

Handout: Maths Masterclass
Bletchley Park
March 2010

Toby O'Neil
<http://mcs.open.ac.uk/tcon2/>
t.c.oneil@open.ac.uk

Version 12/03/2010

1 Warm up

Let S_n denote the sum of the first n whole numbers. So

$$\begin{aligned}S_1 &= 1, \\S_2 &= 1 + 2 = 3, \\S_3 &= 1 + 2 + 3 = 6\end{aligned}$$

and

$$S_n = 1 + 2 + 3 + \cdots + (n - 2) + (n - 1) + n. \quad (*)$$

(The ' \cdots ' just means 'and so on'.)

1. Find

$$S_{10} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10.$$

2. Find

$$S_{100} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \cdots + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 + 100.$$

3. Does

$$S_n = n + (n - 1) + (n - 2) + \cdots + 3 + 2 + 1?$$

4. Can you use this together with the original expression (*) for S_n to find a formula for $2S_n$?
5. Hence find a formula for S_n .
6. What is S_{2010} ?

2 Exponential Growth

1. Bacterial Growth.

At time zero, you have one bacterium. It takes one minute for a bacterium to divide in two. How many bacteria do you have after:

- (a) one minute, (b) five minutes, (c) ten minutes,
(d) half an hour, (e) one hour, (f) n -minutes?

How long will it be until there are about ten million bacteria?

Bonus question. Why do you think it is that bacteria have not taken over the world?

2. Now suppose that you had three bacteria to start with. How would your answers to question 1 change?
3. In fact harmful bacteria such as e-coli and salmonella take about twenty minutes to divide. Given this, if you start with one salmonella bacterium, then how long would it take until you had as many bacteria as in your answer to 1.(e)?

4. Radioactive Decay.

At time zero, you have one unit of a radioactive substance. Each minute, half of the substance decays¹.

How much remains after:

- (a) 1 minute, (b) 2 minutes, (c) 10 minutes, (d) n minutes?

5. Debt.

Suppose I lend my friend £1000 and each month that they fail to pay me back, I charge them 2% of the amount they *currently* owe and add it to their debt.

For example, if they don't pay me back at the end of the first month, then they will owe

$$£1000 + 2\% \text{ of } £1000 = £1020$$

¹Emitting extremely dangerous gamma radiation that causes death to anybody exposed to too much of it — this is the death part of today's session!

and if they don't pay me back by the end of the following month, then they will owe 2% of £1020 on top of the £1020 that they already (now) owe.

- (a) How much will they owe me if they don't pay me back for two months?
- (b) How much will they owe me if they don't pay me back for three months?
- (c) How much will they owe me if they don't pay me back for four months?
- (d) How much will they owe me if they don't pay me back for a year?
- (e) **General Case.** If they don't pay me back for n months, can you find a formula that to tell me how much they owe? How would you show that your formula is correct?

6. APR/AER.

When you look at adverts for loans, mortgages or credit cards, they will often talk about something called the annual percentage rate (APR) ('**Only 12% APR!**') or annual equivalent rate (AER). This is the rate of interest that one would be paying over a period of one year. Thus in the case of my friend who has borrowed £1000, we work out how much they owe after one year if they pay nothing back, and then work out what percentage *increase* in the debt this is.

Do you expect this percentage increase to be more or less than 24%? Why?

Now calculate the APR of the loan I made to my friend.

7. The General Case for exponential growth.

Let A and r be fixed numbers. Suppose that $x_0 = A$ and $x_n = rx_{n-1}$ for $n = 1, 2, 3, \dots$

Thus $x_0 = A$,

$$x_1 = r \times x_0 = r \times A = Ar$$

and

$$x_2 = r \times x_1 = r \times (r \times A) = Ar^2.$$

Can you find a formula for x_n in terms of A , r and n ? How do you know that your formula is correct?

8. **Bonus question.**

- (a) Suppose now that $x_0 = 2$ and we have $x_n = 3x_{n-1} + 4$ for $n = 1, 2, 3, \dots$
- Find x_n for $n = 1, 2, 3, 4, 5, 6$.
 - Find a formula for x_n in terms of n .
- (b) Let a, b and r be fixed numbers with $r \neq 1$. Suppose that $x_0 = a$ and $x_n = rx_{n-1} + b$ for $n = 1, 2, 3, \dots$
- Can you work out the rule for finding x_n in terms of n ?
 - How can you show that your rule is correct?
 - Test your rule for $x_0 = 3$ and $x_n = 4x_{n-1} - 1$.

3 Collatz Sequence Problem

For positive whole numbers n , define

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ 3n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Now consider the sequence that you obtain when, having been given x_0 , you define $x_{n+1} = f(x_n)$.

If $x_0 = 3$, then we find

$$x_1 = 10, \quad x_2 = 5, \quad x_3 = 16, \quad x_4 = 8, \dots$$

and the sequence is

$$3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, \dots 1$$

Find the sequence we obtain when:

1. $x_0 = 64$
2. $x_0 = 20$
3. $x_0 = 21$
4. $x_0 = 13$
5. $x_0 = 7$.

