^{(5)}] = 0$.

By a suitable translation of μ (and hence the corresponding $B^{(i)}$) we may suppose without loss of generality that $0 \in B^{(5)}$ and it is a density point of $B^{(5)}$. Thus we can find a $0 < r''' \le r''$ so that for all $0 < r < r'''$

$$\mu[\mathrm{B}(0, r)] > (1 - (\epsilon^m/4))\mu \mathrm{B}(0, r). \tag{4.6}$$

By (4.3) we can find an $\omega_0 \in \mathrm{Tan}_m(\mu, 0) \cap \mathcal{G}(n, m)$ such that

$$\chi \le \mathrm{D}_m(\omega_0, 0) \le \chi(1 + \epsilon).$$

Thus as for almost every $V \in \mathrm{G}(n, m)$, $\mathcal{H}^m[\mathrm{P}_V B^{(5)}] = 0$ and for infinitely many $0 < \rho \le r'''$

$$\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{0,\rho}}{\rho^m},\, \omega_0\right) < \alpha(m) l \epsilon^{m(m+3)}/2$$

we can find a $\nu_0 \in \mathcal{G}(n, m)$ with $\mathrm{Spt}\,\nu_0 = V_0$, say, such that

$$\mathcal{H}^m[\mathrm{P}_{V_0} B^{(5)}] = 0,$$

$$\chi \le \mathrm{D}_m(\nu_0, 0) \le \chi(1 + \epsilon)$$

and

$$\mathrm{F}_{R+3+\epsilon^{-1}}(\omega_0,\, \nu_0) < \alpha(m) l \epsilon^{m(m+3)}/2.$$

Thus, in view of the density estimates for μ at 0, there is an $r_0 \le r'''$ such that

$$\mu\partial\mathrm{B}(0, r_0) = 0 \text{ and } \mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{0,r_0}}{r_0^m},\, \nu_0\right) < \alpha(m) l \epsilon^{m(m+3)}$$

Hence we have, in summary:

$$\left.\begin{aligned} \mathcal{H}^m[\mathrm{P}_{V_0} B^{(5)}] &= 0, \\ \mu\partial\mathrm{B}(0, r_0) &= 0, \\ \chi \le \mathrm{D}_m(\nu_0, 0) &\le \chi(1 + \epsilon) \text{ and} \\ \mathrm{F}_{R+3+\epsilon^{-1}}\left(\tfrac{\mu_{0,r_0}}{r_0^m},\, \nu_0\right) &< \alpha(m) l \epsilon^{m(m+3)} \end{aligned}\right\} \tag{4.7}$$

and if we set $F := B^{(5)} \cap \mathrm{B}(0, r_0)$ then F is compact and, by (4.6), for $0 \le r \le r_0$

$$\mu[F \cap \mathrm{B}(0, r)] > (1 - (\epsilon^m/4))\mu \mathrm{B}(0, r). \tag{4.8}$$

Henceforth let P denote orthogonal projection onto V_0. Let

$$r_1 := (1 - \alpha) r_0, \qquad L := l r_0^m / \mu \mathrm{B}(0, r_0),$$

$$\Lambda_1 := \alpha(m) 2^{-1} 5^{-m} m^{-m/2} L K \qquad \text{and} \qquad \Lambda_2 := \alpha(m) 3^m 2^{-3m-1} L \delta^m.$$

For $u \in V_0$ and $s \ge 0$ define

$$\mathrm{S}(u, s) := \{y \in \mathbf{R}^n : \|P(y) - u\| \le s\} \cap B(0, r_0)$$

and

$$S^o(u,s) := \{y \in \mathbf{R}^n : \|P(y) - u\| < s\} \cap B(0, r_0).$$

We now define a real-valued function on points of $V_0 \cap \mathrm{B}(0, r_1)$:

For $u \in V_0 \cap \mathrm{B}(0, r_1)$ define $s(u)$ to be the least (non-negative) number such that if

$$s(u) < s \leq r_0 - \|u\|$$

then both

[I] for all $v \in V_0 \cap \mathrm{B}(u, (1-\delta)s)$,

$$F \cap \mathrm{S}(v, \delta s) \neq \emptyset$$

and

[II] there is a $W \in \mathrm{G}(n,m)$ and $t \in \mathbf{R}^n$ such that

$$F \cap \mathrm{S}(u,s) \subset \mathrm{B}(t + W, \delta s)$$

and if $x, y \in W$ then

$$K\|\mathrm{P}x - \mathrm{P}y\| \geq \|x - y\|.$$

Let

$$A := \{u \in V_0 \cap \mathrm{B}(0, r_1) : s(u) = 0\},$$

$$A_1 := \left\{ u \in [V_0 \cap \mathrm{B}(0, r_1)] \setminus A : \mu \mathrm{S}^o(u, s(u)) \geq \Lambda_1 \left[\frac{s(u)}{r_0}\right]^m \mu \mathrm{B}(0, r_0) \right\},$$

$$A_2 := \left\{ u \in [V_0 \cap \mathrm{B}(0, r_1)] \setminus A : \mu [\mathrm{S}(u, s(u)) \setminus F] \geq \Lambda_2 \left[\frac{s(u)}{r_0}\right]^m \mu \mathrm{B}(0, r_0) \right\}$$

and let

$$A_3 := \Big\{ u \in [V_0 \cap \mathrm{B}(0, r_1)] \setminus [A \cup A_1 \cup A_2] : \text{[I] holds for } s(u) \text{ at } u \text{ and}$$

$$\delta s(u) \leq \operatorname{diam}[\mathrm{P}^{\perp}[F \cap \mathrm{S}(u, s(u))]] \leq 2K(1+\delta)s(u) \Big\}.$$

As F is compact, if $u \in A$ then $\mathrm{P}^{-1}(u) \cap F \neq \emptyset$ and hence (from (4.7)) we conclude that $\mathcal{H}^m A = 0$.

The function $s(u)$ provides us with a tool to investigate the properties of the set F. Our next task is to establish some of the elementary properties of $s(u)$ and the sets associated with it. We shall say that a positive real number s is good for a point $u \in V_0 \cap \mathrm{B}(0, r_1)$ if it satisfies both [I] and [II]. It is bad if at least one of [I] and [II] fails to hold!

We should first verify that $s(u)$ is well defined:

Lemma 4.1 *For all $u \in V_0 \cap \mathrm{B}(0, r_1)$, $s(u) \leq \epsilon r_0/\delta$.*

Proof: Fix $u \in V_0 \cap \mathrm{B}(0, r_1)$ and $\epsilon r_0/\delta \leq s \leq r_0 - \|u\|$. As $r_0 - \|u\| \geq \alpha r_0$ and $\epsilon/\delta < \alpha$ this is a non-trivial interval. As $\nu_0 \in \mathcal{M}_C^m(\chi, \chi(1+\epsilon)) \cap \mathcal{G}(n,m)$ has $\operatorname{Spt} \nu_0 = V_0$ and $\mathrm{F}_{R+3}(\mu_{0,r_0}/r_0, \nu_0) < \alpha(m) l \epsilon^{m+3}$ then for $v \in V_0 \cap \mathrm{B}(u, (1-\delta)s)$ we have by Lemma 3.8, since $\delta s \in [\epsilon r_0, r_0]$, that

$$\begin{aligned} \mu \mathrm{B}(v, \delta s) \geq \alpha(m)\chi(1 - 3m\epsilon^m)(\delta s)^m &\geq \alpha(m)\chi(1 - 3m\epsilon^m)(\epsilon r_0)^m \\ &\geq \left[\frac{1 - 3m\epsilon^m}{1 + 3m\epsilon^m}\right](1+\epsilon)^{-1}\epsilon^m \mu \mathrm{B}(0, r_0) \\ &\geq (\epsilon^m/4)\mu \mathrm{B}(0, r_0) > \mu[\mathrm{B}(0, r_0) \setminus F]. \end{aligned}$$

Thus $\mathrm{S}(v, \delta s) \cap F \neq \emptyset$ and so [I] is satisfied. I claim that [II] holds with $W = V_0$ and $t = 0$. For suppose there is an $x \in [F \setminus \mathrm{B}(V_0, \delta s)] \cap \mathrm{S}(u, s)$ and consider $\mathrm{B}(x, \delta s)$ (which is disjoint from V_0). Let $\zeta = x/r_0$ and $\rho = \delta s/r_0$. By the definition of ϵ^m we know that $\delta s \in [\epsilon^m r_0, r_0]$. Moreover as $\|x\| \leq r_0$ and $\delta s \leq \delta r_0$ then $\mathrm{B}(\zeta, \rho) \subset \mathrm{B}(0, R+3+\epsilon^{-1})$ and so by Lemma 2.3 (with $E = \mathrm{B}(\zeta, \rho(1-\epsilon^m))$ and $\tau = \epsilon^m \rho$)

$$
\begin{aligned}
\frac{1}{\epsilon^m \rho} \mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{0,r_0}}{r_0^m}, \nu_0\right) &\geq \frac{1}{r_0^m} \mu_{0,r_0} \mathrm{B}(\zeta, (1-\epsilon^m)\rho) - \nu_0 \mathrm{B}(\zeta, \rho) \\
&= \frac{1}{r_0^m} \mu \mathrm{B}(x, (1-\epsilon^m)\delta s) \\
&\geq \alpha(m) l (1-2m\epsilon^m)(\delta s/r_0)^m \quad \geq \quad \alpha(m) l (1-2m\epsilon^m)\epsilon^{m^2}
\end{aligned}
$$

and so

$$
\begin{aligned}
\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{0,r_0}}{r_0^m}, \nu_0\right) &\geq \alpha(m) l \epsilon^{m(m+1)}(1-2m\epsilon^m)\rho \\
&\geq \alpha(m) l \epsilon^{m(m+2)}(1-2m\epsilon^m) \\
&> \alpha(m) l \epsilon^{m(m+3)} \text{ — a contradiction.}
\end{aligned}
$$

