N-FOLD SUMS OF CANTOR SETS

KATHRYN E. HARE AND TOBY C. O'NEIL

Abstract. We generalize the Newhouse gap lemma by finding a geometric condition which ensures that N-fold sums of compact sets, which might even have thickness zero, are intervals. We also obtain a new proof of a lower bound on the thickness of the sum of two Cantor sets.

1. Introduction

By a Cantor set we mean a compact, totally disconnected, perfect subset of the real line. All Cantor sets can be constructed in a similar fashion to the classical middle-third Cantor set, but rather than using the ratio $1/3$ at each step, we allow the removed intervals to be variable in length and not necessarily centred.

The question of whether the arithmetic sum of two or more Cantor sets contains an interval has arisen in number theory, harmonic analysis and dynamical systems. Recent papers studying the problem include [6], [2], [1] and [3] where it was shown that generically the sum of two regular Cantor sets contains an interval if and only if the Hausdorff dimension of the two Cantor sets sum is greater than one.

In contrast, deterministic results are less clearcut. A useful geometric condition which allows one to conclude that the sum of two Cantor sets contains an interval is provided by the Newhouse gap lemma [4] (see also [5], p. 53). By obtaining a lower bound on the thickness (see Definition 2.4) of the sum of two Cantor sets, Astels [1] was able to extend this to N-fold sums of Cantor sets of positive thickness. Here we show that a weaker geometric condition, which allows for the possibility of sets of thickness zero, suffices to prove that N-fold sums of compact sets contain an interval. We also give a simpler proof of Astels' thickness formula.

2. Results and definitions

Definitions 2.1. For a compact set $C \subseteq \mathbf{R}$, a **gap** of C is a non-empty connected component of the complement of C. A **bridge** of C is a non-empty, closed interval whose left and right endpoints are left-isolated and right-isolated, respectively, in C. A bridge and gap which share a boundary point will be called a **bridge-gap pair**. We will call a bridge-gap pair (B, G) **maximal** if the bridge B is a maximal interval which contains no point of a gap whose length is at least the length of G. We let $g(B)$ be the length (possibly infinite) of the smallest component of the complement of C adjacent to the bridge B. Notice that when (B, G) is a maximal bridge-gap pair, then $g(B) = |G|$ (where $|\cdot|$ denotes the diameter).

Definition 2.2. For $N \geq 1$ we say that non-empty compact sets $C_1, ..., C_N$ satisfy the **N-fold bridge-gap condition** if whenever (B_i, G_i) are maximal bridge-gap

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pairs of C_i for $i = 1, ..., N$ with $\min\{|G_i| : 1 \le i \le N\} < \infty$, then

$$\min_k \{|B_k| + |G_k|\} \le \sum_{1\le i\le N} |B_i| \, .$$

Our main result is:

Theorem 2.3. *Suppose that $C_1, \ldots, C_N$ are non-empty compact sets for which the N-fold bridge-gap condition holds. Then $C_1 + \cdots + C_N$ is an interval.*

Earlier work on this problem, estimated the size of the sum sets in terms of the thickness of the constituent sets.

Definition 2.4. The **thickness** of a non-empty compact set C is defined as

$$\tau(C) \equiv \inf \left\{ \frac{|B|}{|G|} : (B, G) \text{ maximal bridge-gap pairs with } |G| < \infty \right\}.$$

Note that if $\tau(C)$ is infinite, then C is an interval (possibly degenerate), and if C has an isolated point, then $\tau(C) = 0$ except if C is only a single point. The Newhouse gap lemma states that if two compact sets, C_1 and C_2, satisfy $\tau(C_1)\tau(C_2) \ge 1$, and neither set is contained in a translate of a gap of the other, then $C_1 + C_2$ is an interval. Very recently, Astels [1] found a lower bound on the thickness of $C_1 + C_2$ in terms of $\tau(C_1)$ and $\tau(C_2)$, and as a consequence he was able to extend the gap lemma to N-fold sums of Cantor sets: our result above extends Astels' theorem (as we will show in section 3).

