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Singular sets in the calculus of variations

1 Introduction

We present some results of work carried out with members of the UCL analysis seminar and to appear in [2].

Given a class of real-valued functions on the real line, $\mathcal{F}$ (such as Lipschitz functions, absolutely continuous functions, etc.), the canonical problem in the 1-dimensional calculus of variations is to find conditions on an $L\colon \mathbb{R}^3 \to \mathbb{R}$ to guarantee that for any $a, b, A, B \in \mathbb{R}$, there is a function $u \in \mathcal{F}$ with $u(a) = A$ and $u(b) = B$ so that

$$\int_a^b L(x, u(x), u'(x))\, dx = \inf_{v(a)=A, v(b)=B, v\in\mathcal{F}} \int_a^b L(x, v(x), v'(x))\, dx.$$

Moreover, one would like to know that such minimizing functions u have some additional regularity on $[a, b]$ (such as a continuous derivative) beyond that automatically guaranteed by being in $\mathcal{F}$.

We say that a function $L\colon \mathbb{R}^3 \to \mathbb{R}$ is a *Lagrangian* if:

- L is bounded from below and locally bounded from above;
- L is Borel measurable;
- there is a superlinear function $\omega\colon \mathbb{R} \to \mathbb{R}$ such that $L(x, y, p) \geq \omega(p)$ for all $(x, y, p) \in \mathbb{R}^3$. (Superlinearity of ω means that $\lim_{|p|\to\infty} \omega(p)/|p| = \infty$.)

In 1915, Tonelli [5] gave sufficient conditions to ensure that a Lagrangian $L\colon \mathbb{R}^3 \to \mathbb{R}$ has an absolutely continuous minimizer under fixed boundary conditions. He also proves in this paper that any such minimizer is *regular* in

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that it has a continuous derivative provided we allow values in the extended real line: $u' \in C([a,b], \mathbb{R} \cup \{-\infty, \infty\})$. Thus the singular set of such a minimizer, $\{x : |u'(x)| = \infty\}$, is closed, and since u is absolutely continuous, has linear measure zero.

In the converse direction, Davie shows in [3] that for any compact null set $E \subset \mathbb{R}$, there is a smooth Lagrangian and an appropriate choice of boundary conditions for which any minimizer has infinite derivatives exactly on E.

In [1], Ball and Nadirashvili introduce the notion of the *universal singular set* of a Lagrangian L. A point $(x,y) \in \mathbb{R}^2$ is in the universal singular set of L if there is a choice of boundary conditions so that there is a corresponding minimizer u for which $u(x) = y$ and $|u'(x)| = +\infty$. They show that for Lagrangians of class C^3 the universal singular set is of the first Baire category. In [4], Sychëv lowers the smoothness assumption to $L \in C^1$ and, more importantly, shows that the universal singular set is of zero (2-dimensional) Lebesgue measure.

In light of these results, the question about the "true" size of universal singular sets naturally arises: for example, one can ask whether a universal singular set may have positive length or even Hausdorff dimension larger than one.

2 Results

Our main result concerning the geometric size of universal singular sets is the following.

Theorem 2.1 *Let $\omega\colon \mathbb{R} \to \mathbb{R}$ be even, convex and superlinear. Suppose that an absolutely continuous curve $\gamma(t) = (x(t), y(t))\colon [a,b] \to \mathbb{R}^2$ is such that for almost all $t \in [a,b]$, either*

$$\limsup_{s\to t} \left| \frac{y(s) - y(t)}{x(s) - x(t)} \right| < \infty \tag{1}$$

or

$$\liminf_{s\to t} |x(s) - x(t)|\, \omega\left(\frac{y(s) - y(t)}{x(s) - x(t)} \right) > 0. \tag{2}$$

Then $\{\gamma(t) : t \in [a,b]\}$ meets the universal singular set of any Lagrangian L for which $L(x,y,p) \geq \omega(p)$ in a set of linear measure zero. (When $x(s) - x(t) = 0$, we take $\left|\frac{y(s)-y(t)}{x(s)-x(t)}\right|$ to be zero and $|x(s) - x(t)|\omega\left(\frac{y(s)-y(t)}{x(s)-x(t)}\right)$ to be ∞.)

As an immediate corollary we obtain:

Corollary 2.2 *Graphs of absolutely continuous functions and vertical lines meet the universal singular set of any Lagrangian in a set of linear measure zero.*

In the opposite direction, we construct 'nice' Lagrangians whose universal singular sets are essentially as large as the above results allow. In fact, the Lagrangians have the following special form: we assume that we are given a strictly convex superlinear function $\omega \in C^\infty(\mathbb{R})$ for which $\omega(0) = 0$, and we construct Lagrangians L for which

($\star$) $L(x, y, p) = \omega(p) + F(x, y, p)$ where F satisfies:

- ($\star_1$) $F \in C^\infty(\mathbb{R}^3)$;
- ($\star_2$) $F \geq 0$ and for all $x, y \in \mathbb{R}$, $F(x, y, 0) = 0$;
- ($\star_3$) $p \mapsto F(x, y, p)$ is convex for each fixed (x, y).

Our main result in this direction is given by the following theorem.

Theorem 2.3 *Fix a strictly convex superlinear function $\omega \in C^\infty(\mathbb{R})$ for which $\omega(p) \geq \omega(0) = 0$, and let $S \subset \mathbb{R}^2$ be a purely unrectifiable compact set. Then there is a Lagrangian satisfying ($\star$) whose universal singular set contains S.*

In particular, there are Lagrangians whose universal singular sets have Hausdorff dimension two and contain non-trivial continua. So, in spite of Theorem 2.2, universal singular sets may be rather large.

We complement this result by a more particular example showing that, even when one restricts to compact sets, Theorem 2.3 does not provide a complete answer.

Theorem 2.4 *Fix a strictly convex superlinear function $\omega \in C^\infty(\mathbb{R})$ for which $\omega(p) \geq \omega(0) = 0$. Then there is a rectifiable compact set $S \subset \mathbb{R}^2$ of positive linear measure that is contained in the universal singular set of some Lagrangian satisfying ($\star$).*

The proofs of these results are given in [2]. In this paper we also show that Tonelli's regularity result is stable: absolutely continuous functions that are 'almost' minimizers satisfy a form of Tonelli's regularity — the energy of an 'almost' minimizer u over the set where u has large derivative is controlled by how 'far' u is from being an actual minimizer. We also show that many of our results still hold when one relaxes the notion of what it means to be a minimizer.

References

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