Hence [II] holds. ∎

Lemma 4.2 *For all $u \in V_0 \cap \mathrm{B}(0, r_1)$, if $s(u) > 0$ then it is good for u.*

Proof: This is just an exercise in using the compactness of F. If $s \in (s(u), 2s(u))$ it is good for u. Hence for all v in $V_0 \cap \mathrm{B}(u, (1-\delta)s(u))$, $F \cap \mathrm{S}(v, \delta s)$ is a compact non-empty set . Intersecting these compact sets over s gives that $F \cap \mathrm{S}(v, \delta s(u))$ is not empty and so [I] holds.

[II] follows in a similar manner using the compactness of $\mathrm{G}(n, m)$. ∎

Lemma 4.3 *For all $u \in V_0 \cap \mathrm{B}(0, r_1)$ if $s(u) > 0$ then there is an $x_u \in \mathrm{S}(u, 0)$ such that*

$$F \cap \mathrm{S}(u, r_0 - \|u\|) \subset \mathrm{X}(x_u, s(u)/2, 2K(1+2\delta)s(u), V_0).$$

Also

$$\operatorname{diam}\left(\mathrm{P}^{\perp}[F \cap \mathrm{S}(u, s(u))]\right) \leq 2K(1+\delta)s(u).$$

Proof: The second part of the lemma follows from noticing that, as $s(u)$ is good for u (Lemma 4.2), there is an $x_u \in \mathrm{S}(u, 0)$ and $W \in \mathrm{G}(n, m)$ such that

$$F \cap \mathrm{S}(u, s(u)) \subset \mathrm{B}(x_u + W, \delta s(u))$$

and if $x, y \in W$ then

$$K\|\mathrm{P}(x-y)\| \geq \|x-y\|.$$

Thus if $\zeta \in F \cap \mathrm{S}(u, s(u))$ then

$$\|\mathrm{P}^{\perp}(\zeta - x_u)\| \leq K(1+\delta)s(u).$$

For the main statement suppose that x_u is as defined above and fix $\zeta \in F \cap \mathrm{S}(u, r_0 - \|u\|)$. If $\zeta \in F \cap \mathrm{S}(u, s(u))$ then by the above it follows that it is in $\mathrm{X}(x_u, s(u)/2, 2K(1+2\delta), V_0)$ as required. So suppose $\zeta \in F \cap [\mathrm{S}(u, r_0 - \|u\|) \setminus \mathrm{S}(u, s(u))]$. As $s(u)$ is good for u we can find $X \in F \cap \mathrm{S}(u, \delta s(u))$ and

$$\|x_u - X\| \leq K\delta s(u) + K\delta s(u).$$

Now consider $\|\mathrm{P}(\zeta) - u\|$ which is good for u and so we can find $W \in \mathrm{G}(n, m)$ and $y_\zeta \in \mathrm{S}(u, 0)$ such that

$$F \cap \mathrm{S}(u, \|\mathrm{P}(\zeta) - u\|) \subset \mathrm{B}(y_\zeta + W, \delta\|\mathrm{P}(\zeta) - u\|)$$

and for all $x, y \in W$

$$K\|\mathrm{P}(x-y)\| \geq \|x-y\|.$$

In particular both X and ζ are in $\mathrm{B}(y_\zeta + W, \delta\|\mathrm{P}(\zeta) - u\|)$. Hence

$$\begin{aligned} \|X - \zeta\| &\leq K[\|\mathrm{P}(X - y_\zeta)\| + \|\mathrm{P}(\zeta - y_\zeta)\|] + 2K\delta\|\mathrm{P}(\zeta) - u\| \\ &\leq K\delta s(u) + K(1+2\delta)\|\mathrm{P}(\zeta) - u\| \end{aligned}$$

and so as $\delta \leq 1/4$

$$\begin{aligned} \|x_u - \zeta\| &\leq K(\delta + 2\delta)s(u) + K(1+2\delta)\|\mathrm{P}(\zeta - x_u)\| \\ &\leq K(1+2\delta)s(u)/2 + K(1+2\delta)\|\mathrm{P}(\zeta - x_u)\| \end{aligned}$$

as required. ∎

Lemma 4.4 $\mathcal{H}^m(A_2) \leq 4^{m-1}\alpha(m)\epsilon^m \Lambda_2^{-1} r_0^m$.

Proof: Let $D_2 \subset A_2$ be a countable set such that $\{\mathrm{B}(u, s(u)) : u \in D_2\}$ is disjoint and $\{\mathrm{B}(u, 4s(u)) : u \in D_2\}$ covers A_2 [Fed69, 2.8.4]. Then as $D_2 \subset A_2$

$$\mu[\mathrm{B}(0, r_0) \setminus F] \geq \sum_{u \in D_2} \mu[\mathrm{S}(u, s(u)) \setminus F] \geq \Lambda_2 \mu[\mathrm{B}(0, r_0)] r_0^{-m} \sum_{D_2} [s(u)]^m.$$

Hence

$$\sum_{u \in D_2} [s(u)]^m \leq \Lambda_2^{-1} r_0^m \frac{\mu[\mathrm{B}(0, r_0) \setminus F]}{\mu \mathrm{B}(0, r_0)} \leq (\epsilon^m/4) \Lambda_2^{-1} r_0^m.$$

Thus

$$\mathcal{H}^m(A_2) \leq \alpha(m) \sum_{D_2} [4s(u)]^m \leq \alpha(m) 4^{m-1} \epsilon^m \Lambda_2^{-1} r_0^m$$

as required. ∎

Lemma 4.5

$$V_0 \cap \mathrm{B}(0, r_1) \subset A \cup A_1 \cup A_2 \cup A_3.$$

Proof: Suppose that $u \in [V_0 \cap \mathrm{B}(0, r_1)] \setminus [A \cup A_1 \cup A_2]$ and so $s(u) > 0$.

By the definition of $s(u)$ it is possible to find an $s \in ((3/4)s(u), s(u))$ such that it is bad for u— that is either [I] or [II] fails for s at u. Thus there are two cases to consider for s:

(1) Either [I] fails for s at u or

(2) [I] holds for s at u but [II] fails to hold.

Case 1: [I] fails.

In this situation there is a $v \in V_0 \cap \mathrm{B}(u, (1-\delta)s)$ such that $\mathrm{S}(v, \delta s) \cap F = \emptyset$. In this case let $t = (4/3)s$ and so $s(u) \leq t < \alpha r_0$ and thus t is good for u.

Consider

$$\mathcal{C} := \{\mathrm{B}(w, 5\delta t/4) : x \in V_0 \text{ and } \|x - v\| = 4\delta t\}$$

and let $\mathcal{B} = \{\mathrm{B}(v_i, 5\delta t/4)\}_{i=1}^M$ be a maximal disjoint subfamily of $\mathcal{C}$ (recall that M was defined so that (4.2) holds.) Since $\delta \leq 1/17$ then for all $i = 1, \ldots M$

$$\|u - v\| + \|v - v_i\| \leq (1-\delta)s + 4\delta t \leq (1-\delta)t$$

and since t is good for u we conclude that for all i

$$\mathrm{S}(v_i, \delta t) \cap F \neq \emptyset.$$

So for $i = 1, \ldots, M$ choose $x_i \in \mathrm{S}(v_i, \delta t) \cap F$. Then if $i \neq j$ we have

$$\delta t/2 \leq \|\mathrm{P}(x_i - x_j)\| \leq 10\delta t.$$

Since t is good for u there is a $Y \in \mathrm{S}(u, 0)$ and a $W \in \mathrm{G}(n, m)$ such that

$$F \cap \mathrm{S}(u, t) \subset \mathrm{B}(Y + W, \delta t)$$

and for $x, y \in W$

$$K\|\mathrm{P}(x - y)\| \geq \|x - y\|.$$

Hence for all i and j

$$\begin{aligned} \|x_i - x_j\| &\leq 2K\delta t + K\|\mathrm{P}(x_i - x_j)\| \\ &\leq 2K[\delta + 5\delta]t \;=:\; \rho, \text{ say.} \end{aligned}$$

