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The composition of two derivatives has a fixed point.

Abstract

We show that if $f, g\colon [0,1] \to [0,1]$ are both Darboux Baire-1 functions, then their composition, $f \circ g$, possesses a fixed point.

In [2], Gibson and Natkaniec refer to a problem of K. C. Ciesielski who asked whether the composition of two derivative functions from the unit interval to the unit interval necessarily possesses a fixed point. In this note we show that this is the case.

Recalling that a Baire-1 function is the pointwise limit of a sequence of continuous functions and that a Darboux function is one for which the image of any interval in its domain is connected, we can formulate our main result as follows.

Theorem 1 *If $f, g\colon [0,1] \to [0,1]$ are both Darboux Baire-1 functions, then there is an $x \in [0,1]$ for which $(f \circ g)(x) = x$.*

Since derivative functions are examples of Darboux Baire-1 functions, this answers Ciesielski's question. The rest of the paper consists of a proof of this theorem.

Key Words: Darboux, Baire-1, fixed points
Mathematical Reviews subject classification: 26A99, 26A21
*Partially supported by the Hungarian National Foundation for Scientific Research, grant # F029768.
†The authors wish to thank the Erdös Center for their support of the Summer School on Geometric and Dynamical Aspects of Measure Theory in Révfülöp during which this paper was written.

1 Proof of Theorem

Fix two Darboux Baire-1 functions f, $g\colon [0,1] \to [0,1]$. We may assume without loss of generality that

$$f(0) = 0,\ f(1) = 1$$

and

$$g(0) = 1,\ g(1) = 0$$

for, by considering the square $[-1,2] \times [-1,2]$ and extending the sets F and G as indicated in Figure 1, and then rescaling, we can define two new Darboux Baire-1 functions $\tilde{f}, \tilde{g}\colon [0,1] \to [0,1]$ with $\tilde{f}(0) = 0$, $\tilde{f}(1) = 1$, $\tilde{g}(0) = 1$ and $\tilde{g}(1) = 0$ whose composition possesses a fixed point if and only if the original functions did.

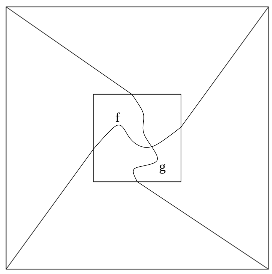

Figure 1: Ensuring that $f(0) = 0, f(1) = 1$ and $g(0) = 1, g(1) = 0$

For $\phi\colon I \to \mathbf{R}$, I an interval, we define

$$\mathrm{graph}_X(\phi) = \{(x, \phi(x)) : x \in I\}$$

and

$$\mathrm{graph}_Y(\phi) = \{(\phi(y), y) : y \in I\}$$

and given $a, b \in \mathbf{R}$, we let $[a,b], (a,b)$ denote the closed, and open intervals connecting them, respectively.

Set

$$F = \mathrm{graph}_X(f) = \{(x, f(x)) \in [0,1]^2 : x \in [0,1]\}$$

and

$$G = \mathrm{graph}_Y(g) = \{(g(y), y) \in [0,1]^2 : y \in [0,1]\},$$

then in order to prove the theorem it is sufficient to show that $F \cap G \neq \emptyset$.

Throughout this note, by *rectangle* we understand a rectangle whose sides are parallel to the usual coordinate axes. Topological notions like open, closed, etc., will be considered relatively to $[0,1]^2$.

Definition 1 *We define a crossing-configuration, $\mathcal{R} = (A, B)$ to be an ordered pair consisting of non-empty finite subsets A and B of F and G, respectively, such that whenever I and J are closed intervals with $A \cup B \subset I \times J$ and $\phi\colon I \to \mathbf{R}$ and $\psi\colon J \to \mathbf{R}$ are continuous functions with:*

