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The composition of two derivatives has a fixed point.

Abstract

We show that if $f, g\colon [0,1] \to [0,1]$ are both Darboux Baire-1 functions, then their composition, $f \circ g$, possesses a fixed point.

In [2], Gibson and Natkaniec refer to a problem of K. C. Ciesielski who asked whether the composition of two derivative functions from the unit interval to the unit interval necessarily possesses a fixed point. In this note we show that this is the case.

Recalling that a Baire-1 function is the pointwise limit of a sequence of continuous functions and that a Darboux function is one for which the image of any interval in its domain is connected, we can formulate our main result as follows.

Theorem 1 *If $f, g\colon [0,1] \to [0,1]$ are both Darboux Baire-1 functions, then there is an $x \in [0,1]$ for which $(f \circ g)(x) = x$.*

Since derivative functions are examples of Darboux Baire-1 functions, this answers Ciesielski's question. The rest of the paper consists of a proof of this theorem.

Key Words: Darboux, Baire-1, fixed points

Mathematical Reviews subject classification: 26A99, 26A21

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1 Proof of Theorem

Fix two Darboux Baire-1 functions f, $g\colon [0,1] \to [0,1]$. We may assume without loss of generality that

$$f(0) = 0,\ f(1) = 1$$

and

$$g(0) = 1,\ g(1) = 0$$

for, by considering the square $[-1,2] \times [-1,2]$ and extending the sets F and G as indicated in Figure 1, and then rescaling, we can define two new Darboux Baire-1 functions $\tilde{f}, \tilde{g}\colon [0,1] \to [0,1]$ with $\tilde{f}(0) = 0$, $\tilde{f}(1) = 1$, $\tilde{g}(0) = 1$ and $\tilde{g}(1) = 0$ whose composition possesses a fixed point if and only if the original functions did.

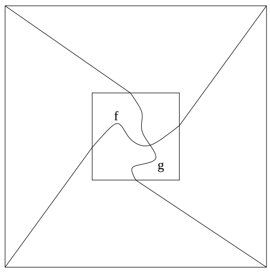

Figure 1: Ensuring that $f(0) = 0, f(1) = 1$ and $g(0) = 1, g(1) = 0$

For $\phi\colon I \to \mathbf{R}$, I an interval, we define

$$\mathrm{graph}_X(\phi) = \{(x, \phi(x)) : x \in I\}$$

and

$$\mathrm{graph}_Y(\phi) = \{(\phi(y), y) : y \in I\}$$

and given $a, b \in \mathbf{R}$, we let $[a,b], (a,b)$ denote the closed, and open intervals connecting them, respectively.

Set

$$F = \mathrm{graph}_X(f) = \{(x, f(x)) \in [0,1]^2 : x \in [0,1]\}$$

and

$$G = \mathrm{graph}_Y(g) = \{(g(y), y) \in [0,1]^2 : y \in [0,1]\},$$

then in order to prove the theorem it is sufficient to show that $F \cap G \neq \emptyset$.

Throughout this note, by *rectangle* we understand a rectangle whose sides are parallel to the usual coordinate axes. Topological notions like open, closed, etc., will be considered relatively to $[0,1]^2$.

Definition 1 *We define a crossing-configuration, $\mathcal{R} = (A, B)$ to be an ordered pair consisting of non-empty finite subsets A and B of F and G, respectively, such that whenever I and J are closed intervals with $A \cup B \subset I \times J$ and $\phi\colon I \to \mathbf{R}$ and $\psi\colon J \to \mathbf{R}$ are continuous functions with:*

$$A \subset \mathrm{graph}_X(\phi) \quad \text{and} \tag{1}$$

$$B \subset \mathrm{graph}_Y(\psi), \tag{2}$$

then

$$\mathrm{graph}_X(\phi) \cap \mathrm{graph}_Y(\psi) \neq \emptyset.$$

Remark 1 *If $\mathcal{R} = (A, B)$ is a crossing configuration, and if ϕ, $\psi\colon I, J \to [0,1]$ are continuous functions satisfying (1) and (2) respectively, then for any rectangle $R \subset [0,1]^2$ which contains $A \cup B$, we know that*

$$\mathrm{graph}_X(\phi) \cap \mathrm{graph}_Y(\psi) \cap R \neq \emptyset.$$

Since, if the intersection were empty, we could modify ϕ and ψ outside a closed rectangle $S \subset R$ with $A \cup B \subset S$ to form two new continuous functions ϕ' and ψ' for which $\mathrm{graph}_X(\phi')$ and $\mathrm{graph}_Y(\psi')$ do not intersect.