Theorem 2.5 (Astels [1])**.** *Suppose $C_1, ..., C_N$ are Cantor sets having thicknesses $\tau_1, ..., \tau_N$ respectively. Suppose that O_j is a bounded gap of C_j of maximal diameter, and*

$$|C_k + \cdots + C_N| \ge |O_{k-1}| \ \text{ and } \ |C_{k-1}| \ge \max\{|O_k|, ..., |O_N|\} \tag{2.1}$$

for all $2 \le k \le N$. Let $\gamma_i = \tau_i/(\tau_i + 1)$.

(1) *If $\sum_{i=1}^N \gamma_i < 1$, then*

$$\tau(C_1 + \cdots + C_N) \ge \frac{\sum_{i=1}^N \gamma_i}{1 - \sum_{i=1}^N \gamma_i}.$$

(2) *If $\sum_{i=1}^N \gamma_i \ge 1$, then $C_1 + \cdots + C_N$ is an interval.*

We outline briefly how Astel's second condition implies the N-fold bridge-gap condition for bounded gaps.

Suppose that (B_i, G_i) are maximal bridge-gap pairs for C_i with $\max\{|G_i| : 1 \le i \le N\} < \infty$. Then

$$\frac{\tau_i}{\tau_i + 1} \le \frac{|B_i|}{|G_i| + |B_i|},$$

and so the assumption that $\sum_{i=1}^N \gamma_i \ge 1$ implies that

$$1 \le \sum_{i=1}^N \frac{|B_i|}{|G_i| + |B_i|}.$$

Hence

$$\min_k \{|B_k| + |G_k|\} \le \sum_{1\le i\le N} |B_i|$$

and the N-fold bridge-gap condition holds for (B_i, G_i), $i = 1, \ldots, N$.

3. Proof of the Main Theorem

We will proceed by examining the relationship between a gap in the sum of two sets and gaps in the original sets .

Lemma 3.1. *Let C_1 and C_2 be non-empty compact sets. Let $B = [b, c]$ be a bridge of $C_1 + C_2$. Then the collection of pairs of gaps (a_i, b_i) of C_i which satisfy any of the following four conditions is both non-empty and finite:*

(1) $a_1 + a_2 < b \leq \min\{a_1 + b_2, b_1 + a_2\} \leq c$;

(2) $b \leq \max\{a_1 + b_2, b_1 + a_2\} \leq c < b_1 + b_2$;

(3) $b \leq a_1 + a_2 \leq c < \min\{a_1 + b_2, b_1 + a_2\}$;

(4) $\max\{a_1 + b_2, b_1 + a_2\} < b \leq b_1 + b_2 \leq c$.

(We use the convention that if, for a pair of gaps, $(-\infty + \infty)$ occurs in one of the expressions (1)–(4), then the corresponding inequality is not satisfied.)

Proof. We can find right-isolated points $a_i \in C_i$ for $i = 1, 2$ with $c = a_1 + a_2$. The gaps of C_i with left endpoints a_i, $i = 1, 2$, clearly satisfy (3). Hence the collection is non-empty.

In order to show finiteness, we first note that if $g(B) = \infty$, then B contains $C_1 + C_2$ and the only gap pairs which can possibly satisfy (1)–(4) are the unbounded gap pairs of C_1 and C_2; there are only four of these.

If $g(B) < \infty$, then, by symmetry, it suffices to show that the collection of pairs of gaps satisfying either (1) or (3) is finite. We note that we always have $a_i < \infty$ and $b_i > -\infty$ for $i = 1, 2$.

Gap pairs satisfying (1): In this case, $a_1 + a_2 \leq b - g(B)$, and both $b \leq a_1 + b_2$ and $b \leq a_2 + b_1$. In particular, neither a_1 nor a_2 are $-\infty$. Rearranging these inequalities gives $\min\{b_1 - a_1, b_2 - a_2\} \geq g(B) > 0$. Hence there are only finitely many gaps satisfying (1).