$$A \subset \mathrm{graph}_X(\phi) \quad \textit{and} \tag{1}$$

$$B \subset \mathrm{graph}_Y(\psi), \tag{2}$$

then

$$\mathrm{graph}_X(\phi) \cap \mathrm{graph}_Y(\psi) \neq \emptyset.$$

Remark 1 *If $\mathcal{R} = (A, B)$ is a crossing configuration, and if ϕ, $\psi\colon I, J \to [0,1]$ are continuous functions satisfying (1) and (2) respectively, then for any rectangle $R \subset [0,1]^2$ which contains $A \cup B$, we know that*

$$\mathrm{graph}_X(\phi) \cap \mathrm{graph}_Y(\psi) \cap R \neq \emptyset.$$

Since, if the intersection were empty, we could modify ϕ and ψ outside a closed rectangle $S \subset R$ with $A \cup B \subset S$ to form two new continuous functions ϕ' and ψ' for which $\mathrm{graph}_X(\phi')$ and $\mathrm{graph}_Y(\psi')$ do not intersect.

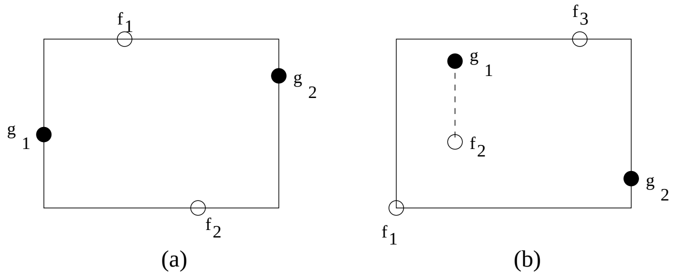

Figure 2: Two examples of crossing configurations: points denoted by $\circ$ lie in F and points denoted by $\bullet$ lie in G.

Lemma 1 *The configurations illustrated in Figure* **??** *are crossing configurations.*

Proof: Figure **??**(a): Here f_1, f_2 are points from F lying on the top and bottom edges of a closed rectangle, and g_1, g_2 are points of G lying on the left and right edges of the rectangle, respectively. Suppose that ϕ and ψ are continuous functions with $\{f_1, f_2\} \subset \text{graph}_X(\phi)$ and $\{g_1, g_2\} \subset \text{graph}_Y(\psi)$. Notice that the part of $\text{graph}_X(\phi)$ lying within the vertical strip whose edges contain f_1 and f_2 may be extended to form a Jordan curve separating g_1 and g_2 in such a way that the added curve does not intersect $\text{graph}_Y(\psi) \cap (\mathbf{R} \times [g_1, g_2])$. (See Figure 2.)

Since $\text{graph}_Y(\psi) \cap (\mathbf{R} \times [g_1, g_2])$ connects g_1 with g_2, we conclude that

$$(\text{graph}_Y(\psi) \cap (\mathbf{R} \times [g_1, g_2])) \cap (\text{graph}_X(\phi) \cap ([f_1, f_2] \times \mathbf{R})) \neq \emptyset,$$

as required.

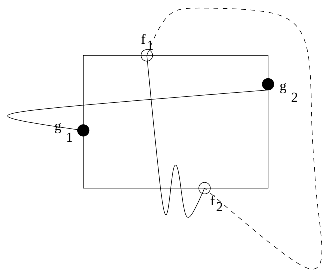

Figure 3: $\text{graph}_X(\phi) \cap ([f_1, f_2] \times \mathbf{R})$ may be extended to form a Jordan curve separating g_1 and g_2.

Figure **??**(b): In this situation we have three points $f_1, f_2, f_3 \in F$ and two points $g_1, g_2 \in G$ with $(f_1)_x < (f_2)_x = (g_1)_x < (f_3)_x$, $(f_1)_y \leq (f_2)_y \leq (g_1)_y \leq (f_3)_y$ and $(f_1)_y \leq (g_2)_y \leq (f_3)_y$. (Here $(\cdot)_x$ and $(\cdot)_y$ denote the x and y coordinates of the point, respectively.) Suppose that ϕ and ψ are continuous functions with $\{f_1, f_2, f_3\} \subset \text{graph}_X(\phi)$ and $\{g_1, g_2\} \subset \text{graph}_Y(\psi)$. Observe that the part of $\text{graph}_X(\phi)$ lying within the vertical strip whose edges contain f_1 and f_3 may be extended to form a Jordan curve separating g_1 and g_2 in such a way that the added curve does not intersect $\text{graph}_Y(\phi) \cap (\mathbf{R} \times [g_1, g_2])$. (See Figure 3.)