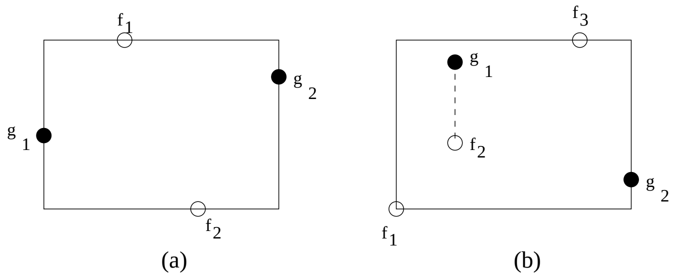

Figure 2: Two examples of crossing configurations: points denoted by $\circ$ lie in F and points denoted by $\bullet$ lie in G.

Lemma 1 *The configurations illustrated in Figure* **??** *are crossing configurations.*

Proof: Figure **??**(a): Here f_1, f_2 are points from F lying on the top and bottom edges of a closed rectangle, and g_1, g_2 are points of G lying on the left and right edges of the rectangle, respectively. Suppose that ϕ and ψ are continuous functions with $\{f_1, f_2\} \subset \mathrm{graph}_X(\phi)$ and $\{g_1, g_2\} \subset \mathrm{graph}_Y(\psi)$. Notice that the part of $\mathrm{graph}_X(\phi)$ lying within the vertical strip whose edges contain f_1 and f_2 may be extended to form a Jordan curve separating g_1 and g_2 in such a way that the added curve does not intersect $\mathrm{graph}_Y(\psi) \cap (\mathbf{R} \times [g_1, g_2])$. (See Figure 2.)

Since $\mathrm{graph}_Y(\psi) \cap (\mathbf{R} \times [g_1, g_2])$ connects g_1 with g_2, we conclude that

$$(\mathrm{graph}_Y(\psi) \cap (\mathbf{R} \times [g_1, g_2])) \cap (\mathrm{graph}_X(\phi) \cap ([f_1, f_2] \times \mathbf{R})) \neq \emptyset,$$

as required.

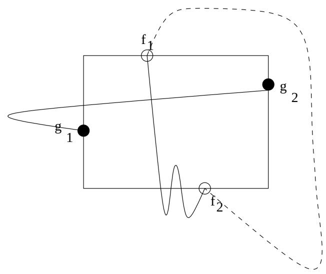

Figure 3: $\mathrm{graph}_X(\phi) \cap ([f_1, f_2] \times \mathbf{R})$ may be extended to form a Jordan curve separating g_1 and g_2.

Figure **??**(b): In this situation we have three points $f_1, f_2, f_3 \in F$ and two points $g_1, g_2 \in G$ with $(f_1)_x < (f_2)_x = (g_1)_x < (f_3)_x$, $(f_1)_y \leq (f_2)_y \leq (g_1)_y \leq (f_3)_y$ and $(f_1)_y \leq (g_2)_y \leq (f_3)_y$. (Here $(\cdot)_x$ and $(\cdot)_y$ denote the x and y coordinates of the point, respectively.) Suppose that ϕ and ψ are continuous functions with $\{f_1, f_2, f_3\} \subset \mathrm{graph}_X(\phi)$ and $\{g_1, g_2\} \subset \mathrm{graph}_Y(\psi)$. Observe that the part of $\mathrm{graph}_X(\phi)$ lying within the vertical strip whose edges contain f_1 and f_3 may be extended to form a Jordan curve separating g_1 and g_2 in such a way that the added curve does not intersect $\mathrm{graph}_Y(\phi) \cap (\mathbf{R} \times [g_1, g_2])$. (See Figure 3.)