Gap pairs satisfying (3): In this case, since $b \leq a_1 + a_2$, neither a_1 nor a_2 are $-\infty$. Moreover, $a_1 + a_2 \leq c$, and both $a_1 + b_2 \geq c + g(B)$ and $a_2 + b_1 \geq c + g(B)$. Rearranging gives $\min\{b_1 - a_1, b_2 - a_2\} \geq g(B) > 0$ and again there are only finitely many gaps. □

The following proposition is the main step in deriving our results.

Proposition 3.2. *Let C_1 and C_2 be non-empty compact sets. Let B be a bridge of $C_1 + C_2$. Then there are bridges B_i of C_i satisfying $|B_1| + |B_2| \leq |B|$ and*

$$g(B) \leq \min\{g(B_1) - |B_2|, g(B_2) - |B_1|\}.$$

(Note that both sides of this expression may be infinite.)

Proof. If $g(B) = +\infty$, then B contains $C_1 + C_2$, and we set B_i to be the unique bridge containing C_i for $i = 1, 2$. Clearly, $|B_1| + |B_2| = |B|$ and, as $g(B_i) = +\infty$, the inequality is also satisfied and we are done.

We now suppose that $g(B) < \infty$.

Assume $B = [b, c]$ and consider the collection, $\mathcal{C}$, of gap pairs (G_1, G_2) (G_i a gap of C_i) satisfying one of (1)–(4) of Lemma 3.1 and which minimise the sum $|G_1| + |G_2|$.

We note that $\mathcal{C}$ cannot consist only of pairs of unbounded gaps: For if it does, then the sum is infinite and, as $g(B) < \infty$, one of the endpoints of B is the endpoint of a bounded gap. Suppose it is c (the argument for b is similar). Then we can find right-isolated points $a_i \in C_i$ for which $a_1 + a_2 = c$. Let b_i be the right endpoints (possibly $+\infty$) of the corresponding gaps. If a_i is the right endpoint of C_i for $i = 1, 2$, then c cannot be the left endpoint of a bounded gap, thus one of the (a_i, b_i) is a bounded gap. But then the pair $((a_1, b_1), (a_2, b_2))$ satisfies (3) of Lemma 3.1 and we have a contradiction.

Choose $(G_1, G_2) = ((a_1, b_1), (a_2, b_2)) \in \mathcal{C}$ such that for any other pair $(G_1', G_2') \in \mathcal{C}$,

$$\min\{|G_1|, |G_2|\} \leq \min\{|G_1'|, |G_2'|\}.$$

(Notice that, by the preceding, at least one of G_1, G_2 is a bounded gap.)

We will use the following two constructions.

(A) If $a_1 + a_2 < b$ and $b \leq a_1 + b_2 \leq c$ we construct a bridge B_1 of C_1 such that $|B_1| \leq a_1 + b_2 - b$, $|G_2| \geq |B_1| + g(B)$ and $g(B_1) = |G_1|$: To do this first note that $-\infty < a_1, b_2 < \infty$, and define

$$\beta_1 = \min\{x \in C_1 : x + b_2 \geq b\};$$

by assumption $\beta_1 \leq a_1$. Set $B_1 = [\beta_1, a_1]$. Then

$$|B_1| = a_1 + b_2 - (\beta_1 + b_2) \leq a_1 + b_2 - b.$$

If $a_2 = -\infty$, then clearly $a_1 + a_2 \leq b - g(B)$. If $a_2 > -\infty$, then $a_1 + a_2 \in C_1 + C_2$ and still $a_1 + a_2 \leq b - g(B)$. Thus