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$$(\text{graph}_Y(\psi) \cap (\mathbf{R} \times [g_1, g_2])) \cap (\text{graph}_X(\phi) \cap ([f_1, f_2] \times \mathbf{R})) \neq \emptyset,$$

as required. ∎

Remark 2 *Since* $(0, 0)$ *and* $(1, 1) \in F$*, and* $(0, 1), (1, 0) \in G$*, we conclude that* $\mathcal{R}_0 = (\{(0, 0),\ (1, 1)\}, \{(0, 1), (1, 0)\})$ *is a crossing configuration.*

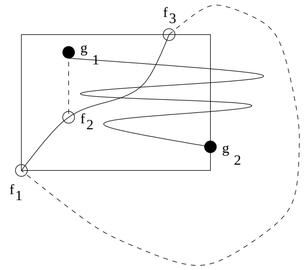

Figure 4: $\mathrm{graph}_X(\phi) \cap ([f_1, f_3] \times \mathbf{R})$ may be extended to form a Jordan curve separating g_1 and g_2.

The key part of our argument is the following proposition.

Proposition 2 *For all crossing-configurations $\mathcal{R} = (A, B)$ and for all open rectangles $R \supset A \cup B$ and open sets $U \supset F$ (or $V \supset G$), we can find a crossing-configuration $\mathcal{R}' = (A', B')$ and a closed rectangle R' with $A' \cup B' \subset R' \subset U \cap R$ (or $V \cap R$).*

Before proving this, we show how it immediately leads to a proof of Theorem 1: Since f and g are Darboux, Baire-1 functions,

$$F = \bigcap_{n=1}^{\infty} U_n, \quad \text{where } U_1 \supset U_2 \supset \cdots \text{ are open sets}$$

and

$$G = \bigcap_{n=1}^{\infty} V_n, \quad \text{where } V_1 \supset V_2 \supset \cdots \text{ are open sets.}$$

We recall, from remark 2, that

$$\mathcal{R}_0 = (\{(0,0),(1,1)\},\{(0,1),(1,0)\})$$

is a crossing-configuration. We now use the proposition to find a sequence of crossing-configurations $\mathcal{R}_i = (A_i, B_i)$ and open rectangles $R_i \supset A_i \cup B_i$ such that

$$\mathrm{cl}(R_{2i+1}) \subset R_{2i} \cap U_{i+1}$$

and

$$\mathrm{cl}(R_{2(i+1)}) \subset R_{2i+1} \cap V_{i+1}$$

for $i = 0, 1, 2, \ldots$. Hence

$$\mathrm{cl}(R_0) = R_0 \supset \mathrm{cl}(R_1) \supset R_2 \supset \mathrm{cl}(R_3) \supset \cdots,$$

$$\emptyset \neq \bigcap_{n=1}^{\infty} \mathrm{cl}(R_n) \subset \bigcap_n U_n = F$$

and

$$\emptyset \neq \bigcap_{n=1}^{\infty} \mathrm{cl}(R_n) \subset \bigcap_n U_n = G,$$

which together imply that $F \cap G \neq \emptyset$ as required.

Proof: We prove the proposition for the case when $U \supset F$, the case when $V \supset G$ is similar.

Suppose that $\mathcal{R} = (A, B)$ is a crossing configuration, $R \supset A \cup B$ is an open rectangle, and let $S = [x_1, x_2] \times [y_1, y_2] \subset R$ be a closed rectangle whose (relative) interior contains $A \cup B$.

