

The theoretical basis of iterative distributed solutions to the biomagnetic inverse problem

Hasson, R. and Swithenby, S.J.

The Open University, Milton Keynes, UK.

Introduction

This paper aims to clarify the theoretical basis of iterative distributed solutions [1,2,3] to the biomagnetic inverse problem. A rationale for employing this technique is that it should produce more focal activity than the standard minimum norm algorithm [4]. We try to clarify the meaning of the images produced by deriving a precise functional that is minimised.

We will now review the algorithm for the generation of an iterated distributed current density image. There are five inputs to the algorithm. Firstly, we have the measurement data from a single time instant, collected into a vector m_k with $k = 1, 2, \dots, N$. Secondly, a volume Q in which the image current density, $\vec{j}(\vec{r})$, is to be reconstructed (called the 'source space'). Thirdly we need to know the geometry of the sensors and of the head which are represented in the lead field distributions $\vec{L}_k(\vec{r})$ defined by:

$$m_k = \int_Q \vec{L}_k(\vec{r}) \cdot \vec{j}(\vec{r}) \, d\vec{r} \quad (1)$$

Fourthly we can include a weighting distribution $w(\vec{r})$ defined on the source space Q which is a consistent way of inputting prior knowledge into the algorithm. Lastly we have a positive number ζ which reflects prior knowledge of the signal to noise ratio. The algorithm consists of carrying out the following steps:

1. The Gram-Schmidt matrix is formed by computing the weighted overlap between the lead fields

$$P_{ij} = \int_Q \vec{L}_i(\vec{r}) \cdot \vec{L}_j(\vec{r}) w(\vec{r}) \, d\vec{r} \quad (2)$$

2. The matrix equation $(P + \zeta I)a = m$ is solved for the coefficient vector a . The minimum norm solution is then given by

$$\vec{j}(\vec{r}) = \sum_{i=1}^N a_i w(\vec{r}) \vec{L}_i(\vec{r}) \quad (3)$$

3. A new weight distribution is defined by $w_1(\vec{r}) = w(\vec{r}) \times |\vec{j}(\vec{r})|^q$ where the parameter q is positive.
4. Steps 1 and 2 are repeated with the new weight distribution w_1 to find a new current density $\vec{j}_1(\vec{r})$.
5. A new weight distribution is defined by $w_2(\vec{r}) = w(\vec{r}) \times |\vec{j}_1(\vec{r})|^q$ (using the same value of q).
6. Steps 1 and 2 are repeated with the new weight distribution w_2 to find a new current density $\vec{j}_2(\vec{r})$ and the whole process repeats until the algorithm converges.

Derivation

Rather than derive the iterated algorithm alone we will show how the iterated algorithm arises as a natural extension of the more standard algorithm. We start by minimising the functional

$$\sum_{i=1}^N \left[m_i - \int_Q \vec{j}(\vec{r}) \cdot \vec{L}_i(\vec{r}) \, d\vec{r} \right]^2 + \zeta \int_Q \frac{\vec{j}(\vec{r}) \cdot \vec{j}(\vec{r})}{w(\vec{r})} \, d\vec{r} \quad (4)$$

The Euler-Lagrange equations for this functional are:

$$-2 \sum_{i=1}^N m_i \vec{L}_i(\vec{r}) + 2 \sum_{i=1}^N \left(\vec{L}_i(\vec{r}) \int_Q \vec{j}(\vec{r}) \cdot \vec{L}_i(\vec{r}) \, d\vec{r} \right) + 2\zeta \frac{\vec{j}(\vec{r})}{w(\vec{r})} = \vec{0} \quad (5)$$

which holds pointwise for every vector \vec{r} in the source space. This simplifies to:

$$\sum_{i=1}^N \vec{L}_i(\vec{r}) \int_Q \vec{j}(\vec{r}) \cdot \vec{L}_i(\vec{r}) \, d\vec{r} + \zeta \frac{\vec{j}(\vec{r})}{w(\vec{r})} = \sum_{i=1}^N m_i \vec{L}_i(\vec{r}) \quad (6)$$

This equation has been used directly to solve the biomagnetic inverse problem using an approach based on Tikhonov regularisation [5]. This equation has many solutions so we impose the very natural restriction that the solution current density $\vec{j}(\vec{r})$ lies in the subspace spanned by the sensitivities of the detectors (arguing that the measurement can contain no information about silent sources). To allow some flexibility in the method we include the weighting parameter $w(\vec{r})$ in the basis of the reconstruction space $\{w(\vec{r})\vec{L}_i(\vec{r}) : i = 1, 2, \dots, N\}$. So we can write the reconstructed current density in terms of this basis as:

$$\vec{j}(\vec{r}) = \sum_{i=1}^N a_i \vec{L}_i(\vec{r}) w(\vec{r}) \quad (7)$$

Substituting this into equation 6 gives:

$$\sum_{i=1}^N \left(\vec{L}_i(\vec{r}) \int_Q \left[\sum_{k=1}^N a_k \vec{L}_k(\vec{r}) w(\vec{r}) \right] \cdot \vec{L}_i(\vec{r}) \, d\vec{r} \right) + \frac{\zeta}{w(\vec{r})} \sum_{i=1}^N a_i \vec{L}_i(\vec{r}) w(\vec{r}) = \sum_{i=1}^N m_i \vec{L}_i(\vec{r}) \quad (8)$$