$$|G_2| - |B_1| = \beta_1 + b_2 - (a_1 + a_2) \geq g(B).$$

If $z \in C_1$ and $z + b_2 < b$, then $z + b_2 \leq b - g(B)$ and therefore β_1 is left-isolated. Let $H_1 = (\alpha_1, \beta_1)$ be the adjacent gap of C_1. As $\alpha_1 < \beta_1$, $\alpha_1 + b_2 < b$, and thus the pair of gaps (H_1, G_2) satisfies (4). If $|G_1| + |G_2| < \infty$, then their minimality ensures that $|H_1| \geq |G_1|$. If $|G_1| + |G_2| = \infty$ and $|H_1| < \infty$, then $|G_1| < \infty$ and $|G_2| = \infty$, otherwise we have a contradiction, and again, by minimality, $|H_1| \geq |G_1|$. If $|H_1| = \infty$, then it is clear that $|H_1| \geq |G_1|$. Thus $g(B_1) = |G_1|$.

(B) If $b \leq a_1 + a_2 \leq c$ and $a_1 + b_2 > c$ we construct a bridge B_1 of C_1 such that $B_1 = [\beta_1, a_1]$, $\beta_1 + a_2 \geq b$, $|G_2| \geq |B_1| + g(B)$ and $g(B_1) = |G_1|$: To this end first note that $-\infty < a_1, a_2 < \infty$ and set

$$\beta_1 = \min\{x \in C_1 : x + a_2 \geq b,\ x + b_2 \geq c + g(B)\},$$

(if $b_2 = \infty$, then the second condition is vacuous). Let $B_1 = [\beta_1, a_1]$ and observe, since β_1 is left-isolated in C_1, we have a gap $H_1 = (\alpha_1, \beta_1)$ of C_1 with right-endpoint β_1. Our definition of β_1 ensures that

$$b \leq \beta_1 + a_2 \leq a_1 + a_2 \leq c < \beta_1 + b_2,$$

Hence $|G_2| - |B_1| \geq g(B)$. If $|H_1| < \infty$, then $-\infty < \alpha_1 < \infty$ and we have various possibilities: if $\alpha_1 + b_2 \leq \beta_1 + a_2$, then (2) holds for the pair (H_1, G_2); if, instead, $\alpha_1 + b_2 > \beta_1 + a_2$ but $\alpha_1 + a_2 < b$, then (1) holds for (H_1, G_2); finally, if $\alpha_1 + a_2 \geq b$, then as $\alpha_1 < \beta_1$ the choice of β_1 ensures that $\alpha_1 + b_2 \leq c$ and (2) is satisfied for the pair (H_1, G_2). If $|G_1| + |G_2| < \infty$, then minimality ensures that $|H_1| \geq |G_1|$. If $|G_1| + |G_2| = \infty$, then $|G_2| = \infty$ and $|G_1| < \infty$, as otherwise minimality implies that $\infty = |G_1| + |G_2| \leq |H_1| + |G_2| < \infty$, a contradiction. Thus, by the choice of (G_1, G_2), $|H_1| = \min\{|H_1|, |G_2|\} \geq \min\{|G_1|, |G_2|\} = |G_1|$. If $|H_1| = +\infty$, then $|H_1| \geq |G_1|$. Hence in all cases we conclude that $g(B_1) = |G_1|$.

We will also use symmetric versions of these constructions, obtained by exchanging the indices and/or changing direction. For example, a version of *(A)* may assume that $b_1 + b_2 > c$ and $b \leq a_1 + b_2 \leq c$ and construct a bridge B_2 of C_2 such that $|B_2| \leq c - (a_1 + b_2)$, $|G_1| \geq |B_2| + g(B)$ and $g(B_2) = |G_2|$.

We now find the required bridges B_1, B_2 depending upon which of (1)–(4) holds. By symmetry it suffices to treat (1) and (3). First, we split (1) into two cases.