As $\rho \leq r_0 \leq r''$ then, by (4.5), there is a $\nu \in \mathcal{M}_C^m(l, u)$ so that

$$\mathrm{F}_{R+3+\epsilon^{-1}}(\mu_{x_1,\rho}/\rho^m,\, \nu) < \alpha(m) l \epsilon^{m(m+3)}$$

and so, by Lemma 3.5, we can find for each i a $z_i \in [x_1 + \rho \mathrm{Spt}\, \nu]$ with

$$\|x_i - z_i\| \leq \rho \epsilon^m.$$

As $\rho\epsilon^m \leq \delta t/8$ we conclude that

$$\mathrm{P}(z_i) \in \mathrm{B}(v_i, 9\delta t/8)$$

and so if $i \neq j$ then

$$\delta t/4 \leq \|\mathrm{P}(z_i - z_j)\| \leq 10\delta t + 2\epsilon^m \rho \leq (41/4)\delta t.$$

In addition

$$\|z_i - z_j\| \leq \rho(1 + 2\epsilon^m).$$

Thus

$$\begin{aligned} \min_{i \neq j}\{\|\mathrm{P}(z_i - z_j)\|\} &\geq (\delta t/4)/[\rho(1 + 2\epsilon^m)] \max_{i,j}\{\|z_i - z_j\|\} \\ &= \delta\,(8K[\delta + 5\delta][1 + 2\epsilon^m])^{-1} \max_{i,j}\{\|z_i - z_j\|\} \quad \geq \quad \gamma \max_{i,j}\{\|z_i - z_j\|\}. \end{aligned}$$

I claim that v is in the convex hull of $\{\mathrm{P}z_1, \ldots, \mathrm{P}z_M\}$: For if not then there is a unit vector $e \in V_0$ such that if

$$H^- := \{y \in V_0 : \langle y,\, e\rangle < 0\}$$

then

$$v + H^- \supset \mathrm{conv}\,\{\mathrm{P}z_1, \ldots, \mathrm{P}z_M\}.$$

But consider the point $v + 4\delta t e$: If $z \in v + H^-$ then $z = v + \zeta$ for some $\zeta \in H^-$. Thus, as $\zeta \in H^-$,

$$\|z - (v + 4\delta t e)\| = \|\zeta - 4\delta t e\| \geq |\langle \zeta - 4\delta t e,\, e\rangle| > \quad 4\delta t.$$

Hence $\mathrm{B}(v + 4\delta t e, 5\delta t/4) \cap \mathrm{B}(v + H^-, 5\delta t/2) = \emptyset$ and so $\mathrm{B}(v + 4\delta t e, 5\delta t/4) \in \mathcal{C}$ which contradicts the maximality of $\mathcal{B}$ and thus the claim holds.

Hence $\rho^{-1}(v - x_1) \in \mathrm{conv}\,\{\rho^{-1}(\mathrm{P}z_1 - x_1), \ldots, \rho^{-1}(\mathrm{P}z_M - x_1)\}$ and for all i, $\rho^{-1}(z_i - x_1) \in \mathrm{Spt}\,\nu$. Thus, from (4.2), we can find a

$$z \in (\mathrm{Spt}\,\nu) \cap \mathrm{B}(\rho^{-1}(z_1 - x_1), R\rho^{-1} \max\|z_i - z_j\|) \subset \mathrm{Spt}\,\nu \cap \mathrm{B}(\rho^{-1}(z_1 - x_1), R(1 + 2\epsilon^m))$$

such that

$$\mathrm{P}z \in \mathrm{B}(\rho^{-1}(v - \mathrm{P}x_1), \xi\rho^{-1} \min_{i \neq j} \|z_i - z_j\|).$$

Hence on setting $\zeta = x_1 + \rho z$ we conclude that

$$\zeta \in \mathrm{B}(z_1, R\rho(1 + 2\epsilon^m)) \cap [x_1 + \rho \mathrm{Spt}\, \nu]$$

and

$$\mathrm{P}\zeta \in \mathrm{B}(v, \xi(41/4)\delta t).$$

Thus as

$$(41/4)\xi\delta t \leq \delta s/2$$

and

$$\max_i\{\|x_i\|\} + \rho\epsilon^m + R\rho(1 + 2\epsilon^m) + \delta s/2 \leq r_0$$

we have that

$$\mathrm{B}(\zeta, \delta s/2) \subset \mathrm{S}(v, \delta s)\, (\subset [\mathrm{S}(u, s) \setminus F])\,.$$

Hence

$$\mu[\mathrm{S}(u,s) \setminus F] \quad \geq \quad \mu\mathrm{B}(\zeta, \delta s/2)$$

and as $\delta s/2 \in [\epsilon^m \rho, \rho]$, we can apply Lemma 3.8 to conclude that

$$\begin{aligned} &\geq \alpha(m) l (1 - 3m\epsilon^m)(\delta s/2)^m \\ &= \alpha(m) 2^{-m} (1 - 3m\epsilon^m) L \delta^m (s/r_0)^m \mu\mathrm{B}(0, r_0) \end{aligned}$$

and so as $s(u) \geq s > 3s(u)/4$

$$\begin{aligned} \mu[\mathrm{S}(u, s(u)) \setminus F] &\geq \mu[\mathrm{S}(u,s) \setminus F] \\ &\geq \alpha(m) 2^{-m} (1 - 3m\epsilon^m) L \delta^m (s/r_0)^m \mu\mathrm{B}(0, r_0) \\ &\geq \alpha(m) 3^m 2^{-3m} L (1 - 3m\epsilon^m) \delta^m \left[\frac{s(u)}{r_0}\right]^m \mu\mathrm{B}(0, r_0). \end{aligned}$$

But this implies that $u \in A_2$ which is impossible and so Case (1) cannot occur.
Case 2:[I] holds but [II] fails.
Hence either

(i) there is a $W \in \mathrm{G}(n,m)$ and $x_u \in \mathrm{S}(u, 0)$ such that

$$F \cap \mathrm{S}(u, s) \subset \mathrm{B}(x_u + W, \delta s)$$

and there are $x, y \in W$ with

$$K\|\mathrm{P}x - \mathrm{P}y\| < \|x - y\|.$$

Or

(ii) for all $W \in \mathrm{G}(n,m)$ and all $x_u \in \mathrm{S}(u, 0)$

$$[F \cap \mathrm{S}(u,s)] \setminus \mathrm{B}(x_u + W, \delta s) \neq \emptyset.$$

So suppose we have case *(i)* for some W and x_u. Then [Fed69, 1.7.3] enables us to find an orthonormal basis for W, $\{e_1, \ldots, e_m\}$, such that if $i \neq j$ then

$$\langle \mathrm{P}e_i, \mathrm{P}e_j \rangle = 0.$$

First observe that [I] holding implies that $\{\mathrm{P}e_1, \ldots, \mathrm{P}e_m\}$ is an orthogonal basis for V_0 and so, in particular, $\|\mathrm{P}e_i\| \neq 0$ for all i.

Now observe that there is an i such that $\|\mathrm{P}e_i\| < 1/K$: For if $\|\mathrm{P}e_i\| \geq 1/K$ for all i and $x, y \in W$ are such that

$$K\|\mathrm{P}(x-y)\| < \|x-y\|$$

then

$$\begin{aligned} \|\mathrm{P}(x-y)\|^2 &= \langle \mathrm{P}(x-y), \mathrm{P}(x-y) \rangle \\ &= \sum_i \langle x-y, e_i \rangle^2 \|\mathrm{P}e_i\|^2 \\ &\geq K^{-2} \sum_i \langle x-y, e_i \rangle^2 = (\|x-y\|/K)^2 \end{aligned}$$

which contradicts the choice of x and y. So we may suppose without loss of generality that $\|\mathrm{P}e_1\| < 1/K$.

Consider open cuboids in V_0 with sides parallel to $\mathrm{P}e_1, \ldots, \mathrm{P}e_m$ and with sidelength equal to $4s/(5Km)$ in the $\mathrm{P}e_1$ direction and s/m in all the others. Let $\mathcal{C}$ be a maximal disjoint family of such cuboids contained in $\mathrm{PB}(u, (5m-1)s/(5m))$. Then

$$\operatorname{card}(\mathcal{C}) \geq \left\lfloor \frac{2^m(5m-1)K}{4\sqrt{m}} \left(\frac{5m-1}{5\sqrt{m}} \right)^{m-1} \right\rfloor \geq 2^{m-3} K m^{-m/2} (5m-1)^m 5^{1-m}.$$

Suppose that $C \in \mathcal{C}$ and c is the centre of C then there is an $x_C \in F \cap \mathrm{S}(u, s)$ such that

$$\|\mathrm{P}x_C - c\| \leq \delta s$$

and so, as $\delta < 2/(5Km)$, $\mathrm{P}x_C \in C$. Consider the family of balls in $\mathbf{R}^n$ given by

$$\mathcal{B} = \{\mathrm{B}(x_C, (5m)^{-1}s) : C \in \mathcal{C}\}.$$

I claim that this is a disjoint family. In order to verify this we need to show that if x, x' are distinct centres of balls in $\mathcal{B}$ then $\|x - x'\| > 2s/(5m)$. So suppose that x and x' are two such distinct centres and let c and c' be the centres of the corresponding cuboids in $\mathcal{C}$. Notice that c and c' are also distinct.