4 Death and Debt

4.1 Loans

1. **Debt.** Following my loan of £1000 to my friend, they decide that they can pay back £100 of their debt per month. Assuming that I still charge them interest of 2% per month on the outstanding debt, the amount they owe at the end of each month will be given by the following formula:

$$\text{debt}_{\text{new}} = 1.02 * \text{debt}_{\text{old}} - 100.$$

(In the final month, if they owe less than £100, then they just pay what they owe.)

- (a) How much will they owe me after one month? Two months? Three months?
 - (b) How many months will it take them to pay off the debt?
 - (c) How much will they pay me in total?
2. My friend is paying off his debt too quickly and I want it to take much longer. What is the minimum interest rate that I should charge to guarantee that they can never pay the debt off if they only pay £100 per month? What APR is this equivalent to?
 3. **Optional — more debt!** For this question, you will need to look at my credit card statement. By looking at the information on my statement, can you work out:
 - (a) How long will it take me to pay off my bill if I only pay off £100 each month? How much will I pay in total?
 - (b) How long it will take me to pay off my bill if I only pay the minimum amount each month? How much will I pay in total in this case?
 - (c) Do you think this is an easy calculation to make?

4.2 A more sophisticated population model

So far our model of bacterial growth gives that the number of bacteria just grows forever. In reality, other factors intervene to prevent this happening. (Can you think of three factors that may act to restrict bacterial growth?)

We are now going to look at another model of population growth that tries to take these factors into account. It is known as the logistic map.

Before we start, we make some simplifying assumptions. Instead of counting how many bacteria there are, we suppose that we already know in advance the maximum possible number. (This can usually be estimated by looking at how much food is available.) Our model will be set up to produce a number between zero and one that represents the proportion of the maximum possible population alive at that step. (So if the model gives the value 0.5, then this means that the number of bacteria is half the maximum possible number of bacteria.)

The model is as follows. We are given a fixed parameter value K that is somewhere between 0 and 4 — this is supposed to represent how quickly the bacteria can divide, and a starting population fraction x_0 that is somewhere between 0 and 1. For $n = 1, 2, 3, \dots$, the fraction of population alive at generation n is given by the formula

$$x_n = Kx_{n-1}(1 - x_{n-1}). \quad (\text{L})$$

The idea is that as the proportion of bacteria alive gets nearer to 1, then there are fewer resources available and bacteria are more likely to die than reproduce. (Indeed, if the proportion alive were to equal 1, then there would be no resource (food) available and all the bacteria would die.)

4.3 Initial investigation

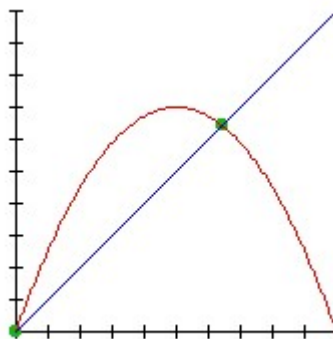
1. Before we do proceed any further, we should check that x_n always lies between 0 and 1 (provided that $0 \leq K \leq 4$). Let f be the function given by $f(x) = Kx(1 - x)$ (so $x_n = f(x_{n-1})$).
 - (a) Show that for $0 \leq x \leq 1$, $f(x) \geq 0$.
 - (b) Find the maximum value of f .
 - (c) Hence deduce that for $0 \leq K \leq 4$, $0 \leq x_n \leq 1$.
2. Let $x_0 = 0.5$. Calculate x_n for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots$ when:

- (a) $K = 0.5$,
- (b) $K = 0.8$,
- (c) $K = 1.2$,
- (d) $K = 2$,
- (e) $K = 2.5$,
- (f) $K = 3.0$

What appears to be happening?

3. **Fixed points.** A number X is said to be a fixed point of the logistic map if it satisfies the equation $X = KX(1 - X)$. If $x_0 = X$, then $x_n = X$ for all n — hence the terminology.

If you plot the graphs of $y = x$ and $y = Kx(1 - x)$, then the values of x where the graphs meet correspond to fixed points.



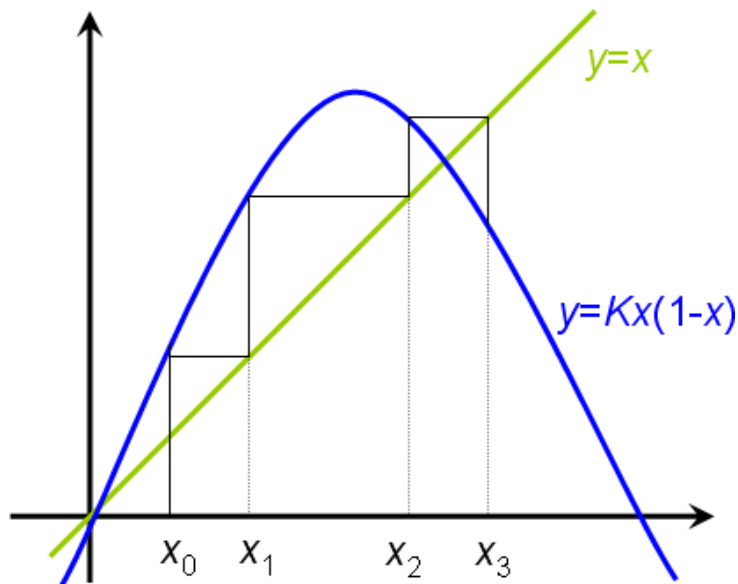
- (a) Which number is *always* a fixed point of the logistic map?
 - (b) For which values of K are there fixed points greater than zero?
4. Calculate x_n for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots$ and $x_0 = 0.5$ when:
- (a) $K = 3.1$,
 - (b) $K = 3.3$,
 - (c) $K = 3.5$.