Observe that if $\phi\colon [x_1, x_2] \to \mathbf{R}$ were a continuous function for which $A \subset \mathrm{graph}_X(\phi)$ and $\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset$, then $G \subset [0,1]^2 \setminus (\mathrm{graph}_X(\phi) \cap S)$ would be a relatively open set, and we would be able to construct a continuous function $\psi\colon [0,1] \to \mathbf{R}$ such that $\mathrm{graph}_Y(\psi) \supset B$ and $\mathrm{graph}_Y(\psi)$ would be a subset of $[0,1]^2 \setminus (\mathrm{graph}_X(\phi) \cap S)$ and so $\mathrm{graph}_X(\phi) \cap \mathrm{graph}_Y(\psi) \cap S = \emptyset$. (See [1].) But (A, B) is a crossing configuration — a contradiction.

We now show that if there were no crossing-configurations (A', B') and closed rectangles with $A' \cup B' \subset R' \subset U \cap R$, then we would be able to construct a continuous function $\phi\colon [x_1, x_2] \to \mathbf{R}$ with $A \subset \mathrm{graph}_X(\phi)$ and $\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset$ giving us our required contradiction.

We do this via the method of regular intervals: we say an interval $I \subset [x_1, x_2]$ is *regular*, if for all $s, t \in I$, $s < t$, we can find a continuous function $\phi\colon [s, t] \to \mathbf{R}$ for which

$$\phi(s) = f(s), \qquad \phi(t) = f(t)$$

and

$$\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset.$$

(Note that regular intervals need neither be open nor closed.) If we show that $[x_1, x_2]$ is itself regular, then we are done.

It is easy to see that:

(1) If I and J are regular intervals and $I \cap J \neq \emptyset$, then $I \cup J$ is regular;

(2) If I is an interval which is the (finite or infinite) union of *open* regular intervals, then I is regular.

It is slightly trickier to verify:

(3) If I is regular, then $\mathrm{cl}(I)$ is regular.

Proof of (3): Let r be the left endpoint of I. (The proof for the right endpoint is similar.) It is enough to show that we can find $r' > r$ arbitrarily close to r for which there is a continuous function $\phi\colon [r, r'] \to \mathbf{R}$ such that $\phi(r) = f(r)$, $\phi(r') = f(r')$ and $\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset$.

Choose $r_i \in I$ such that $r_1 > r_2 > \cdots > r$, $r_n \to r$ and for which $f(r_n) \to f(r)$ (the Darboux property for f ensures we can find such a sequence). For each $k \in \mathbf{N}$, we can find n_k such that

$$f(r_{n_k}) \in (f(r) - 2^{-k}, f(r) + 2^{-k})$$

and both

$$g(f(r) - 2^{-k}) \text{ and } g(f(r) + 2^{-k}) \notin (r, r_{n_k}).$$

Fix a sequence $n_1 < n_2 < n_3 < \cdots$ with this property. Since I is regular we can find a sequence of continuous functions $\phi_k\colon [r_{n_{k+1}}, r_{n_k}] \to \mathbf{R}$ for which

$$\phi_k(r_{n_k}) = f(r_{n_k}),\ \phi_k(r_{n_{k+1}}) = f(r_{n_{k+1}}) \text{ and } \mathrm{graph}_X(\phi_k) \cap G \cap S = \emptyset.$$

Then the function $\tilde{\phi}\colon [r, r_{n_1}] \to \mathbf{R}$ defined by

$$\tilde{\phi}(x) = \begin{cases} f(r) & \text{if } x = r \\ \max\{\min\{\phi_k(x), f(r) + 2^{-k}\}, f(r) - 2^{-k}\} & \text{if } x \in [r_{n_{k+1}}, r_{n_k}] \end{cases}$$

is a well-defined continuous function for which $\tilde{\phi}(r_{n_k}) = f(r_{n_k})$ for all k, $\tilde{\phi}(r) = f(r)$ and $\mathrm{graph}_X(\tilde{\phi}) \cap G \cap S = \emptyset$. ■

Suppose that $[x_1, x_2]$ is not a regular interval and let

$$P = [x_1, x_2] \setminus \bigcup \{I \subset [x_1, x_2] : I \text{ is relatively open and regular}\}.$$

Then P is closed, and observations (2) and (3) imply that $P \cap (x_1, x_2)$ is non-empty and has no isolated points. Thus we can choose $r \in P \cap (x_1, x_2)$ such that

- $f|_P$ is continuous at r; and
- r is not the endpoint of any interval in $[x_1, x_2] \setminus P$ which is contiguous to P.