Since $F \cap \mathrm{S}(u, s) \subset \mathrm{B}(x_u + W, \delta s)$ we can find X and X' in $x_u + W$ such that

$$\|x - x'\| \geq \|X - X'\| \geq \|x - x'\| - 2\delta s$$

and

$$\max\{\|\mathrm{P}(X - x)\|, \|\mathrm{P}(X' - x')\|\} \leq \delta s.$$

Hence

$$\max\{\|\mathrm{P}X - c\|, \|\mathrm{P}X' - c'\|\} \leq (\delta + \delta)s < 2s/(5Km)$$

and thus $\mathrm{P}X$ and $\mathrm{P}X'$ lie in different cuboids.

As c and c' are the centres of distinct cuboids in $\mathcal{C}$ there is an i such that

$$|\langle c - c', \mathrm{P}e_i/\|\mathrm{P}e_i\| \rangle| > 0.$$

If $i \geq 2$ then

$$|\langle c - c', \mathrm{P}e_i/\|\mathrm{P}e_i\|\rangle| \geq s/m$$

and hence

$$\|\mathrm{P}(x - x')\| \geq (m^{-1} - 2\delta)s.$$

Thus

$$\|x - x'\| \geq (m^{-1} - 2\delta)s > 2s/(5m)$$

and we are done.

If $i = 1$ then

$$|\langle c - c', \mathrm{P}e_1/\|\mathrm{P}e_1\|\rangle| \geq 4s/(5Km)$$

and so

$$|\langle \mathrm{P}(X - X'), \mathrm{P}e_1/\|\mathrm{P}e_1\|\rangle| \geq [4(5Km)^{-1} - 2(\delta + \delta)]s$$

but

$$\begin{aligned} |\langle \mathrm{P}(X - X'), \mathrm{P}e_1/\|\mathrm{P}e_1\|\rangle| &= |\langle X - X', e_1\rangle|\|\mathrm{P}e_1\| \\ &< |\langle X - X', e_1\rangle|/K \end{aligned}$$

and thus

$$\begin{aligned} \|X - X'\| \geq |\langle X - X', e_1\rangle| &\geq K[4(5Km)^{-1} - 2(\delta + \delta)]s \\ &> 2s/(5m) \end{aligned}$$

hence

$$\|x - x'\| > 2s/(5m)$$

as required. Thus $\mathcal{B}$ is a disjoint collection of balls with centres in F and as for all $C \in \mathcal{C}$

$$\|x_C\| + s/(5m) \leq r_0$$

we can conclude that all balls in $\mathcal{B}$ are contained in $\mathrm{S}(u, s)$. Thus

$$\begin{aligned} \mu\mathrm{S}(u, s) \geq \sum_{B \in \mathcal{B}} \mu(B) &\geq \alpha(m)l(s/(5m))^m \operatorname{card}(\mathcal{B}) \\ &\geq \alpha(m)l(5m)^{-m}2^{m-3}Km^{-m/2}(5m - 1)^m 5^{1-m}s^m \\ &\geq \alpha(m)(2/5)^{m-1}m^{-m/2}LK(s/r_0)^m \mu\mathrm{B}(0, r_0) \end{aligned}$$

and so

$$\mu\mathrm{S}^o(u, s(u)) \geq 2^{-1}\alpha(m)5^{-m}m^{-m/2}LK\left[\frac{s(u)}{r_0}\right]^m \mu\mathrm{B}(0, r_0)$$

which implies that $u \in A_1$ which is impossible. Hence *(ii)* must hold. So for $3s(u)/4 \leq s < s(u)$ we have that [I] holds for s at u and for all $W \in \mathrm{G}(n, m)$ and all $x_u \in \mathrm{S}(u, 0)$

$$[F \cap \mathrm{S}(u, s)] \setminus \mathrm{B}(x_u + W, \delta s) \neq \emptyset.$$

Hence, in particular, for all $x_u \in \mathrm{S}(u, 0)$

$$[F \cap \mathrm{S}(u, s)] \setminus \mathrm{B}(x_u + V_0, \delta s) \neq \emptyset$$

and so there are an x_s and y_s in $\mathrm{S}(u, s) \cap F$ such that

$$\|\mathrm{P}^{\perp}(x_s - y_s)\| \geq \delta s.$$

Thus, as $F \cap \mathrm{S}(u, s(u))$ is compact and contains $F \cap \mathrm{S}(u, s)$ for $s < s(u)$, we conclude that there exist x and y in $F \cap \mathrm{S}(u, s(u))$ with

$$\|\mathrm{P}^{\perp}(x - y)\| \geq \delta s.$$

Hence, as $s(u)$ is good for u (Lemma 4.2), we may use Lemma 4.3 to deduce that

$$\mathrm{diam}\left[\mathrm{P}^{\perp}\left(F \cap \mathrm{S}(u, s(u))\right)\right] \leq 2K(1+\delta)s(u)$$

and so $u \in A_3$ as required. ■

Lemma 4.6 *Let*

$$\eta := (1 + 3m\epsilon^m)\left[1 - 4^{m-1}\epsilon^m \Lambda_2^{-1}\right]^{-1} - 1$$

and suppose that

$$1 \leq T \leq \epsilon^{(2/(3m))-1}.$$

Then there is a $u \in A_3$ and a Borel set J contained in $V_0 \cap \mathrm{B}(0, r_0)$ such that

$$V_0 \cap \mathrm{B}(u, Ts(u)) \subset J \subset V_0 \cap \mathrm{B}(u, 4Ts(u)),$$

$$\mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] \leq \chi(1+\eta)\mathcal{H}^m(J)$$

and if $0 < \theta < [2/(3m)]^{m+1}$ then

$$\mathcal{H}^m\left[\mathrm{B}(\partial_{V_0} J, \theta s(u)) \cap V_0\right] \leq c(m)\theta^{1/(m+1)}[Ts(u)]^m$$

(where $c(m)$ is the constant from Proposition 3.4 .)

Proof: Consider

$$\mathcal{C} := \{\mathrm{PB}(u, s(u)) : u \in A_1\} \cup \{\mathrm{PB}(u, Ts(u)) : u \in A_3\}$$

which is a cover of $A_1 \cup A_3$. As $4Ts(u) \leq \alpha r_0$ it follows that for all $u \in A_1 \cup A_3$

$$\mathrm{PB}(u, 4Ts(u)) \subset V_0 \cap \mathrm{B}(0, r_0).$$

By Proposition 3.4 we can find a disjoint subcollection, $\mathcal{J}$ of Borel sets contained in V_0, which may be written as a disjoint union, $\mathcal{J}_1 \cup \mathcal{J}_3$, such that

(a) $A_1 \subset \cup\mathcal{J}_1$ and $A_3 \subset \cup\mathcal{J}_3$,

(b) for all $J \in \mathcal{J}_1$ there is a $u \in A_1$ such that

$$\mathrm{B}(u, s(u)) \cap V_0 \subset J \subset \mathrm{B}(u, 4s(u)) \cap V_0$$

and for all $J \in \mathcal{J}_3$ there is a $u \in A_3$ such that

$$\mathrm{B}(u, Ts(u)) \cap V_0 \subset J \subset \mathrm{B}(u, 4Ts(u)) \cap V_0,$$

(c) for all $J \in \mathcal{J}_1$ if $0 < \theta < [2/(3m)]^{m+1}$ and if u is as determined in (b) then

$$\mathcal{H}^m\left[\mathrm{B}(\partial_{V_0} J, \theta s(u)) \cap V_0\right] \leq c(m)\theta^{1/(m+1)}[s(u)]^m.$$

and for all $J \in \mathcal{J}_3$ if $0 < \theta < [2/(3m)]^{m+1}$ and if u is as determined in (b) then

$$\mathcal{H}^m\left[\mathrm{B}(\partial_{V_0} J, \theta Ts(u)) \cap V_0\right] \leq c(m)\theta^{1/(m+1)}[Ts(u)]^m.$$

Thus $\mathcal{H}^m(\cup\mathcal{J}) \geq \mathcal{H}^m(A_1 \cup A_3)$. If for all $J \in \mathcal{J}$, $\mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] > \chi(1+\eta)\mathcal{H}^m J$ then as

$$\sum \mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] \leq \mu\mathrm{B}(0, r_0) \leq \alpha(m)\chi(1+3m\epsilon^m)r_0^m$$

we find that

$$\begin{aligned} \alpha(m)\chi(1+3m\epsilon^m)r_0^m &> \sum \chi(1+\eta)\mathcal{H}^m J \\ &\geq \chi(1+\eta)\mathcal{H}^m(A_1 \cup A_3) \end{aligned}$$

which, by our estimate for the size of A_2 (Lemma 4.4), is

$$\begin{aligned} &\geq \alpha(m)\chi(1+\eta)r_0^m[1 - 4^{m-1}\epsilon^m\Lambda_2^{-1}] \\ &= \alpha(m)\chi(1+3m\epsilon^m)r_0^m \text{ — a contradiction.} \end{aligned}$$