Since $\mathrm{graph}_Y(\psi) \cap (\mathbf{R} \times [g_1, g_2])$ connects g_1 and g_2, we conclude that

$$(\mathrm{graph}_Y(\psi) \cap (\mathbf{R} \times [g_1, g_2])) \cap (\mathrm{graph}_X(\phi) \cap ([f_1, f_2] \times \mathbf{R})) \neq \emptyset,$$

as required. ■

Remark 2 *Since* $(0, 0)$ *and* $(1, 1) \in F$*, and* $(0, 1), (1, 0) \in G$*, we conclude that* $\mathcal{R}_0 = (\{(0, 0),\ (1, 1)\}, \{(0, 1), (1, 0)\})$ *is a crossing configuration.*

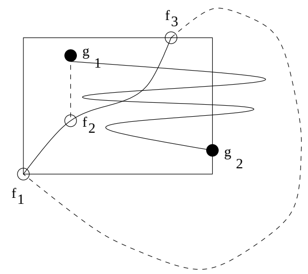

Figure 4: $\text{graph}_X(\phi) \cap ([f_1, f_3] \times \mathbf{R})$ may be extended to form a Jordan curve separating g_1 and g_2.

The key part of our argument is the following proposition.

Proposition 2 *For all crossing-configurations $\mathcal{R} = (A, B)$ and for all open rectangles $R \supset A \cup B$ and open sets $U \supset F$ (or $V \supset G$), we can find a crossing-configuration $\mathcal{R}' = (A', B')$ and a closed rectangle R' with $A' \cup B' \subset R' \subset U \cap R$ (or $V \cap R$).*

Before proving this, we show how it immediately leads to a proof of Theorem 1: Since f and g are Darboux, Baire-1 functions,

$$F = \bigcap_{n=1}^{\infty} U_n, \quad \text{where } U_1 \supset U_2 \supset \cdots \text{ are open sets}$$

and

$$G = \bigcap_{n=1}^{\infty} V_n, \quad \text{where } V_1 \supset V_2 \supset \cdots \text{ are open sets.}$$

We recall, from remark 2, that

$$\mathcal{R}_0 = (\{(0,0), (1,1)\}, \{(0,1), (1,0)\})$$

is a crossing-configuration. We now use the proposition to find a sequence of crossing-configurations $\mathcal{R}_i = (A_i, B_i)$ and open rectangles $R_i \supset A_i \cup B_i$ such that

$$\text{cl}(R_{2i+1}) \subset R_{2i} \cap U_{i+1}$$

and

$$\text{cl}(R_{2(i+1)}) \subset R_{2i+1} \cap V_{i+1}$$

for $i = 0, 1, 2, \ldots$. Hence

$$\mathrm{cl}(R_0) = R_0 \supset \mathrm{cl}(R_1) \supset R_2 \supset \mathrm{cl}(R_3) \supset \cdots,$$

$$\emptyset \neq \bigcap_{n=1}^{\infty} \mathrm{cl}(R_n) \subset \bigcap_n U_n = F$$

and

$$\emptyset \neq \bigcap_{n=1}^{\infty} \mathrm{cl}(R_n) \subset \bigcap_n U_n = G,$$

which together imply that $F \cap G \neq \emptyset$ as required.
Proof: We prove the proposition for the case when $U \supset F$, the case when $V \supset G$ is similar.

Suppose that $\mathcal{R} = (A, B)$ is a crossing configuration, $R \supset A \cup B$ is an open rectangle, and let $S = [x_1, x_2] \times [y_1, y_2] \subset R$ be a closed rectangle whose (relative) interior contains $A \cup B$.