(1) holds and $a_1 + a_2 < b \leq a_1 + b_2,\ b_1 + a_2,\ b_1 + b_2 \leq c$

Here $-\infty < a_1, a_2, b_1, b_2 < \infty$ and we define B_1 using Construction *(A)* and B_2 via *(A)* but with interchanged indices. Then

$$g(B_1) = |G_1| \geq |B_2| + g(B),\ g(B_2) = |G_2| \geq |B_1| + g(B)$$

and

$$|B_1| + |B_2| \leq (a_1 + b_2 - b) + (a_2 + b_1 - b) \leq b_1 + b_2 - b \leq |B|\,.$$

(1) holds and $a_1 + a_2 < b \leq a_1 + b_2 \leq c < b_1 + b_2$

Define B_1 using Construction *(A)* and B_2 with Construction *(A)* in the alternative version described above. It is easy to check that the desired properties hold.

(3) holds: $b \leq a_1 + a_2 \leq c < a_1 + b_2,\ b_1 + a_2$

Define B_1 by Construction *(B)* and B_2 by Construction *(B)* with interchanged indices. Then

$$g(B_1) = |G_1| \geq |B_2| + g(B) \text{ and } g(B_2) = |G_2| \geq |B_1| + g(B),$$

so we need only check that $|B_1| + |B_2| \leq |B|$. Assume otherwise, i.e.,

$$a_1 + a_2 - (\beta_1 + \beta_2) > c - b.$$

Since $a_1 + a_2 \leq c$ this implies that $\beta_1 + \beta_2 < b$. Choose $\gamma_i \in B_i \bigcap C_i$ such that $\gamma_i \geq \beta_i$, $\gamma_1 + \gamma_2 \leq b - g(B)$ and $\gamma_1 + \gamma_2$ is maximal. Construction *(B)* ensures that both $\beta_1 + a_2,\ \beta_2 + a_1 \geq b$, thus $\gamma_i < a_i$. The choice of γ_i implies that there exist (bounded) gaps $D_i = (\gamma_i, \delta_i)$ of C_i such that $\delta_i \leq a_i$ and $\delta_1 + \gamma_2,\ \delta_2 + \gamma_1 \geq b$. Hence the gaps D_1, D_2 satisfy (1) of Lemma 3.1. But $D_i \subseteq B_i$ and thus

$$|D_1| + |D_2| \leq |B_1| + |B_2| \leq |G_2| - g(B) + |G_1| - g(B),$$

contradicting the minimality of $|G_1| + |G_2|$.

This concludes the proof of the proposition. □

Corollary 3.3. *Suppose that $C_1, \ldots, C_N$ are non-empty compact sets and B is a bridge of $C_1 + \cdots + C_N$. Then we can find bridges B_i of C_i for $i = 1, \ldots, N$ such that $\sum_{i=1}^{N} |B_i| \leq |B|$ and*

$$g(B) \leq \min \left\{ g(B_k) - \sum_{i \neq k} |B_i| : 1 \leq k \leq N \right\}. \tag{3.1}$$

Proof. Note if $N = 1$, the sum on the righthand side of the inequality is zero, and the result is obvious. For $N \geq 2$, we apply Proposition 3.2 iteratively to estimate: the size of a bridge of $C_1 + \cdots + C_N$ in terms of bridges of C_1 and $C_2 + \cdots + C_N$; a bridge of $C_2 + \cdots + C_N$ in terms of bridges of C_2 and $C_3 + \cdots + C_N$;. . . ; and a bridge of $C_{N-1} + C_N$ in terms of bridges of C_{N-1} and C_N. □

We are now in a position to prove Theorem 2.3.