Hence there is a $J \in \mathcal{J}$ such that

$$\mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] \leq \chi(1+\eta)\mathcal{H}^m(J).$$

If $J \in \mathcal{J}_1$ and u is the associated point of A_1 then we find that

$$\begin{aligned} \alpha(m)\chi(1+\eta)[4s(u)]^m \geq \chi(1+\eta)\mathcal{H}^m(J) &\geq \mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] \\ &\geq \mu\mathrm{S}^0(u, s(u)) \end{aligned}$$

since $u \in A_1$

$$\begin{aligned} &\geq \Lambda_2[s(u)/r_0]^m \mu\mathrm{B}(0, r_0) \\ &\geq \alpha(m)\chi(1-3m\epsilon^m)\Lambda_2 s(u)^m \end{aligned}$$

but then

$$(1+\eta)4^m \geq (1-3m\epsilon^m)\Lambda_1$$

which is impossible and so $J \in \mathcal{J}_3$ which implies the Lemma. ■

Lemma 4.7 *Suppose that $1 \leq T \leq \epsilon^{(2/(3m))-1}$. Then there are a $u \in A_3$, $X \in F \cap \mathrm{S}(u, \delta s(u))$, $\nu \in \mathrm{Tan}_m(\mu, X)\,(\subset \mathcal{M}_C^m(l, u))$ and a closed set $I \subset V_0$ such that:*

(1) $\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{X,s(u)}}{s(u)^m}, \nu\right) < \alpha(m)l\epsilon^{m(m+3)}$,

(2) there is a

$$Y \in \mathrm{Spt}\,\nu \cap \mathrm{P}^{-1}\left[\mathrm{B}(0, 1+\delta+\epsilon^m)\right] \cap \mathrm{X}(0, 2+\epsilon^m, 2K(1+2\delta), V_0)$$

with

$$\|\mathrm{P}^{\perp}Y\| \geq \delta/2 - \epsilon^m,$$

(3) for all $w \in V_0 \cap \mathrm{B}(0, 5T)$

$$\mathrm{P}^{-1}[\mathrm{B}(w, (5T+\delta)(1-\delta)^{-1}\delta + \epsilon^m)] \cap \mathrm{Spt}\,\nu \cap \mathrm{X}(0, 2+\epsilon^m, 2K(1+2\delta), V_0) \neq \emptyset,$$

(4) $\mathrm{B}(0, T(1-(2\epsilon^m)^{m+1})) \cap V_0 \subset I \subset \mathrm{B}(0, 5T) \cap V_0$,

(5) $\nu[\mathrm{P}^{-1}(I) \cap \mathrm{B}(0, 30K\epsilon^{(2/(3m))-1})] \leq \chi(1+\eta)(\mathcal{H}^m(I) + 2c(m)\epsilon^m\epsilon^{(2/(3m))-1}) + \alpha(m)l\epsilon^{2m}$,

(6) for all $0 < \theta \leq [3m]^{-(m+1)}$

$$\mathcal{H}^m\left[\mathrm{B}(\partial_{V_0} I, \theta T) \cap V_0\right] \leq c(m)[(2\epsilon^m)^{m+1} + \theta]^{1/(m+1)}T^m.$$

Proof: Fix $1 \le T \le \epsilon^{(2/(3m))-1}$. From Lemma 4.6 we can find a $u \in A_3$ and a Borel set $J \subset V_0$ such that

$$\mathrm{PB}(u, Ts(u)) \subset J \subset \mathrm{PB}(u, 4Ts(u)), \tag{4.9}$$

$$\mu[\mathrm{P}^{-1}(J) \cap \mathrm{B}(0, r_0)] \le \chi(1+\eta)\mathcal{H}^m(J) \tag{4.10}$$

and if $0 < \theta \le [2/(3m)]^{m+1}$ then

$$\mathcal{H}^m\left[\mathrm{B}(\partial_{V_0} J, \theta T s(u)) \cap V_0\right] \le c(m)\theta^{1/(m+1)}T^m. \tag{4.11}$$

Since $u \in A_3$ we can find a $y, y' \in F \cap \mathrm{S}(u, s(u))$ such that

$$\|\mathrm{P}^{\perp}(y - y')\| \ge \delta s(u).$$

Thus, as $s(u)$ is good for u (Lemma 4.2), $F \cap \mathrm{S}(u, \delta s(u)) \neq \emptyset$ and so we can find an $X \in F \cap \mathrm{S}(u, \delta s(u))$ such that

$$\max\{\|\mathrm{P}^{\perp}(y - X)\|, \|\mathrm{P}^{\perp}(y' - X)\|\} \ge \delta s(u)/2.$$

We may assume without loss of generality that

$$\|\mathrm{P}^{\perp}(y - X)\| \ge s(u)/2$$

and, as $(5T+\delta)(1-\delta)^{-1}s(u) \le \alpha r_0 \le r_0 - \|u\|$, we may use Lemma 4.3 to conclude that there is a $t \in \mathrm{S}(u, 0)$ such that

$$F \cap \mathrm{P}^{-1}\left[\mathrm{B}(u, (5T+\delta)(1-\delta)^{-1}s(u))\right] \subset \mathrm{X}(t, s(u)/2, 2K(1+2\delta), V_0).$$

Hence

$$F \cap \mathrm{P}^{-1}\left[\mathrm{B}(u, (5T+\delta)(1-\delta)^{-1}s(u))\right] \subset \mathrm{X}(X, 2s(u), 2K(1+2\delta), V_0).$$

As $s(u) < r_0$ we may use (4.5) of Section 4.2 to find a

$$\nu \in \mathrm{Tan}_m(\mu, X)\,(\subset \mathcal{M}^m_C(l, u))$$

such that

$$\mathrm{F}_{R+3+\epsilon^{-1}}\left(\frac{\mu_{X,s(u)}}{s(u)^m}, \nu\right) < \alpha(m) l \epsilon^{m(m+3)}$$

which is 4.7(1).

If $z \in F \cap \mathrm{P}^{-1}\left[\mathrm{B}(u, (5T+\delta)(1-\delta)^{-1}s(u))\right]$ then

$$\|z - X\| \le ((5T+\delta)(1-\delta)^{-1}s(u) + 2\epsilon r_0\delta/\delta \le 3 + \epsilon^{-1}$$

and so we may use Lemma 3.5 to conclude that there is a $\zeta \in \mathrm{Spt}\,\nu$ such that

$$\|\zeta - (z - X)/s(u)\| \le \epsilon^m.$$

In particular, as

$$y \in \mathrm{P}^{-1}\mathrm{B}(u, s(u)) \cap F \cap \mathrm{X}(X, 2s(u), 2K(1+2\delta), V_0),$$

there is a

$$Y \in \mathrm{Spt}\,\nu \cap \mathrm{P}^{-1}\left[\mathrm{B}(0, 1+\delta+\epsilon^m)\right] \cap \mathrm{X}(0, 2+\epsilon^m, 2K(1+2\delta), V_0)$$

such that

$$\|Y - (y - X)/s(u)\| \le \epsilon^m.$$

Hence by our estimate for y and as $0 \in \operatorname{Spt}\nu$ (since $\nu \in \operatorname{Tan}_m(\mu, x)$) we have

$$\operatorname{diam}\left[\mathrm{P}^{\perp}\left(\operatorname{Spt}\nu \cap \mathrm{P}^{-1}(\mathrm{B}(0, 1+\delta+\epsilon^m)) \cap \mathrm{X}(0, 2+\epsilon^m, 2K(1+2\delta), V_0)\right)\right] \qquad \geq \frac{\delta}{2} - \epsilon^m$$

which verifies 4.7(2).

Suppose $w \in \mathrm{B}(0, 5T) \cap V_0$ and let $v = s(u)w + \mathrm{P}X$ and so $v \in \mathrm{B}(u, (5T+\delta)s(u)) \cap V_0$. As $(5T+\delta)(1-\delta)^{-1}s(u)$ is good for u it follows (by [I]) that there is a

$$z \in F \cap \mathrm{P}^{-1}\mathrm{B}(v, (5T+\delta)(1-\delta)^{-1}\delta s(u)).$$

Hence we may use Lemma 3.5 again to conclude that there is a $\zeta \in \operatorname{Spt}\nu$ with

$$\|\zeta - (z - X)/s(u)\| \leq \epsilon^m.$$

But $\|v - \mathrm{P}z\| \leq (5T+\delta)(1-\delta)^{-1}\delta s(u)$ and so

$$\|w - \mathrm{P}\zeta\| \leq (5T+\delta)(1-\delta)^{-1}\delta + \epsilon^m.$$

On observing that $z \in \mathrm{X}(X, 2s(u), 2K(1+2\delta), V_0)$ we conclude that

$$\zeta \in \mathrm{X}(0, 2+\epsilon^m, 2K(1+2\delta), V_0)$$

and so 4.7(3) holds.