Observe that if $\phi\colon [x_1, x_2] \to \mathbf{R}$ were a continuous function for which $A \subset \mathrm{graph}_X(\phi)$ and $\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset$, then $G \subset [0,1]^2 \setminus (\mathrm{graph}_X(\phi) \cap S)$ would be a relatively open set, and we would be able to construct a continuous function $\psi\colon [0,1] \to \mathbf{R}$ such that $\mathrm{graph}_Y(\psi) \supset B$ and $\mathrm{graph}_Y(\psi)$ would be a subset of $[0,1]^2 \setminus (\mathrm{graph}_X(\phi) \cap S)$ and so $\mathrm{graph}_X(\phi) \cap \mathrm{graph}_Y(\psi) \cap S = \emptyset$. (See [1].) But (A, B) is a crossing configuration — a contradiction.

We now show that if there were no crossing-configurations (A', B') and closed rectangles with $A' \cup B' \subset R' \subset U \cap R$, then we would be able to construct a continuous function $\phi\colon [x_1, x_2] \to \mathbf{R}$ with $A \subset \mathrm{graph}_X(\phi)$ and $\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset$ giving us our required contradiction.

We do this via the method of regular intervals: we say an interval $I \subset [x_1, x_2]$ is *regular*, if for all $s, t \in I$, $s < t$, we can find a continuous function $\phi\colon [s, t] \to \mathbf{R}$ for which

$$\phi(s) = f(s), \qquad \phi(t) = f(t)$$

and

$$\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset.$$

(Note that regular intervals need neither be open nor closed.) If we show that $[x_1, x_2]$ is itself regular, then we are done.

It is easy to see that:

(1) If I and J are regular intervals and $I \cap J \neq \emptyset$, then $I \cup J$ is regular;

(2) If I is an interval which is the (finite or infinite) union of *open* regular intervals, then I is regular.

It is slightly trickier to verify:

(3) If I is regular, then $\mathrm{cl}(I)$ is regular.
Proof of (3): Let r be the left endpoint of I. (The proof for the right endpoint is similar.) It is enough to show that we can find $r' > r$ arbitrarily close to r for which there is a continuous function $\phi\colon [r, r'] \to \mathbf{R}$ such that $\phi(r) = f(r)$, $\phi(r') = f(r')$ and $\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset$.

Choose $r_i \in I$ such that $r_1 > r_2 > \cdots > r$, $r_n \to r$ and for which $f(r_n) \to f(r)$ (the Darboux property for f ensures we can find such a sequence). For each $k \in \mathbf{N}$, we can find n_k such that

$$f(r_{n_k}) \in (f(r) - 2^{-k}, f(r) + 2^{-k})$$

and both

$$g(f(r) - 2^{-k}) \text{ and } g(f(r) + 2^{-k}) \notin (r, r_{n_k}).$$

Fix a sequence $n_1 < n_2 < n_3 < \cdots$ with this property. Since I is regular we can find a sequence of continuous functions $\phi_k\colon [r_{n_{k+1}}, r_{n_k}] \to \mathbf{R}$ for which

$$\phi_k(r_{n_k}) = f(r_{n_k}),\ \phi_k(r_{n_{k+1}}) = f(r_{n_{k+1}}) \text{ and } \mathrm{graph}_X(\phi_k) \cap G \cap S = \emptyset.$$

Then the function $\tilde{\phi}\colon [r, r_{n_1}] \to \mathbf{R}$ defined by

$$\tilde{\phi}(x) = \begin{cases} f(r) & \text{if } x = r \\ \max\{\min\{\phi_k(x), f(r) + 2^{-k}\}, f(r) - 2^{-k}\} & \text{if } x \in [r_{n_{k+1}}, r_{n_k}] \end{cases}$$

is a well-defined continuous function for which $\tilde{\phi}(r_{n_k}) = f(r_{n_k})$ for all k, $\tilde{\phi}(r) = f(r)$ and $\mathrm{graph}_X(\tilde{\phi}) \cap G \cap S = \emptyset$. ■

Suppose that $[x_1, x_2]$ is not a regular interval and let

$$P = [x_1, x_2] \setminus \bigcup\{I \subset [x_1, x_2] : I \text{ is relatively open and regular}\}.$$

Then P is closed, and observations (2) and (3) imply that $P \cap (x_1, x_2)$ is non-empty and has no isolated points. Thus we can choose $r \in P \cap (x_1, x_2)$ such that

- $f|_P$ is continuous at r; and
- r is not the endpoint of any interval in $[x_1, x_2] \setminus P$ which is contiguous to P.

