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The dimension of visible sets

1 Introduction

Given a subset E of the plane, Urysohn [7, 8] defined the notion of linear accessibility for a point $p \in E$: p is *linearly accessible* if there is a non-degenerate line segment L that only meets E at the point p . In a sequence of papers, Nikodym [2, 3, 4] investigated the relationship between the set theoretic complexity of E and the set of linearly accessible points.

We consider a variant of this notion: for a compact set in \mathbb{R}^2 , K , and $x \in \mathbb{R}^2$ we define the visible part of K from the point x by

$$K_x = \{u \in K : [x, u] \cap K = \{u\}\},$$

where $[x, u]$ denotes the closed line segment joining x to u . (This notion also makes sense in \mathbb{R}^n , and may also be extended to define visibility from affine subspaces of \mathbb{R}^n .)

Clearly, since K_x is always a subset of K , the (Hausdorff) dimension of K_x is always at most the dimension of K . Our interest lies in determining whether any stronger relationship holds between $\dim_H(K)$ and $\dim_H(K_x)$ for ‘typical’ (in the sense of measure) points x in the plane.

Our first result (from [1]) shows that for sets of small enough Hausdorff dimension, visible sets usually have the same Hausdorff dimension as the original set.

Theorem 1.1 *Let K be a compact subset of the plane. If $\dim_H(K) \leq 1$, then for (Lebesgue) almost every $x \in \mathbb{R}^2$,*

$$\dim_H(K_x) = \dim_H(K).$$

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A version of this result also holds for visibility from points in \mathbb{R}^n : in this case 1 is replaced by $n - 1$.

The basic idea underlying the proof of this theorem is the observation that around a point x in the complement of K it is possible to project K radially onto a small circle around x that is disjoint from K . On observing that K_x projects to the same set on this circle and denoting this projection by $\pi_x(K)$, we easily verify that

$$\dim_H(\pi_x(K)) \leq \dim_H(K_x) \leq \dim_H(K).$$

We now use the generalised projection framework developed by Peres and Schlag in [6] to show that $\dim_H(\pi_x(K)) = \dim_H(K)$ for almost every x . This implies the theorem.

In the same paper [1], we investigate some particular classes of compact connected sets in the plane and show that their visible sets almost surely have dimension one. For example, if K is an embedding into the plane of the graph of a continuous function from S^1 to \mathbb{R} , then K_x has dimension one except for at most a single point in the plane. Similarly, if K is a quasicircle, then for *all* points lying in the complement of K , $\dim_H(K_x)$ is one. (A set K is a quasicircle if it is a closed Jordan curve and there is a constant m such that whenever $x, y \in K$, there is an arc of K connecting x and y lying entirely within the ball with centre x and radius $m|x - y|$.)

These examples motivated a general investigation of connected compact sets in the plane that culminated in the proof of the following theorem in [5].

Theorem 1.2 *Let $K \subset \mathbb{R}^2$ be a compact connected set with $\dim_H(K) > 1$. Then for $\frac{1}{2} + \sqrt{\dim_H(K) - \frac{3}{4}} < s \leq \dim_H(K)$,*

$$\dim_H\{x \in \mathbb{R}^2 : \dim_H(K_x) > s\} \leq \frac{\dim_H(K) - s}{s - 1}.$$

In particular, this theorem shows that if $K \subset \mathbb{R}^2$ is a compact connected set containing at least two points, then for (Lebesgue) almost every $x \in \mathbb{R}^2$,

$$\dim_H(K_x) \leq \frac{1}{2} + \sqrt{\dim_H(K) - \frac{3}{4}}.$$

The proof of Theorem 1.2 relies on the idea that two distinct visible subsets of K , both of dimension strictly larger than one, must be ‘separated’ in some sense. Much of the paper [5] is concerned with proving such a result.

The first step in proving such a separation property is to observe that if the visible set K_x has dimension larger than $s > 1$, then there is a measure ν_x living on K_x with $\nu_x(B(u, r)) \leq r^s$ whenever $u \in K_x$ and $0 \leq r \leq 1$, such that

for ν_x -almost every point u in K_x , there are points of K_x lying approximately at a distance of at least r^{2-s} above and below u inside the cone with vertex x and just containing $B(u, r)$. After an appropriate uniformisation procedure this allows us to show that in a sufficiently small ball around a ν_x -typical point of K_x , points visible from y are concentrated in a thin strip — this implies that the corresponding ν_y mass of a (small) ball centred on K_x is small. This is the required ‘separation’ property, and it is now a routine (but long) argument involving energies to use this observation to prove the Theorem.

2 Open Problems

There are many directions for further investigation, and I only give a few of them here.

Problem 2.1 *Is there a version of Theorem 1.2 that holds for totally disconnected compact sets? That is, if K is a totally disconnected set in the plane with Hausdorff dimension strictly larger than one, then is there a positive number c (depending upon $\dim_H(K)$) such that for (Lebesgue) almost every x in the plane,*

$$\dim_H(K_x) \leq \dim_H(K) - c?$$

Problem 2.2 *Is there a compact subset of the plane with Hausdorff dimension larger than one such that for a set of points of positive area, the visible sets each have dimension strictly larger than one?*

Problem 2.3 *Is there a version of Theorem 1.2 that holds in higher dimensions? The only proof that I know of Theorem 1.2 uses the connectedness properties of the plane in an essential way.*

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