Let

$$I := \operatorname{Clos}\left\{(x - \mathrm{P}X)/s(u) : x \in J \setminus \mathrm{B}(\partial_{V_0}J, (2\epsilon^m)^{m+1}Ts(u))\right\}$$

and so

$$\mathrm{B}(I, \epsilon^{m(m+1)}T) \subset \{(x - \mathrm{P}X)/s(u) : x \in J\}$$

and, for $\theta > 0$,

$$\mathrm{B}(\partial_{V_0}I, \theta T) \subset \mathrm{B}(\partial_{V_0}\{(x - \mathrm{P}X)/s(u) : x \in J\}, (\theta + (2\epsilon^m)^{m+1})T).$$

Moreover from (4.11) we have that

$$\mathcal{H}^m(I) \geq [s(u)]^{-m}\mathcal{H}^m(J) - 2c(m)\epsilon^m T^m$$

and, since $\epsilon^m < [3m]^{-(m+1)}$, for $0 < \theta < [3m]^{-(m+1)}$ we have

$$\mathcal{H}^m[\mathrm{B}(\partial_{V_0}I, \theta T) \cap V_0] \leq c(m)\left[(2\epsilon^m)^{m+1} + \theta\right]^{1/(m+1)} T^m$$

which verifies 4.7(6).

From (4.9) we conclude that

$$\mathrm{B}(0, T(1 - (2\epsilon^m)^{m+1} - \delta)) \cap V_0 \subset I \subset \mathrm{B}(0, 5T) \cap V_0$$

which is 4.7(4).

It only remains to verify 4.7(5). Since

$$\max\{30K\epsilon^{(2/(3m))-1} + \epsilon^{m(m+1)}T,\ 5T\} \leq R + 3 + \epsilon^{-1}$$

we may use Lemma 2.3 (with $E = \mathrm{P}^{-1}(I) \cap \mathrm{B}(0, 30K\epsilon^{(2/(3m))-1})$ and $\tau = \epsilon^{m(m+1)}T$) to conclude that

$$\begin{aligned} &\nu\left[\mathrm{P}^{-1}(I) \cap \mathrm{B}(0, 30K\epsilon^{(2/(3m))-1})\right] \\ &\quad\leq \frac{\mu\left[\mathrm{P}^{-1}(J) \cap \mathrm{B}(X, (30K\epsilon^{(2/(3m))-1} + \epsilon^{m(m+1)}T)s(u))\right]}{[s(u)]^{-m}} + \frac{\alpha(m)l\epsilon^{m(m+3)}}{\epsilon^{m(m+1)}T} \end{aligned}$$

which as $\|X\| + (30K\epsilon^{(2/(3m))-1} + \epsilon^{m(m+1)}T)s(u) \leq r_0$

$$\leq \quad [s(u)]^{-m}\mu\left[\mathrm{P}^{-1}(J)\cap \mathrm{B}(0,r_0)\right]+\frac{\alpha(m)l\epsilon^{m(m+3)}}{\epsilon^{m(m+1)}T}$$

which, by (4.10), is

$$\begin{aligned}
&\leq \quad \chi(1+\eta)[s(u)]^{-m}\mathcal{H}^m(J)+\alpha(m)l\epsilon^{2m}/T \\
&\leq \quad \chi(1+\eta)\left[\mathcal{H}^m(I)+2c(m)\epsilon^m\epsilon^{(2/(3m))-1^m}\right]+\alpha(m)l\epsilon^{2m}
\end{aligned}$$

verifying 4.7(5) as required. ∎

4.3 Properties of $B^{(1)}$ independent of ϵ

Lemma 4.8 *For all $T\geq 1$ there is an $x\in B^{(2)}$, $\nu\in \mathrm{Tan}_m(\mu,x)$ and $V\in \mathrm{G}(n,m)$ such that*

(1) there is a $Y\in \mathrm{Spt}\,\nu\cap \mathrm{X}(0,2,2K(1+2\delta),V)\cap \mathrm{P}_V^{-1}(\mathrm{B}(0,1))$ with $\|\mathrm{P}_V^{\perp}Y\|\geq \delta/2$,

(2) $V\cap \mathrm{B}(0,5T)\subset \mathrm{P}_V[\mathrm{Spt}\,\nu\cap \mathrm{X}(0,2,2K(1+2\delta),V)]$,

(3) for all Borel sets C contained in $V\cap \mathrm{int}\,(\mathrm{B}(0,T))$

$$\nu[\mathrm{P}_V^{-1}(C)]=\chi\mathcal{H}^m(C),$$

(4) $\mathrm{Spt}\,\nu\cap \mathrm{P}_V^{-1}[\mathrm{int}\,(\mathrm{B}(0,T))]\subset \mathrm{X}(0,2,2K(1+2\delta),V)$.

Proof: Fix $T\geq 1$ and suppose we have a sequence of positive real numbers $\Xi_i\to 0$ and sequences $x_i\in B^{(2)}$, $\nu_i\in \mathrm{Tan}_m(\mu,x_i)\,(\subset \mathcal{M}_C^m(l,u))$ and $V_i\in \mathrm{G}(n,m)$ such that

(a) the points $x_i\to x\in B^{(2)}$, $\nu_i\to\nu$ and so, by the definition of $B^{(1)}\left(\supset B^{(2)}\right)$, $\nu\in \mathrm{Tan}_m(\mu,x)\subset \mathcal{M}_C^m(l,u)$,

(b) the m-planes $V_i\to V$ and if $\mathrm{P}_i:=\mathrm{P}_{V_i}$, $\mathrm{P}_i^{\perp}:=\mathrm{P}_{V_i}^{\perp}$ then we can find a $Y_i\in \mathrm{Spt}\,\nu_i\cap \mathrm{P}_i^{-1}[\mathrm{B}(0,1+\Xi_i)]\cap \mathrm{X}(0,2+\Xi_i,2K(1+2\delta),V)$ with $\|\mathrm{P}_i^{\perp}Y_i\|\geq \delta/2-\Xi_i$ and $Y_i\to Y$,

(c) there is a compact set $I_i\subset V_i$ such that $I_i\to I$ in the Hausdorff metric (denoted by d_H) and

$$\mathrm{B}(0,T(1-\Xi_i))\cap V\subset I_i\subset \mathrm{B}(0,4T+\Xi_i)\cap V,$$

$$\nu_i[\mathrm{P}_i^{-1}(I_i)\cap \mathrm{B}(0,Xi_i^{-1})]\leq \chi\mathcal{H}^m(I_i)+\Xi_i$$

and if $0<\theta<[3m]^{-(m+1)}$ then

$$\mathcal{H}^m[\mathrm{B}(\partial_i I_i,\theta T)\cap V_i]\leq c(m)\left[\Xi_i+\theta\right]^{1/(m+1)}T^m$$

(where $\partial_i:=\partial_{V_i}$.)

(d) For all $v\in V_i\cap \mathrm{B}(0,5T)$ there is a $\zeta\in \mathrm{Spt}\,\nu_i\cap \mathrm{X}(0,2+\Xi_i,2K(1+2w),V_i)$ such that

$$\|\mathrm{P}_i(\zeta)-v\|\leq \Xi_i T.$$

Then I claim that ν, x, Y and V would satisfy the Lemma. (We shall verify the existence of such sequences Ξ_i, ν_i, x_i, I_i and V_i later.)

From Lemma 3.14 (with $L=T$) we conclude that ν and Y satisfy 4.8(1,2).

It is clear that I (defined in (c) above) satisfies

$$\mathrm{B}(0,T)\cap V\subset I\subset \mathrm{B}(0,5T)\cap V.$$

Suppose that $0<\delta\le[3m]^{-(m+1)}$ and j is chosen so that for all $i>j$

$$I\subset \mathrm{B}(I_i,\delta T) \text{ and } I_i\subset \mathrm{B}(I,\delta T).$$

Thus $I\subset V\cap \mathrm{B}(I_i,\delta T)$ and, clearly,

$$\mathcal{H}^m(V\cap \mathrm{B}(I_i,\delta T))\le \mathcal{H}^m(V_i\cap \mathrm{B}(I_i,\delta T)).$$

Hence

$$\begin{aligned}\mathcal{H}^m(I)\le\mathcal{H}^m(V\cap \mathrm{B}(I_i,\delta T)) &\le \mathcal{H}^m(V_i\cap \mathrm{B}(I_i,\delta T))\\ &= \mathcal{H}^m\left[(I_i\cup \mathrm{B}(\partial_i I_i,\delta T))\cap V_i\right]\\ &\le \mathcal{H}^m(I_i)+c(m)\left[\Xi_i+\delta\right]^{1/(m+1)}T^m.\end{aligned}$$

Now observe that as $I_i\subset V_i\cap \mathrm{B}(I,\delta T)$ we have

$$\mathcal{H}^m(I_i)\le\mathcal{H}^m(V_i\cap \mathrm{B}(I,\delta T))\le\mathcal{H}^m(V\cap \mathrm{B}(I,\delta T)).$$

