Geometric Measure Theory*

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Geometric measure theory is an area of analysis concerned with solving geometric problems via measure theoretic techniques. The canonical motivating physical problem is probably that investigated experimentally by Plateau in the nineteenth century [3]: given a boundary wire, how does one find the (minimal) soap film which spans it? Slightly more mathematically, given a boundary curve, find the surface of minimal area spanning it. The many different approaches to solving this problem have found utility in most areas of modern mathematics and geometric measure theory is no exception: techniques and ideas from geometric measure theory have been found useful in the study of partial differential equations, the calculus of variations, harmonic analysis, and fractals.

Successes in the field include: classifying the structure of singularities in soap films (see [18], together with the fine descriptive article [4]); showing that the standard 'double bubble' is the optimal shape for enclosing two prescribed volumes in space [13], and developing powerful computer software for modelling the evolution of surfaces under the action of physical forces [7].

The main reference text for the subject is still Federer's comprehensive book [11]. It is very densely written and Morgan's book [15] serves as a useful guide through it. Federer's colloquium lectures [10] provide a comprehensive overview of the subject and contain a summary of the main results in his book [11]. More recent books include Simon [17], which contains an introduction to the theory of varifolds and Allard's regularity theorem, and Mattila's book [14] which includes information about tangent measures and their uses. Both of these books are also suitable as introductions to the area. For a slightly different slant, the book by Evans and Gariepy [9], discusses applications of some of the ideas of geometric measure theory in the theory of Sobolev Spaces and functions of bounded variation.

Many variational problems are solved by enlarging the allowed class of solutions, showing that in this enlarged class a solution exists, and then showing that the solution possesses more regularity than an arbitrary element of the enlarged class. Much of the work in geometric measure theory has been directed towards placing this informal description on a formal footing appropriate for the study of surfaces.

The key concept underlying the whole theory is that of rectifiability: this is a measure theoretic notion of smoothness. A set E in Euclidean n-space, $\mathbf{R}^n$, is

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(countably) *m-rectifiable* if there is a sequence of C^1 maps, $f_i\colon \mathbf{R}^m \to \mathbf{R}^n$, such that

$$\mathcal{H}^m\left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbf{R}^m)\right) = 0.$$

It is *purely m-unrectifiable*, if for all C^1 maps $f\colon \mathbf{R}^m \to \mathbf{R}^n$,

$$\mathcal{H}^m(E \cap f(\mathbf{R}^m)) = 0.$$

(Here, $\mathcal{H}^m$ denotes the m-dimensional Hausdorff (outer) measure defined by

$$\mathcal{H}^m(E) = \sup_{\delta>0} \inf \left\{ c_m \sum_i |E_i|^m : E \subset \cup_i E_i,\ |E_i| < \delta \text{ for all } i \right\},$$

where $|\cdot|$ denotes the diameter and the constant, c_m, is chosen so that, when $m = n$, Hausdorff measure is just the usual Lebesgue measure.)

A basic decomposition result states that any set $E \subset \mathbf{R}^n$ of finite m-dimensional Hausdorff measure may be written as the union of an m-rectifiable set and a purely m-unrectifiable set, with the intersection necessarily having $\mathcal{H}^m$-measure zero.

In practice, the definition of rectifiability is usually used with Lipschitz maps replacing C^1 maps: it may be shown that this doesn't change anything, see [14, Theorem 15.21].

A standard example of a 1-rectifiable set in the plane is a countable union of circles whose centres are dense in the unit square and with radii having a finite sum; the closure of the resulting set contains the unit square, and yet, as indicated below, the set itself still has 'tangents' at $\mathcal{H}^1$-almost every point. An example of a purely 1-unrectifiable set is given by taking the cross-product of the 1/4-Cantor set with itself. (The 1/4-Cantor set is formed by removing 2^k intervals of diameter 4^{-k} (rather than 3^{-k}) at each stage of its construction.)

The main importance of the class of rectifiable sets is that it possesses many of the nice properties of the smooth surfaces which we are seeking to generalise. For example, although, in general, classical tangents may not exist (consider the circle example above), an m-rectifiable set will possess a unique approximate tangent at $\mathcal{H}^m$-almost every point: an m-dimensional linear subspace, V, of $\mathbf{R}^n$ is an *approximate m-tangent plane* for E at x if

$$\limsup_{r\to 0} \frac{\mathcal{H}^m(E \cap B(x,r))}{r^m} > 0$$

and for all $0 < s < 1$

$$\lim_{r\to 0} \frac{\mathcal{H}^m(\{y \in E \cap B(x,r) : \operatorname{dist}(y-x, V) > s|y-x|\})}{r^m} = 0.$$

Conversely, if $E \subset \mathbf{R}^n$ has finite $\mathcal{H}^m$-measure and for $\mathcal{H}^m$-almost every $x \in E$, it has an approximate m-tangent plane, then E is m-rectifiable.