Without loss of generality we can assume that $g(f(r)) > r$. We will show that in this case we can always find $r' > r$ for which (r, r') is regular which contradicts our choice of r.

Since the endpoints of any interval contiguous to P belong to P, and the closure of any regular interval is also regular, then it is enough to find an

$r' > r$ such that for $s, t \in (r, r') \cap P$ we can find continuous $\phi\colon [s,t] \to \mathbf{R}$ with $\phi(s) = f(s)$, $\phi(t) = f(t)$ and $\mathrm{graph}_X(\phi) \cap G \cap S = \emptyset$.

Case 1: $(r, f(r)) \notin S$.

In this case, since $f|_P$ is continuous at r, the existence of r' is trivial.

Case 2: $(r, f(r)) \in S$ and there is no $r^* > r$ for which either $f|_{(r,r^*)} \geq f(r)$ or $f|_{(r,r^*)} \leq f(r)$.

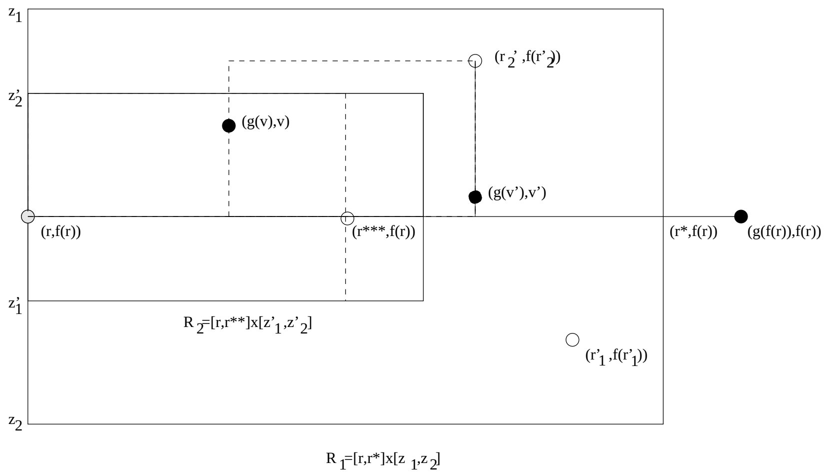

Figure 5: Case 2

Choose $r < r^* < g(f(r))$ and $z_1 < f(r) < z_2$ for which $R_1 = [r, r^*] \times [z_1, z_2] \subset R \cap U$. (See Figure 4.) Then we can find $r_1', r_2' \in (r, r^*)$ such that

$$z_1 < f(r_1') < f(r) < f(r_2') < z_2.$$

Now we choose z_1', z_2' and $r < r^{**} < \min\{r_1', r_2'\}$ such that

$$f(r_1') < z_1' < f(r) < z_2' < f(r_2'),$$

$$R_2 = [r, r^{**}] \times [z_1', z_2'] \subset R_1 \quad \text{and}$$

$$\mathrm{graph}_X(f|_{P \cap [r, r^{**}]}) \subset R_2.$$

Finally by the Darboux property, we can find $r^{***} \in (r, r^{**})$ for which $f(r^{***}) = f(r)$.

Claim: $([r, r^{***}] \times [z_1', z_2']) \cap G = \emptyset$.

Proof of claim: For suppose $(g(v), v) \in [r, r^{***}] \times [z_1', z_2']$, without loss of generality we can assume that $v > f(r)$. By the Darboux property applied to g we can find $v' \in [f(r), v]$ with $g(v') = r_2'$. But then

$$(\{(r^{***}, f(r^{***})),\ (r_2', f(r_2'))\}, \{(g(v), v),\ (g(v'), v')\})$$

is a crossing-configuration of the type (a) illustrated in Figure **??** contained in

$$[g(v), g(v')] \times [f(r), f(r_2')] \subset R \cap U,$$

which is a contradiction. ■

But now clearly the interval (r, r^{***}) is regular.