Hence on sending i to infinity we deduce that

$$\mathcal{H}^m(I)\le\liminf_{i\to\infty}\mathcal{H}^m(I_i)+c(m)\delta^{1/(m+1)}T^m$$

and

$$\limsup_{i\to\infty}\mathcal{H}^m(I_i)\le\mathcal{H}^m(V\cap \mathrm{B}(I,\delta T)).$$

But $I=\cap_{\delta>0}\mathrm{B}(I,\delta T)$ and so sending δ to zero gives

$$\mathcal{H}^m(I)=\lim_{i\to\infty}\mathcal{H}^m(I_i).$$

Now observe that if i and δ are such that $I\subset \mathrm{B}(I_i,\delta T)$ and if $x\in I\setminus \mathrm{B}(\partial_V I,\alpha T)$ (for some $\alpha>\delta$) then

$$V\cap \mathrm{B}(x,\alpha T)\subset I\subset \mathrm{B}(I_i,\delta T)$$

and so $x\in I_i$ and

$$\mathrm{d}(x,\partial_i I_i)>[\alpha-\delta]T$$

which means that

$$I\setminus \mathrm{B}(\partial_V I,\alpha T)\subset \mathrm{B}(I_i,\delta T)\setminus \mathrm{B}(\partial_i I,(\alpha-\delta)T).$$

Fix $0<\delta<1$ and choose j such that if $i\ge j$ then

$$d_H(I,I_i)\le\delta T,\qquad \mathcal{H}^m(I_i)\le\mathcal{H}^m(I)+\delta,$$

$$\Xi_i>6\delta^{-1}T,\qquad \mathrm{F}_{7\delta^{-1}T}(\nu_i,\nu)<\delta^2 T$$

and if $x\in \mathrm{B}(0,6\delta^{-1}T)$ then

$$\|\mathrm{P}_i(x)-\mathrm{P}_V(x)\|<\delta T/2.$$

If $x\in \mathrm{P}_V^{-1}\left[I\setminus \mathrm{B}(\partial_V I,2\delta T)\right]\cap \mathrm{B}(0,6\delta^{-1}T)$ then

$$\begin{aligned}\|\mathrm{P}_i(x)-\mathrm{P}_i\mathrm{P}_V(x)\| &\le \|\mathrm{P}_i(x)-\mathrm{P}_V(x)\|+\|\mathrm{P}_V(x)-\mathrm{P}_i\mathrm{P}_V(x)\|\\ &< \delta T/2+\delta T/2=\delta T.\end{aligned}$$

Thus

$$\mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 2\delta T)\right] \cap \mathrm{B}(0, 6\delta^{-1}T) \subset \mathrm{P}_i^{-1}(I_i) \cap \mathrm{B}(0, 6\delta^{-1}T)$$

and so we may use Lemma 2.3 with $\tau = \delta T$ and $E = \mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 3\delta T)\right] \cap \mathrm{B}(0, 6\delta^{-1}T)$ to deduce that

$$\begin{aligned}
&\nu\left[\mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 3\delta T)\right] \cap \mathrm{B}(0, 5\delta^{-1}T)\right] \\
&\qquad \leq \nu_i\left[\mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 2\delta T)\right] \cap \mathrm{B}(0, 6\delta^{-1}T)\right] + \mathrm{F}_{7\delta^{-1}T}(\nu_i, \nu) \\
&\qquad \leq \nu_i\left[\mathrm{P}_i^{-1}(I_i) \cap \mathrm{B}(0, 6\delta^{-1}T)\right] + \delta
\end{aligned}$$

as $\Xi_i^{-1} > 6\delta^{-1}T$ we may use (c) to deduce

$$\begin{aligned}
&\leq \chi\mathcal{H}^m(I_i) + \Xi_i + \delta \\
&\leq \chi\mathcal{H}^m(I) + \Xi_i + (\chi+1)\delta.
\end{aligned}$$

Hence on sending i to infinity we find that

$$\nu\left[\mathrm{P}_V^{-1}\left[I \setminus \mathrm{B}(\partial_V I, 3\delta T)\right] \cap \mathrm{B}(0, 5\delta^{-1}T)\right] \leq \chi\mathcal{H}^m(I) + (\chi+1)\delta$$

but δ was arbitrary and so we conclude that

$$\nu\left[\mathrm{P}_V^{-1}(\operatorname{int}_V I)\right] \leq \chi\mathcal{H}^m(I).$$

Now recall from (4.4) that as $\nu \in \mathrm{Tan}_m(\mu, x)$ and $x \in B^{(2)}$ then for ν-a.e. ζ

$$\overline{\mathrm{D}}_m(\nu, \zeta) \geq \chi.$$

Thus if $C \subset \mathrm{B}(0, 5T) \cap V$ is a Borel set then from Lemma 2.1 we can deduce that

$$\begin{aligned}
&\nu\left[\mathrm{P}_V^{-1}(C) \cap \mathrm{X}(0, 2, 2K(1+2\delta), V)\right] \\
&\qquad \geq \chi\mathcal{H}^m\left[\mathrm{P}_V^{-1}(C) \cap \operatorname{Spt}\nu \cap \mathrm{X}(0, 2, 2K(1+2\delta), V)\right]
\end{aligned}$$

which, by projecting back onto V, is

$$\geq \chi\mathcal{H}^m(C).$$

Hence if $C \subset \operatorname{int}(\mathrm{B}(0, T)) \cap V$ is a Borel set then

$$\begin{aligned}
\chi\mathcal{H}^m(C) &\leq \nu[\mathrm{P}_V^{-1}(C)] \\
&\leq \nu[\mathrm{P}_V^{-1}(I)] - \nu[\mathrm{P}_V^{-1}(I \setminus C) \cap \mathrm{X}(0, 2, 2K(1+2\delta), V)]
\end{aligned}$$

which by the preceeding

$$\begin{aligned}
&\leq \chi\mathcal{H}^m(I) - \chi[\mathcal{H}^m(I) - \mathcal{H}^m(C)] \\
&= \chi\mathcal{H}^m(C).
\end{aligned}$$

Thus for all Borel sets $C \subset \operatorname{int}(\mathrm{B}(0, T)) \cap V$

$$\nu[\mathrm{P}_V^{-1}(C)] = \chi\mathcal{H}^m(C)$$

which is 4.8(3) of the Lemma.

If there is an $x \in \mathrm{P}_V^{-1}[\operatorname{int}\mathrm{B}(0, T)] \setminus \mathrm{X}(0, 2, 2K(1+2\delta), V)$ such that $x \in \operatorname{Spt}\nu$ then we can find an $r > 0$ such that $\mathrm{P}_V\mathrm{B}(x, r) \subset \operatorname{int}\mathrm{B}(0, T)$, $\nu\mathrm{B}(x, r) > 0$ and

$$\mathrm{B}(x, r) \cap \mathrm{X}(0, 2, 2K(1+2\delta), V) = \emptyset$$

but then

$$\begin{aligned}
\chi\mathcal{H}^m(\mathrm{B}(\mathrm{P}_V x, r)\cap V) &= \nu\left[\mathrm{P}_V^{-1}(\mathrm{B}(\mathrm{P}_V x, r))\right] \\
&= \nu\left[\mathrm{P}_V^{-1}(\mathrm{B}(\mathrm{P}_V x, r))\cap \mathrm{X}(0,2,2K(1+2\delta),V)\right] \\
&\quad + \nu\left[\mathrm{P}_V^{-1}(\mathrm{B}(\mathrm{P}_V x, r))\setminus \mathrm{X}(0,2,2K(1+2\delta),V)\right] \\
&\geq \chi\mathcal{H}^m(\mathrm{B}(\mathrm{P}_V x, r)\cap V)) + \nu[\mathrm{B}(x,r)] \\
&> \chi\mathcal{H}^m(\mathrm{B}(\mathrm{P}_V x, r)\cap V) \text{ — a contradiction}
\end{aligned}$$

and so ν satisfies 4.8(4).

It remains to show that we can find sequences which satisfy (a) through to (d). In order to achieve this it suffices to choose a sequence of positive ϵ tending to zero and use Lemma 4.7 to find associated sequences of measures, points, planes and sets. Upon noting that any sequence of measures $(\omega_i) \subset \mathcal{M}_C^m(l,u)$ possesses a convergent subsequence (this is an application of Lemma 2.2 together with the uniform upper density estimate on the measures ω_i) and that, by compactness, any sequence of points x_i in $B^{(2)}$ possesses a convergent subsequence and similarly for $V_i \in \mathrm{G}(n,m)$ and compact sets $I_i \subset \mathrm{B}(0,5T)$ we deduce that we can, indeed, find a sequence satisfying (a) through (d). Hence the Lemma holds. ∎

Lemma 4.9 *There is an $X \in B^{(2)}$, $\omega \in \mathrm{Tan}_m(\mu, X)$ and $W \in \mathrm{G}(n,m)$ such that*

(1) $\mathrm{Spt}\,\omega \subset \mathrm{X}(0,2,2K(1+2\delta),W)$,

(2) $W = \mathrm{P}_W[\mathrm{Spt}\,\omega]$,

(3) for all Borel sets $I \subset W$
$$\omega[\mathrm{P}_W^{-1}(I)] = \chi\mathcal{H}^m(I),$$

(4) there is a $Y \in \mathrm{Spt}\,\omega \cap \mathrm{P}_W^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,2,2K(1+2\delta),W)$ with
$$\|\mathrm{P}_W^{\perp} Y\| \geq \delta/2,$$

(5) for ω-a.e. ζ, $\overline{\mathrm{D}}_m(\omega,\zeta) \geq \chi$,

(6) ω is m-rectifiable.