What appears to be happening?

4.4 Cobweb diagrams

Suppose that we have plotted $y = Kx(1 - x)$ and $y = x$ on the same graph, and we wish to investigate the sequence $x_n = Kx_{n-1}(1 - x_{n-1})$ for some given starting value x_0 .

We can proceed as follows:



The vertical and horizontal lines represents the cobweb.

We start along the x -axis at our initial point x_0 . Under the map

$$y = x_n = f(x_{n-1})$$

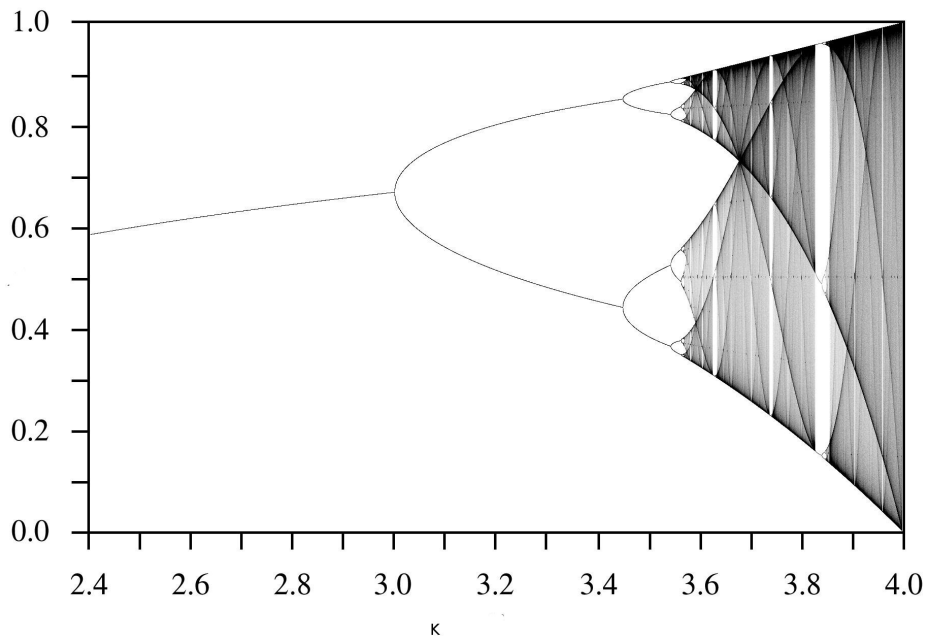
(which in the above diagram is given by the logistic map), the initial point gets mapped to a new point x_1 which we find by drawing a vertical line from the x -axis up to the curve representing f . To find the next point in the orbit we begin by moving horizontally to the curve $y = x$. Our x -coordinate will now be x_1 . We can now find the next point in our orbit, x_2 , by once again drawing a vertical line up (or down) to the curve $f(x)$. We then continue this procedure until we have calculated as many points in the orbit of x_0 as we wish. Recall that the fixed points occur where the graphs $y = x$ and $y = Kx(1 - x)$ meet.

1. Let $K = 2.5$. Draw the graphs of $y = Kx(1 - x)$ and $y = x$ for $0 \leq x \leq 1$ on the same piece of graph paper. Investigate which starting values of x_0 converge to the fixed point of the logistic map for this value of K .

2. Let $K = 3.2$. Draw the graphs of $y = Kx(1-x)$ and $y = x$ for $0 \leq x \leq 1$ on the same piece of graph paper. Investigate which starting values of x_0 converge to the fixed point of the logistic map for this value of K .
3. What happens as K gets nearer to 3.57?

5 Bifurcations of the logistic map

The following diagram shows the periodic points of the logistic map as you vary K .



For $0 \leq K \leq 1$ (not shown), there is a single fixed point at zero, and all starting populations converge to it. (That is, the population becomes extinct.)

For $1 < K \leq 3$ (partially shown), there are two fixed points but, provided the starting population is not zero, it converges to the fixed point at $(K - 1)/K$.

For $3 < K \leq 1 + \sqrt{6} \approx 3.45$, the population eventually oscillates between two possible values.

As you increase K , the population oscillates between more and more values until eventually (at about $K = 3.57$) it ceases to follow any predictable pattern — this is the onset of chaos.

Wikipedia (http://en.wikipedia.org/wiki/Logistic_map) has quite a nice explanation. Well it did when I looked at it on Friday, March 12, 2010 at 11:56 ;-)