Often one is faced with the task of showing that some set, which is a solution to the problem under investigation, is in fact rectifiable, and hence possesses some smoothness. A major concern in geometric measure theory, is finding criteria which guarantee rectifiability. One of the most striking results in this direction is the Besicovitch-Federer projection theorem which illustrates the stark difference between rectifiable and unrectifiable sets. A basic version of it states that if $E \subset \mathbf{R}^n$ is a purely m-unrectifiable set of finite m-dimensional Hausdorff measure, then for almost every orthogonal projection P of $\mathbf{R}^n$ onto an m-dimensional linear subspace, $\mathcal{H}^m(P(E)) = 0$. (It is not particularly difficult to show that in contrast, m-rectifiable sets will have projections of positive measure for almost every projection.) This deep result was first proved for 1-unrectifiable sets in the plane by Besicovitch, and later extended to higher dimensions by Federer. Recently, White [19] has shown how the higher dimensional version of this theorem follows via an inductive argument from the planar version.

It is also possible (and useful) to define a notion of rectifiability for Radon (outer) measures: a Radon measure μ is said to be *m-rectifiable* if it is absolutely continuous with respect to m-dimensional Hausdorff measure and there is an m-rectifiable set E for which $\mu(\mathbf{R}^n \setminus E) = 0$. The complementary notion of a measure μ being *purely m-unrectifiable* is defined by requiring that μ is singular with respect to all m-rectifiable measures. Thus, in particular, a set E is m-rectifiable if and only if $\mathcal{H}^m \lfloor E$ (the restriction of $\mathcal{H}^m$ to E) is m-rectifiable: this allows us to study rectifiable sets through m-rectifiable measures.

It is common in analysis to construct measures as solutions to equations, and we would like to be able to deduce something about the structure of these measures (for example, that they are rectifiable). Often the only *a priori* information we have is some limited metric information about the measure, perhaps we know how the mass of small balls grows with radius. Probably the strongest known result in this direction is Preiss' density theorem [16] (see also [14] for a lucid sketch of the proof). This states that if μ is a Radon measure on $\mathbf{R}^n$ for which $\lim_{r\to 0} \frac{\mu(B(x,r))}{r^m}$ exists and is positive and finite for μ-almost every x, then μ is m-rectifiable.

A natural approach to solving a minimal surface problem would be to take a sequence of approximating sets whose areas are decreasing and finally extract a convergent subsequence with the hope that the limit would possess the required properties. Unfortunately, the usual notions of convergence for sets in Euclidean spaces are not suited to this. The theory of currents, introduced by de Rham and extensively developed by Federer and Fleming in [12] (see [10] for a comprehensive outline of the theory and [11] for details), was developed as a way around this obstacle for oriented surfaces. In essence, currents are generalised surfaces, obtained by viewing an m-dimensional (oriented) surface as defining a continuous linear functional on the space of differential forms with compact support of degree m. Using the duality with differential forms, it is then possible to define many natural operations on currents. For example the *boundary* of an m-current can be defined to be the $(m-1)$-current, ∂S, which is given via the exterior derivative for differential forms by setting

$$\partial S(\phi) = S(d\phi)$$

for a differential form ϕ of degree $(m-1)$.

Of particular importance is the class of *m-rectifiable currents*: this consists of those currents which can be written as

$$S(\phi) = \int \langle \xi(x), \phi(x) \rangle \theta(x) d\mathcal{H}^m \lfloor R(x)$$