Proof of Theorem 2.3. If $C_1+\cdots+C_N$ contains a bounded gap, then we can find a bridge B of $C_1+\cdots+C_N$ for which $0<g(B)<\infty$ and thus $|B|<|C_1+\cdots+C_N|$. Hence, by Corollary 3.3, we can find bridges B_i of C_i for $i=1,\dots,N$, with

$$|B_1|+\cdots+|B_N|\le |B|<|C_1+\cdots+C_N| \tag{3.2}$$

and

$$0<g(B)\le \min\left\{g(B_k)-\sum_{i\neq k}|B_i| : 1\le k\le N\right\}. \tag{3.3}$$

Choose G_i a gap of C_i adjacent to B_i with $g(B_i)=|G_i|$. Then (B_i,G_i) is a maximal bridge-gap pair for C_i, $i=1,\dots,N$. The only way (3.3) cannot contradict the N-fold bridge-gap condition is if $|G_i|=\infty$ for all i, but then $B_i\supset C_i$ which contradicts (3.2). Thus $C_1+\cdots+C_N$ does not contain a bounded gap. □

Proof of Astels' Theorem 2.5. If $N=1$, then there is nothing to prove, so we suppose that $N\ge 2$. We first note that Astels' inequality (2.1) concerning the largest gaps ensures that if we have maximal bridge-gap pairs (B_i,G_i) of C_i for $i=1,\dots,N$ with $\max\{|G_i|:1\le i\le N\}=\infty$ and $\min\{|G_i|:1\le i\le N\}<\infty$, then the N-fold bridge-gap condition holds for these bridge-gap pairs: For, choose $1\le i\le N$ such that $|G_i|=\infty$ and for $j<i$, $|G_j|<\infty$. If for $i\le k\le N$, $|G_k|=\infty$, then $|B_k|=|C_k|$ and $i\ge 2$. Thus

$$|B_{i-1}|+|G_{i-1}|\le |B_{i-1}|+|O_{i-1}|\le |B_{i-1}|+\sum_{k=i}^{N}|C_k|\le \sum_{k=1}^{N}|B_k|.$$

Otherwise, we find $i<k\le N$ with $|G_k|<\infty$ and see that $|B_k|+|G_k|\le |B_k|+|O_k|\le |B_k|+|C_i|$. In either case the N-fold bridge-gap condition holds.

Proof of (1): Consider a maximal bridge-gap pair (B,G) of $C_1+\cdots+C_N$ for which $|G|<\infty$. Then, by Corollary 3.3, we can find bridges B_i of C_i for which $|B_1|+\cdots+|B_N|\le |B|$ and

$$0<|G|=g(B)\le \min\left\{g(B_k)-\sum_{i\neq k}|B_i| : 1\le k\le N\right\}. \tag{3.4}$$

Choose G_i, a gap of C_i, so that (B_i,G_i) is a maximal bridge-gap pair, and thus $g(B_i)=|G_i|$. If any of the G_i are unbounded, then, in view of the above, the N-fold bridge-gap condition will hold which contradicts (3.4). Hence $|G_i|<\infty$ for all i. We conclude that $|G_i|\le |B_i|/\tau_i$ for all i. Hence

$$\begin{aligned}\frac{|G|}{|B|} &\le \min\left\{\frac{|G_k|+|B_k|-\sum_{i=1}^N|B_i|}{\sum_{i=1}^N|B_i|} : 1\le k\le N\right\}\\ &\le \min\left\{(1+\tau_k^{-1})\frac{|B_k|}{\sum_{i=1}^N|B_i|} : 1\le k\le N\right\}-1.\end{aligned}$$

It is easy to verify that this minimum is maximised when

$$\frac{|B_k|}{\sum_{i=1}^N|B_i|}=\frac{(1+\tau_k^{-1})^{-1}}{\sum_{i=1}^N(1+\tau_i^{-1})^{-1}}=\frac{\gamma_k}{\sum_{i=1}^N\gamma_i}.$$

(This is where we use $\sum_i \gamma_i < 1$.) Simplifying gives us the bound

$$\frac{|B|}{|G|} \geq \frac{\sum_i \gamma_i}{1 - \sum_{i=1}^N \gamma_i}$$

from which *(1)* follows.