Without loss of generality we can assume that $g(f(r)) > r$. We will show that in this case we can always find $r' > r$ for which (r, r') is regular which contradicts our choice of r.

Since the endpoints of any interval contiguous to P belong to P, and the closure of any regular interval is also regular, then it is enough to find an

$r' > r$ such that for $s, t \in (r, r') \cap P$ we can find continuous $\phi\colon [s,t] \to \mathbf{R}$ with $\phi(s) = f(s)$, $\phi(t) = f(t)$ and $\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset$.

Case 1: $(r, f(r)) \notin S$.

In this case, since $f|_P$ is continuous at r, the existence of r' is trivial.

Case 2: $(r, f(r)) \in S$ and there is no $r^* > r$ for which either $f|_{(r,r^*)} \geq f(r)$ or $f|_{(r,r^*)} \leq f(r)$.

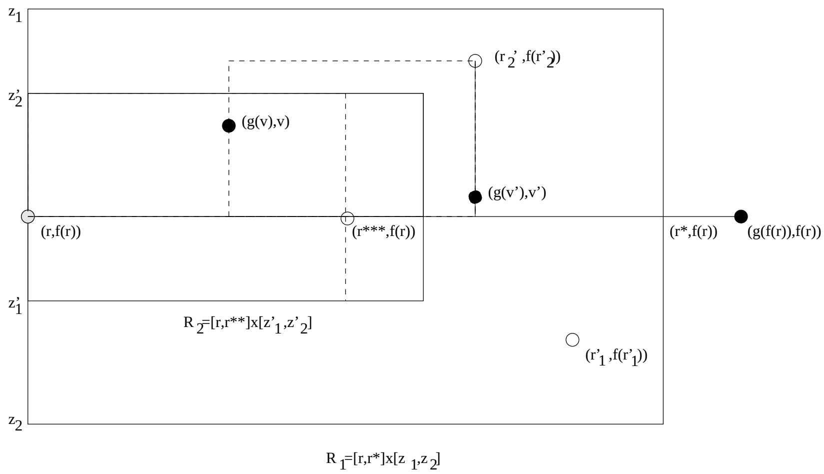

Figure 5: Case 2

Choose $r < r^* < g(f(r))$ and $z_1 < f(r) < z_2$ for which $R_1 = [r, r^*] \times [z_1, z_2] \subset R \cap U$. (See Figure 4.) Then we can find $r_1', r_2' \in (r, r^*)$ such that

$$z_1 < f(r_1') < f(r) < f(r_2') < z_2.$$

Now we choose z_1', z_2' and $r < r^{**} < \min\{r_1', r_2'\}$ such that

$$f(r_1') < z_1' < f(r) < z_2' < f(r_2'),$$

$$R_2 = [r, r^{**}] \times [z_1', z_2'] \subset R_1 \quad \text{and}$$

$$\mathrm{graph}_X(f|_{P \cap [r, r^{**}]}) \subset R_2.$$

Finally by the Darboux property, we can find $r^{***} \in (r, r^{**})$ for which $f(r^{***}) = f(r)$.

Claim: $([r, r^{***}] \times [z_1', z_2']) \cap G = \emptyset$.

Proof of claim: For suppose $(g(v), v) \in [r, r^{***}] \times [z_1', z_2']$, without loss of generality we can assume that $v > f(r)$. By the Darboux property applied to g we can find $v' \in [f(r), v]$ with $g(v') = r_2'$. But then

$$(\{(r^{***}, f(r^{***})),\ (r_2', f(r_2'))\}, \{(g(v), v),\ (g(v'), v')\})$$

is a crossing-configuration of the type (a) illustrated in Figure **??** contained in

$$[g(v), g(v')] \times [f(r), f(r_2')] \subset R \cap U,$$

which is a contradiction. ∎

But now clearly the interval (r, r^{***}) is regular.