Proof: From Lemma 4.8 we may find for all $T \geq 1$ an $x_T \in B^{(1)}$, $\omega_T \in \mathrm{Tan}_m(\mu, x_T)$, $W_T \in \mathrm{G}(n,m)$ and a $Y_T \in \mathrm{Spt}\,\omega_T \cap \mathrm{P}_W^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,2,2K(1+2\delta),W_T)$ such that

$$\left.\begin{array}{c}
\|\mathrm{P}_{W_T}^{\perp} Y_T\| \geq \delta/2 \\
W_T \cap \mathrm{B}(0,5T) \subset \mathrm{P}_{W_T}[\mathrm{Spt}\,\omega_T \cap \mathrm{X}(0,2,2K(1+2\delta),W_T)], \\
\text{for all Borel sets } I \text{ contained in } \mathrm{int}\,(\mathrm{B}(0,T)) \cap W_T,\ \omega_T[\mathrm{P}_{W_T}^{-1}(I)] = \chi\mathcal{H}^m(I), \\
\mathrm{Spt}\,\omega_T \cap \mathrm{P}_{W_T}^{-1}[\mathrm{int}\,(\mathrm{B}(0,T))] \subset \mathrm{X}(0,2,2K(1+2\delta),W_T).
\end{array}\right\} \quad (4.12)$$

By repeated use of compactness, application of Lemma 2.2 and (4.1) we may find a sequence $T(i) \to \infty$ such that

$$\left.\begin{array}{c}
x_i := x_{T(i)} \to X \in B^{(1)}, \\
\omega_i := \omega_{T(i)} \to \omega \in \mathrm{Tan}_m(\mu, X), \\
W_i := W_{T(i)} \to W \in \mathrm{G}(n,m) \\
Y_i := Y_{T(i)} \to Y \in \mathrm{P}_W^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,2,2K(1+2\delta),W).
\end{array}\right\} \quad (4.13)$$

Let $\mathrm{P}_i := \mathrm{P}_{W_{T(i)}}$ and $\mathrm{P}_i^{\perp} := \mathrm{P}^{\perp}_{W_{T(i)}}$. From Lemma 3.14 we may immediately deduce that $Y \in \operatorname{Spt}\omega \cap \mathrm{P}_W^{-1}(\mathrm{B}(0,1)) \cap \mathrm{X}(0,2,2K(1+2\delta),W)$, $\|\mathrm{P}_W^{\perp} Y\| \geq \delta/2$ and

$$\mathrm{P}_W[\operatorname{Spt}\omega \cap \mathrm{X}(0,2,2K(1+2\delta),W)] \supset W. \tag{4.14}$$

Hence 4.9(2) and 4.9(4) hold.

Since $\omega \in \operatorname{Tan}_m(\mu, X)$ for some $X \in B^{(2)}$ we know from (4.4) that for ω-a.e. ζ

$$\overline{\mathrm{D}}_m(\omega,\zeta) \geq \chi$$

and so 4.9(5) holds. Hence if $C \subset W$ is a Borel set then as $\mathrm{P}_W[\operatorname{Spt}\nu \cap \mathrm{X}(0,2,2K(1+2\delta),W)] \supset W$ we deduce from Lemma 2.1 that

$$\begin{aligned}
\omega[\mathrm{P}_W^{-1}(C) \cap \mathrm{X}(0,2,2K(1+2\delta),W)] &\geq \chi\mathcal{H}^m[\mathrm{P}_W^{-1}(C) \cap \mathrm{X}(0,2,2K(1+2\delta),W) \cap \operatorname{Spt}\omega] \\
&\geq \chi\mathcal{H}^m(C).
\end{aligned}$$

Hence in order to verify 4.9(3) it is sufficient to show that for all $T \geq 1$

$$\omega[\mathrm{P}_W^{-1}(\mathrm{B}(0,T))] \leq \alpha(m)\chi T^m.$$

Since then if $C \subset W$ we deduce that, for all $T \geq 1$,

$$\begin{aligned}
\chi\mathcal{H}^m[C \cap \mathrm{B}(0,T)] &\leq \omega[\mathrm{P}_W^{-1}(\mathrm{B}(0,T) \cap C)] \\
&\leq \omega[\mathrm{P}_W^{-1}\mathrm{B}(0,T)] - \omega[\mathrm{P}_W^{-1}(\mathrm{B}(0,T) \setminus (C \cap \mathrm{B}(0,T))] \\
&\leq \alpha(m)\chi T^m - \chi[\alpha(m)T^m - \mathcal{H}^m(C \cap \mathrm{B}(0,T))] \\
&= \chi\mathcal{H}^m(C \cap \mathrm{B}(0,T))
\end{aligned}$$

and 4.9(3) then follows on sending T to infinity. So fix $T \geq 1$ and $0 < \Xi < 1/(2T)$. Choose i so large that

$$\mathrm{F}_{1+1/\Xi}(\omega_i,\omega) \leq \Xi^2 T, \qquad T(i) > T$$

and

$$\mathrm{P}_W^{-1}[\mathrm{B}(0,T)] \cap \mathrm{B}(0,1/\Xi) \subset \mathrm{P}_i^{-1}[\mathrm{B}(0,T(1+\Xi))] \cap \mathrm{B}(0,1/\Xi).$$

Then

$$\omega[\mathrm{P}_i^{-1}[\mathrm{B}(0,T)] \cap \mathrm{B}(0,1/\Xi)] \leq \omega[\mathrm{P}_i^{-1}[\mathrm{B}(0,T(1+\Xi))] \cap \mathrm{B}(0,1/\Xi)]$$

and so as Lemma 2.3 implies that

$$\omega[\mathrm{P}_i^{-1}[\mathrm{B}(0,T(1+\Xi))] \cap \mathrm{B}(0,1/\Xi)] \leq \omega_i[\mathrm{P}_i^{-1}[\mathrm{B}(0,T(1+2\Xi))] \cap \mathrm{B}(0,1/\Xi)] + \Xi$$

which, from (4.12) above

$$= \alpha(m)\chi T^m(1+2\Xi)^m + \Xi$$

and so we may conclude that

$$\omega[\mathrm{P}_i^{-1}[\mathrm{B}(0,T)] \cap \mathrm{B}(0,1/\Xi)] \leq \alpha(m)\chi T^m(1+2\Xi)^m + \Xi$$

and as Ξ was arbitrary we deduce that

$$\omega[\mathrm{P}_i^{-1}(\mathrm{B}(0,T))] \leq \alpha(m)\chi T^m.$$

Thus 4.9(3) holds.

The fact that 4.9(3) holds together with (4.14) implies (using an identical technique to that used in Lemma 4.8) that

$$\operatorname{Spt}\omega \subset \mathrm{X}(0, 2, 2K(1+2\delta), W)$$

and so 4.9(1) holds.

It remains to verify 4.9(6) but as $\omega \in \operatorname{Tan}_m(\mu, X) \subset \mathcal{M}_C^m(l, u)$ and we have already verified 4.9(1,2,3,5) then we may use Proposition 3.12 to deduce that ω is m-rectifiable. Thus the Lemma holds. ∎

4.4 Deriving a contradiction

We are now able to find a contradiction: Let ω be the measure whose existence is guaranteed by Lemma 4.9 and let $X \in B^{(2)}$, $Y \in \mathbf{R}^n$ and $W \in \mathrm{G}(n,m)$ be as in Lemma 4.9. Since $\omega \in \operatorname{Tan}_m(\mu, X)$ we know that $0 \in \operatorname{Spt}\omega$ and as $Y \in \operatorname{Spt}\omega$ has $\|\mathrm{P}_W^\perp Y\| \geq \delta/2$ we conclude that

$$\operatorname{diam}(\mathrm{P}_W^\perp \operatorname{Spt}\omega) \geq \delta/2\,(> 0).$$

In addition ω is m-rectifiable,

$$W = \mathrm{P}_W[\operatorname{Spt}\omega],$$

and for ω-a.e. $x \in \operatorname{Spt}\omega$

$$\overline{\mathrm{D}}_m(\omega, x) \geq \chi.$$

Hence we may apply Lemma 3.13 to conclude that there is a Borel set $B \subset W$ with

$$\omega[\mathrm{P}_W^{-1} B] > \chi \mathcal{H}^m(B)$$

but this contradicts the definition of ω. Thus no such measure ω can exist and so our original measure μ must be m-rectifiable as required.

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