Case 3: $(r, f(r)) \in S$ and there is $r^* > r$ such that $f|_{(r,r^*)} \geq f(r)$. (Or $(r, f(r)) \in S$ and there is $r^* > r$ such that $f|_{(r,r^*)} \leq f(r)$.)

Without loss of generality, we do the case when there is an $r^* > r$ with $f|_{(r,r^*)} \geq f(r)$ and there is no $r' > r$ for which $f|_{(r,r')}$ is constant. Choose $z_2 > f(r)$ and $r^* > r$ for which $f|_{(r,r^*)} \geq f(r)$ and $[r, r^*] \times [f(r), z_2] \subset R \cap U$ and set $R_1 = [r, r^*] \times [f(r), z_2]$.

Since $f|_{(r,r^*)}$ is not constant (and so $f(r) < 1$), then we can find $r < r_2' < r^*$ such that $f(r) < f(r_2') < z_2$. Choose $R_2 = [r, r''] \times [f(r), z_2']$ such that

$$f(r) < z_2' < f(r_2'), \quad r < r'' < r_2' \quad \text{and}$$

$$\text{graph}_X(f|_{P\cap[r,r'']}) \subset R_2.$$

We show that there are no points $(u, f(u))$, $(g(v), v)$ in R_2 for which $u = g(v)$ and $f(u) < v$. For if there were, we could use the fact that g is Darboux to find $w \in (f(r), v)$ for which $g(w) = r_2'$ and then we would have a crossing-configuration, namely $(\{(u, f(u)), (r, f(r)), (r_2', f(r_2'))\}, \{(g(v), v), (g(w), w)\})$ contained in R_1, see Lemma 1 and Figure **??**.

Lemma 2 *There is a rectangle* $R_3 = [r, r'''] \times [f(r), z_2''] \subset R_2$ *such that* $\text{graph}_X(f|_{P\cap[r,r''']}) \subset R_3$ *and there are no points* $(u, f(u)), (g(v), v) \in R_3$ *for which* $g(v) \leq u$, $f(u) \leq v$.

Proof: If $R_3 = R_2$ does not satisfy the lemma, then there is $(u_0, f(u_0))$ and $(g(v_0), v_0) \in R_2$ for which

$$g(v_0) \leq u_0 \quad \text{and} \quad f(u_0) \leq v_0.$$

Choose $f(r) < z_2'' < f(u_0)$ and use the fact that r is a continuity point of $f|_P$ to find $r < r''' < g(v_0)$ for which $\text{graph}_X(f|_{P\cap[r,r''']}) \subset R_3 = [r, r'''] \times [f(r), z_2'']$.

Suppose now that we can find $(u, f(u))$ and $(g(v), v)$ in R_3 for which $g(v) \leq u$ and $f(u) \leq v$. Then by the Darboux property for g, we can find a point

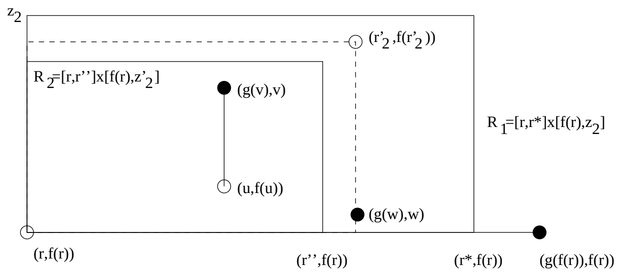

Figure 6: Diagram illustrating occurrence of a crossing-configuration in Case 3 if we can find $u = g(v)$ and $f(u) < r$ in R_2.

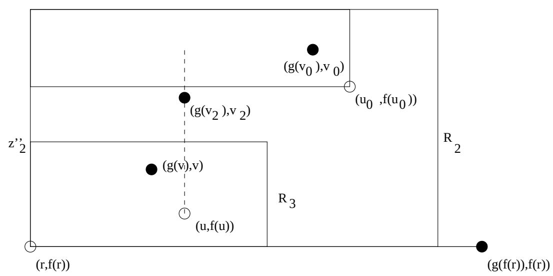

Figure 7: The figure for Lemma 2

$(g(v_2), v_2)$ with $v_2 \in (v, v_0)$ and $g(v_2) = f(u)$ but then $u = g(v_2)$ and $f(u) < v_2$ contradicting the observation made just before this lemma; see Figure **??**. Hence the lemma holds. ■