Proof of (2): We have already shown in section 2 that for bridge-gap pairs with bounded gaps, (2) implies the N-fold bridge-gap condition. If any of the gaps are unbounded, then we already noted that the N-fold bridge-gap condition is satisfied. In either event, the N-fold bridge-gap condition is satisfied and we conclude that $C_1 + \cdots + C_N$ is an interval. □

3.1. **Example.** As we have just shown, the hypotheses in Astels' theorem imply that the N-fold bridge-gap condition is satisfied. The following example shows that the thickness assumption is not necessary. Indeed, Cantor sets can satisfy the bridge-gap condition and yet each have thickness zero.

Define $C_1 = \bigcup_k C_1^{(k)}$ as follows: Set $n_1 = 3$ and choose odd integers $z_k \neq 1$ tending to infinity. Inductively define $n_{k+1} = n_k z_k$. Let $C_1^{(1)} = [0, 1/n_1] \bigcup [1 - 1/n_1, 1]$ and assume inductively that $C_1^{(k)} = \bigcup_i [r_i, r_i + 1/n_k]$ where $r_i + 1/n_k < r_{i+1}$. For k even define

$$C_1^{(k+1)} = \bigcup_i \left([r_i, r_i + \frac{1}{n_{k+1}}] \bigcup [r_i + \frac{1}{n_k} - \frac{1}{n_{k+1}}, r_i + \frac{1}{n_k}] \right)$$

and for k odd define

$$C_1^{(k+1)} = \bigcup_i \bigcup_{j=0}^{(z_k-1)/2} [r_i + \frac{2j}{n_{k+1}}, r_i + \frac{2j+1}{n_{k+1}}].$$

We define C_2 similarly, except that the roles of k even and odd are reversed.

One can easily see that the intervals of $C_1^{(k+1)}$ are of length $1/n_{k+1}$, and the gaps introduced in the Cantor set construction at this step are either the same length again (k odd) or of length $(z_k - 2)/n_{k+1}$ (k even). Thus the thickness of C_1 is $\inf \frac{1}{z_k - 2} = 0$. Similarly the thickness of C_2 is zero.

Notice the length of any bridge of $C_1^{(1)}$ is equal to the length of any gap of $C_2^{(1)}$, thus the 2-fold bridge-gap condition holds at step 1 in the construction. Now assume that it holds for all steps up to k in the construction. Clearly the gaps introduced at step $k+1$ have length less than any interval from step k or earlier. Moreover, any gap from $C_1^{(k+1)}$ for k odd, or $C_2^{(k+1)}$ for k even, is of the same length as any bridge from step $k+1$. Thus the bridge-gap condition is satisfied at all steps. Note also that $|C_k| = 1$ which is more than the length of any gap of either set, so by our theorem $C_1 + C_2 = [0, 2]$.

References

[1] Astels, S., *Cantor sets and numbers with restricted partial quotients*, Trans. Amer. Math. Soc., **352**(2000), 133-170.

[2] Cabrelli, C., Hare, K. and Molter, U., *Sums of Cantor sets*, Ergodic theory and Dynamical systems, **17**(1997), 1299-1313.

[3] Moreira, C. and Yoccoz, J., *Stable intersections of Cantor sets with large Hausdorff dimension*, preprint.

[4] Newhouse, S., *Lectures on dynamical systems*, in Dynamical systems, CIME Lecture, Bressanone, Italy, 1978, Progress in Mathematics, 8, Birkhauser, 1980, 1-114.

[5] Palis, J. and Takens, F. *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge Studies in Adv. Math. 35, 1993.

[6] Palis, J. and Yoccoz, J., *On the arithmetic sum of regular Cantor sets*, Ann. Inst. H. Poincare Anal. Nonlineaire, **14**(1997), 439-456.

Dept of Pure Mathematics, University of Waterloo, Waterloo, Ont. Canada, N2L 3G1

E-mail address: kehare@uwaterloo.ca

Dept of Pure Mathematics, Faculty of Maths and Computing, The Open University, Walton Hall, Milton Keynes, MK7 6AA, UK.

E-mail address: t.c.oneil@open.ac.uk