Case 3: $(r, f(r)) \in S$ and there is $r^* > r$ such that $f|_{(r,r^*)} \geq f(r)$. (Or $(r, f(r)) \in S$ and there is $r^* > r$ such that $f|_{(r,r^*)} \leq f(r)$.)

Without loss of generality, we do the case when there is an $r^* > r$ with $f|_{(r,r^*)} \geq f(r)$ and there is no $r' > r$ for which $f|_{(r,r')}$ is constant. Choose $z_2 > f(r)$ and $r^* > r$ for which $f|_{(r,r^*)} \geq f(r)$ and $[r, r^*] \times [f(r), z_2] \subset R \cap U$ and set $R_1 = [r, r^*] \times [f(r), z_2]$.

Since $f|_{(r,r^*)}$ is not constant (and so $f(r) < 1$), then we can find $r < r_2' < r^*$ such that $f(r) < f(r_2') < z_2$. Choose $R_2 = [r, r''] \times [f(r), z_2']$ such that

$$f(r) < z_2' < f(r_2'), \quad r < r'' < r_2' \quad \text{and}$$

$$\text{graph}_X(f|_{P \cap [r,r'']}) \subset R_2.$$

We show that there are no points $(u, f(u))$, $(g(v), v)$ in R_2 for which $u = g(v)$ and $f(u) < v$. For if there were, we could use the fact that g is Darboux to find $w \in (f(r), v)$ for which $g(w) = r_2'$ and then we would have a crossing-configuration, namely $(\{(u, f(u)), (r, f(r)), (r_2', f(r_2'))\}, \{(g(v), v), (g(w), w)\})$ contained in R_1, see Lemma 1 and Figure **??**.

Lemma 2 *There is a rectangle* $R_3 = [r, r'''] \times [f(r), z_2''] \subset R_2$ *such that* $\text{graph}_X(f|_{P \cap [r,r''']}) \subset R_3$ *and there are no points* $(u, f(u)), (g(v), v) \in R_3$ *for which* $g(v) \leq u,\ f(u) \leq v$.

Proof: If $R_3 = R_2$ does not satisfy the lemma, then there is $(u_0, f(u_0))$ and $(g(v_0), v_0) \in R_2$ for which

$$g(v_0) \leq u_0 \quad \text{and} \quad f(u_0) \leq v_0.$$

Choose $f(r) < z_2'' < f(u_0)$ and use the fact that r is a continuity point of $f|_P$ to find $r < r''' < g(v_0)$ for which $\text{graph}_X(f|_{P \cap [r,r''']}) \subset R_3 = [r, r'''] \times [f(r), z_2'']$.

Suppose now that we can find $(u, f(u))$ and $(g(v), v)$ in R_3 for which $g(v) \leq u$ and $f(u) \leq v$. Then by the Darboux property for g, we can find a point

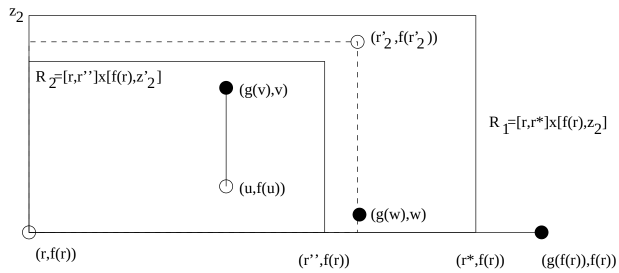

Figure 6: Diagram illustrating occurrence of a crossing-configuration in Case 3 if we can find $u = g(v)$ and $f(u) < r$ in R_2.

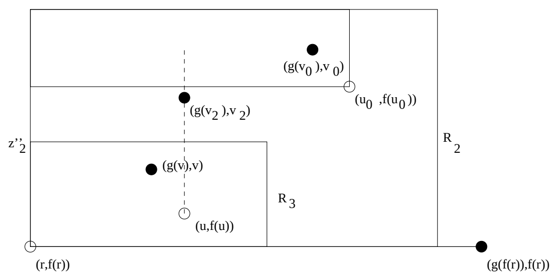

Figure 7: The figure for Lemma 2

$(g(v_2), v_2)$ with $v_2 \in (v, v_0)$ and $g(v_2) = f(u)$ but then $u = g(v_2)$ and $f(u) < v_2$ contradicting the observation made just before this lemma; see Figure **??**. Hence the lemma holds. ■