We now prove that the interval (r, r''') we have constructed is regular; that is, for all $s, t \in P \cap [r, r''']$ with $s < t$, we can find a continuous function $\phi\colon [s, t] \to \mathbf{R}$ for which $\text{graph}_X(\phi) \cap G \cap S = \emptyset$, and $\phi(s) = f(s)$ and $\phi(t) = f(t)$. We know that $(s, f(s))$ and $(t, f(t))$ are in R_3. We can assume that s is not the left endpoint of an interval contiguous to P. If $f(s) \geq f(t)$, then Lemma 2 allows us to choose ϕ to be the affine function joining $f(s)$ and $f(t)$. If $f(s) < f(t)$ we distinguish two cases:

(A): There is $u \in (s, t)$ such that $f(u) \leq f(s)$.

In this case, Lemma 2 implies that G does not meet the shaded region of Figure **??** and we can join $(s, f(s)$, $(u, f(t))$ and $(t, f(t))$ by a piecewise linear function.

(B): There is a sequence (u_n) with $t > u_1 > u_2 > \cdots \to s$ such that $f(t) >$

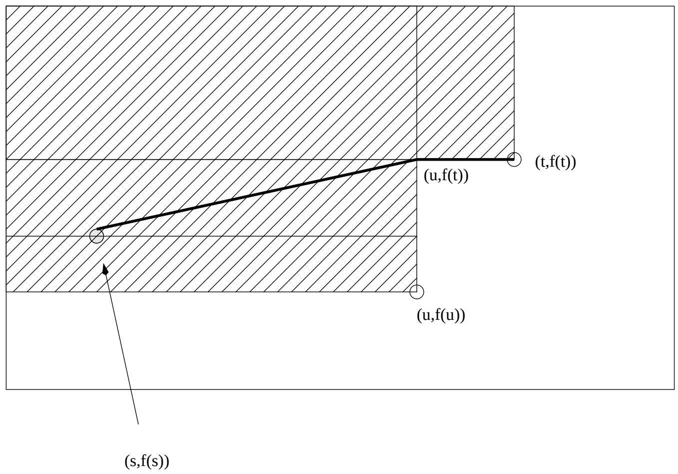

Figure 8: Constructing our continuous function ϕ in Case (A).

$f(u_1) > f(u_2) > \cdots \to f(s)$. In this case we can join the points $(t, f(t))$, $(u_1, f(t))$, $(u_2, f(u_1))$, $(u_3, f(u_2)), \ldots$ piecewise linearly, see Figure **??**. ■

2 Open problems

There are a couple of natural questions suggested by this result:

1. is the graph of the composition of two Darboux Baire-1 functions connected?

2. does a similar result hold for the composition of n Darboux Baire-1 functions when $n \geq 3$?

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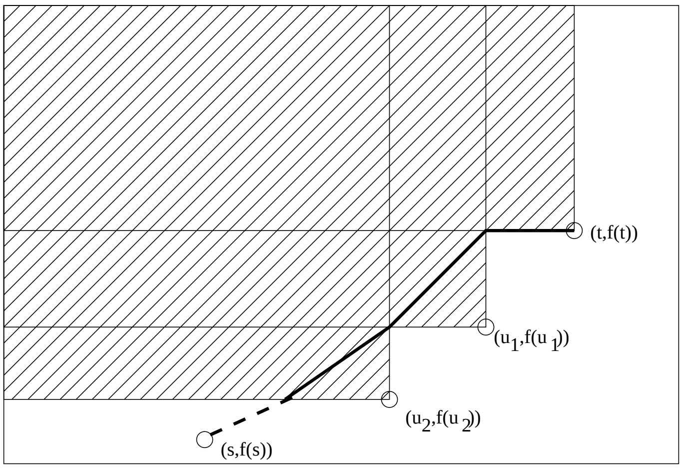

Figure 9: Constructing our continuous function ϕ in Case (B).