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The dimension of visible sets

1 Introduction

Given a subset E of the plane, Urysohn [7, 8] defined the notion of linear accessibility for a point $p \in E$: p is *linearly accessible* if there is a non-degenerate line segment L that only meets E at the point p. In a sequence of papers, Nikodym [2, 3, 4] investigated the relationship between the set theoretic complexity of E and the set of linearly accessible points.

We consider a variant of this notion: for a compact set in $\mathbb{R}^2$, K, and $x \in \mathbb{R}^2$ we define the visible part of K from the point x by

$$K_x = \{u \in K : [x, u] \cap K = \{u\}\},$$

where $[x, u]$ denotes the closed line segment joining x to u. (This notion also makes sense in $\mathbb{R}^n$, and may also be extended to define visibility from affine subspaces of $\mathbb{R}^n$.)

Clearly, since K_x is always a subset of K, the (Hausdorff) dimension of K_x is always at most the dimension of K. Our interest lies in determining whether any stronger relationship holds between $\dim_H(K)$ and $\dim_H(K_x)$ for 'typical' (in the sense of measure) points x in the plane.

Our first result (from [1]) shows that for sets of small enough Hausdorff dimension, visible sets usually have the same Hausdorff dimension as the original set.

Theorem 1.1 *Let K be a compact subset of the plane. If $\dim_H(K) \leq 1$, then for (Lebesgue) almost every $x \in \mathbb{R}^2$,*

$$\dim_H(K_x) = \dim_H(K).$$

Key Words: visible sets, linearly accessible, connectedness, Hausdorff dimension.
Mathematical Reviews subject classification: Primary 28A80. Secondary 28A78, 31A15.

A version of this result also holds for visibility from points in $\mathbb{R}^n$: in this case 1 is replaced by $n-1$.

The basic idea underlying the proof of this theorem is the observation that around a point x in the complement of K it is possible to project K radially onto a small circle around x that is disjoint from K. On observing that K_x projects to the same set on this circle and denoting this projection by $\pi_x(K)$, we easily verify that

$$\dim_H(\pi_x(K)) \leq \dim_H(K_x) \leq \dim_H(K).$$

We now use the generalised projection framework developed by Peres and Schlag in [6] to show that $\dim_H(\pi_x(K) = \dim_H(K)$ for almost every x. This implies the theorem.

In the same paper [1], we investigate some particular classes of compact connected sets in the plane and show that their visible sets almost surely have dimension one. For example, if K is an embedding into the plane of the graph of a continuous function from S^1 to $\mathbb{R}$, then K_x has dimension one except for at most a single point in the plane. Similarly, if K is a quasicircle, then for *all* points lying in the complement of K, $\dim_H(K_x)$ is one. (A set K is a quasicircle if it is a closed Jordan curve and there is a constant m such that whenever $x, y \in K$, there is an arc of K connecting x and y lying entirely within the ball with centre x and radius $m|x-y|$.)

These examples motivated a general investigation of connected compact sets in the plane that culminated in the proof of the following theorem in [5].

Theorem 1.2 *Let $K \subset \mathbb{R}^2$ be a compact connected set with $\dim_H(K) > 1$. Then for $\frac{1}{2} + \sqrt{\dim_H(K) - \frac{3}{4}} < s \leq \dim_H(K)$,*

$$\dim_H\{x \in \mathbb{R}^2 : \dim_H(K_x) > s\} \leq \frac{\dim_H(K) - s}{s-1}.$$

In particular, this theorem shows that if $K \subset \mathbb{R}^2$ is a compact connected set containing at least two points, then for (Lebesgue) almost every $x \in \mathbb{R}^2$,

$$\dim_H(K_x) \leq \frac{1}{2} + \sqrt{\dim_H(K) - \frac{3}{4}}.$$

The proof of Theorem 1.2 relies on the idea that two distinct visible subsets of K, both of dimension strictly larger than one, must be 'separated' in some sense. Much of the paper [5] is concerned with proving such a result.

The first step in proving such a separation property is to observe that if the visible set K_x has dimension larger than $s > 1$, then there is a measure ν_x living on K_x with $\nu_x(B(u,r)) \leq r^s$ whenever $u \in K_x$ and $0 \leq r \leq 1$, such that for ν_x-almost every point u in K_x, there are points of K_x lying approximately at a distance of at least r^{2-s} above and below u inside the cone with vertex x and just containing $B(u, r)$. After an appropriate uniformisation procedure this allows us to show that in a sufficiently small ball around a ν_x-typical point of K_x, points visible from y are concentrated in a thin strip — this implies that the corresponding ν_y mass of a (small) ball centred on K_x is small. This is the required 'separation' property, and it is now a routine (but long) argument involving energies to use this observation to prove the Theorem.

2 Open Problems

There are many directions for further investigation, and I only give a few of them here.

Problem 2.1 *Is there a version of Theorem 1.2 that holds for totally disconnected compact sets? That is, if K is a totally disconnected set in the plane with Hausdorff dimension strictly larger than one, then is there a positive number c (depending upon $\dim_H(K)$) such that for (Lebesgue) almost every x in the plane,*

$$\dim_H(K_x) \leq \dim_H(K) - c?$$

Problem 2.2 *Is there a compact subset of the plane with Hausdorff dimension larger than one such that for a set of points of positive area, the visible sets each have dimension strictly larger than one?*

Problem 2.3 *Is there a version of Theorem 1.2 that holds in higher dimensions? The only proof that I know of Theorem 1.2 uses the connectedness properties of the plane in an essential way.*

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