where R is an m-rectifiable set with $\mathcal{H}^m(R) < \infty$, $\theta(x)$ is a positive integer-valued function with $\int \theta d\mathcal{H}^m \lfloor R < \infty$ and $\xi(x)$ can be written as $v_1 \wedge \cdots \wedge v_m$ with $v_1, \ldots, v_m$ forming an orthonormal basis for the approximate tangent space of R at x for $\mathcal{H}^m$-almost every $x \in R$. (That is, $\xi(x)$ is a unit simple m-vector whose associated m-dimensional vectorspace is the approximate tangent space of R at x for $\mathcal{H}^m$-almost every $x \in R$.) The *mass* of a current given in this way is defined by $\mathbf{M}(S) = \int \theta(x) d\mathcal{H}^m \lfloor R(x)$. If the boundary of an m-rectifiable current is itself an $(m-1)$-rectifiable current, then the m-current is said to be an *integral current*. These are the class of currents suitable for investigating Plateau's problem. Federer and Fleming's celebrated closure theorem says that on a not too wild compact domain (it should be a Lipschitz retract of some open neighbourhood of itself), those integral currents S on the domain which all have the same boundary T, an $(m-1)$-current with finite mass, and for which $\mathbf{M}(S)$ is bounded above by some constant c, forms a compact set. (The topology is that generated by the *integral flat distance* defined for m-integral currents S_1, S_2 by

$$\mathcal{F}_K(S_1, S_2) = \inf\{\mathbf{M}(U) + \mathbf{M}(V) : U + \partial V = S_1 - S_2\}$$

where the infimum is over U and V such that U is an m-rectifiable current on K and V is an $(m+1)$-rectifiable current on K.) In particular, if the constant c is chosen large enough so that this set is non-empty, then we can deduce the existence of a mass-minimising current with the given boundary T.

The theory of currents is ideally suited for investigating oriented surfaces but for unoriented surfaces, problems arise. The theory of varifolds was initiated by Almgren and extensively developed by Allard [1] (see also [2] for a nice survey) as an alternative notion of surface which didn't require an orientation. An *m-varifold* on an open subset of $\mathbf{R}^n$, Ω, is a Radon measure on $\Omega \times G(n,m)$. ($G(n,m)$ denotes the *Grassmanian manifold* consisting of m-dimensional linear subspaces of $\mathbf{R}^n$.) The space of m-varifolds is equipped with the weak topology given by saying that $\nu_i \to \nu$ if and only if $\int f d\nu_i \to \int f d\nu$ for all compactly supported, continuous real-valued functions on $\Omega \times G(n,m)$. Given an m-varifold ν, we associate a Radon measure on Ω, $\|\nu\|$, by setting $\|\nu\|(A) = \nu(A \times G(n,m))$ for $A \subset \Omega$. As a partial converse, given an m-rectifiable measure $\|\mu\|$, we can associate an *m-rectifiable varifold* μ by defining for $B \subset \Omega \times G(n,m)$

$$\mu(B) = \|\mu\|\{x : (x, T_x) \in B\}$$

where T_x is the approximate tangent plane at x. The *first variation* of an m-varifold, ν, is a map from the space of smooth compactly supported vectorfields on Ω to $\mathbf{R}$ defined by

$$\delta\nu(X) = \int \langle X(x), V\rangle d\nu(x, V).$$

If $\delta\nu = 0$, then the varifold is said to be *stationary.* The idea is that the variation measures the rate of change in the 'size' of the varifold if it is perturbed slightly. A key result in the theory of varifolds is Allard's regularity theorem which states that stationary varifolds which satisfy a growth condition (detailed below) are supported on a smooth manifold. More precisely: For all $\epsilon \in (0,1)$, there are constants $\delta > 0$, $C > 0$ such that whenever $a \in \mathbf{R}^n$, $0 < R < \infty$ and ν is an m-dimensional stationary varifold on the open ball $U(a, R)$ with

1. $a \in \mathrm{spt}\nu$;
2. $\lim_{r\to 0} \frac{\|\nu\|(B(a,r))}{c_m r^m}$ existing and at least one for $\|\nu\|$-almost every x; and
3. $\|\nu\|(B(a,R)) \le c_m(1+\delta)R^m$,

then $\mathrm{spt}(\|\nu\|) \cap B(a,(1-\epsilon)R)$ is a continuously diffentiable embedded m-submanifold of $\mathbf{R}^n$, and $\mathrm{dist}(T_x, T_y) \le C(r|x-y|)^{1-\epsilon}$ for points in this submanifold. (The distance between the tangent spaces is given by the distance between their corresponding orthogonal projections.) This is a theorem which gives much more than just rectifiability; it gives information about the degree of smoothness as well. See Simon [17] for some variants and a proof of this result.

Given the success of the theory in Euclidean spaces, it is natural to ask whether a similar theory holds in more general spaces [8]. There are many difficulties to be overcome but recent papers of Ambrosio and Kircheim [5, 6] suggest that it may be possible. Despite the many successes of the subject, there are many problems yet to be resolved and geometric measure theory will be a major subdiscipline of analysis for the foreseeable future.

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