We now prove that the interval (r, r''') we have constructed is regular; that is, for all $s, t \in P \cap [r, r''']$ with $s < t$, we can find a continuous function $\phi\colon [s, t] \to \mathbf{R}$ for which $\text{graph}_X(\phi) \cap G \cap S = \emptyset$, and $\phi(s) = f(s)$ and $\phi(t) = f(t)$. We know that $(s, f(s))$ and $(t, f(t))$ are in R_3. We can assume that s is not the left endpoint of an interval contiguous to P. If $f(s) \geq f(t)$, then Lemma 2 allows us to choose ϕ to be the affine function joining $f(s)$ and $f(t)$. If $f(s) < f(t)$ we distinguish two cases:

(A): There is $u \in (s, t)$ such that $f(u) \leq f(s)$.

In this case, Lemma 2 implies that G does not meet the shaded region of Figure **??** and we can join $(s, f(s)$, $(u, f(t))$ and $(t, f(t))$ by a piecewise linear function.

(B): There is a sequence (u_n) with $t > u_1 > u_2 > \cdots \to s$ such that $f(t) >$

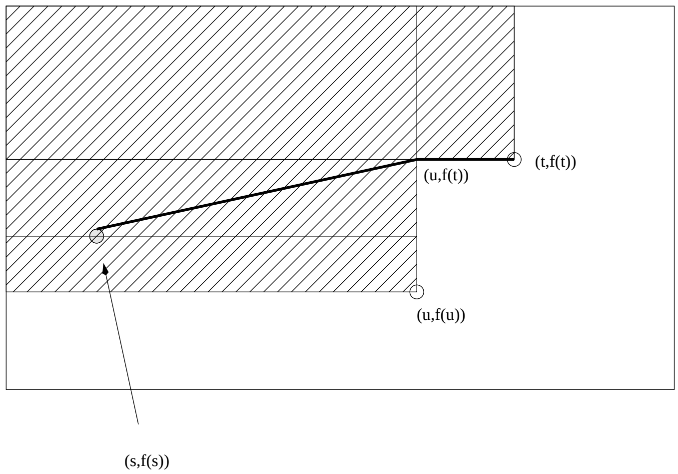

Figure 8: Constructing our continuous function ϕ in Case (A).

$f(u_1) > f(u_2) > \cdots \to f(s)$. In this case we can join the points $(t, f(t))$, $(u_1, f(t))$, $(u_2, f(u_1))$, $(u_3, f(u_2)), \ldots$ piecewise linearly, see Figure **??**. ■

2 Open problems

There are a couple of natural questions suggested by this result:

1. is the graph of the composition of two Darboux Baire-1 functions connected?
2. does a similar result hold for the composition of n Darboux Baire-1 functions when $n \geq 3$?

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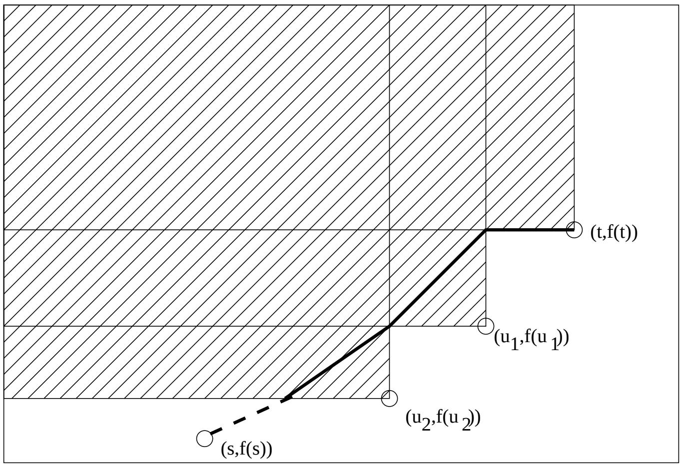

Figure 9: Constructing our continuous function ϕ in Case (B).