Now we can equate the coefficients of the basis element $\vec{L}_i(\vec{r}) w(\vec{r})$ to get

$$\sum_{k=1}^N a_k \int_Q \vec{L}_k(\vec{r}) \cdot \vec{L}_i(\vec{r}) w(\vec{r}) \, d\vec{r} + \zeta a_i = m_i \quad (9)$$

which is usually written in matrix form as:

$$(P + \zeta I)a = m \text{ where } P_{ik} = \int_Q \vec{L}_k(\vec{r}) \cdot \vec{L}_i(\vec{r}) w(\vec{r}) \, d\vec{r} \quad (10)$$

This is a one step algorithm for solving the biomagnetic inverse problem [4,1]. Now we go on to derive the iterative method which has previously been used [1,2,3]. Start with the functional defined by:

$$\sum_{i=1}^N \left[m_i - \int_Q \vec{j}(\vec{r}) \cdot \vec{L}_i(\vec{r}) \, d\vec{r} \right]^2 + \zeta \int_Q \frac{[\vec{j}(\vec{r}) \cdot \vec{j}(\vec{r})]^{(q+2)/2}}{w(\vec{r})} \, d\vec{r} \quad (11)$$

Before we derive an algorithm based on the above functional, we consider its physical significance. The integral in the first term is the forward problem for the current $\vec{j}(\vec{r})$, so the whole term is just the sum of the squared deviations of the predicted measurements from the actual measurements.

The second term is a regularization term to make the problem computationally tractible. It represents a cost associated with the magnitude of the $(q+2)$ th moment of the field (and so $q=0$ corresponds to the minimum power solution). In the denominator is the weighting factor $w(\vec{r})$ which means that there is less cost at points where the weight is large. So if $w(\vec{r})$ is normalized to one it can be interpreted as a probability weight. An obvious extension of the method as written here is to have a tensor weight $W(\vec{r})$ and this has been used to model anisotropies in the cortex. The other parameter in the functional is the scalar ζ which controls the compromise between fitting the data (first term) and smoothness (second term).

Now we follow the same steps as in the above derivation, first we write down the Euler-Lagrange equations and simplify to get:

$$\sum_{i=1}^N \vec{L}_i(\vec{r}) \int_Q \vec{j}(\vec{r}) \cdot \vec{L}_i(\vec{r}) \, d\vec{r} + \frac{\zeta(q+2)}{2w(\vec{r})} \vec{j}(\vec{r}) \int_Q \frac{(\vec{j}(\vec{r}) \cdot \vec{j}(\vec{r}))^{q/2}}{w(\vec{r})} \, d\vec{r} = \sum_{i=1}^N m_i \vec{L}_i(\vec{r}) \quad (12)$$

This equation is not a linear equation unless $q = 0$, which was the case considered previously. So instead of just forming a matrix equation to solve this we must look for an iterative scheme. This scheme is the one outlined in the introduction. First compute the Gramm-Schmidt matrix:

$$P_{ik}^0 = \int_Q \vec{L}_k(\vec{r}) \cdot \vec{L}_i(\vec{r}) w(\vec{r}) d\vec{r} \quad (13)$$

and then use equation 10 to compute a current density \vec{j}^0 . This starts the iteration process. Now at the n th step we use the current density estimate \vec{j}^{n-1} to compute a new Gramm-Schmidt matrix:

$$P_{ik}^n = \int_Q \vec{L}_k(\vec{r}) \cdot \vec{L}_i(\vec{r}) w(\vec{r}) \left(\vec{j}^{n-1}(\vec{r}) \cdot \vec{j}^{n-1}(\vec{r}) \right)^{q/2} d\vec{r} \quad (14)$$

and then we calculate a new current density, \vec{j}^n , by solving equation 10 with the new Gramm-Schmidt matrix P_{ik}^n . Clearly, if this iteration scheme converges then it converges to a solution of equation 12. We have not shown that the above scheme will converge in general, but in practice we find that for typical experimental arrangements it converges very rapidly (usually within three iterations).

Examples and Discussion

The general properties of the iterated current density reconstruction algorithm compared to the minimum norm reconstruction can be deduced from the form of the functional. In the introduction it was mentioned that one of the reasons for using this algorithm is that localised sources are reconstructed as more localised distributions. This may be seen for a 1-d example in Fig. 1. A localised source at the origin is reconstructed as a distributed source for $q = 0$ and $q = 2$ but the distribution is more sharply peaked for the latter. This is a direct consequence of the form of the minimized functional i.e. a quadratic polynomial ($q = 0$) falls off more sharply than a quartic polynomial ($q = 2$).

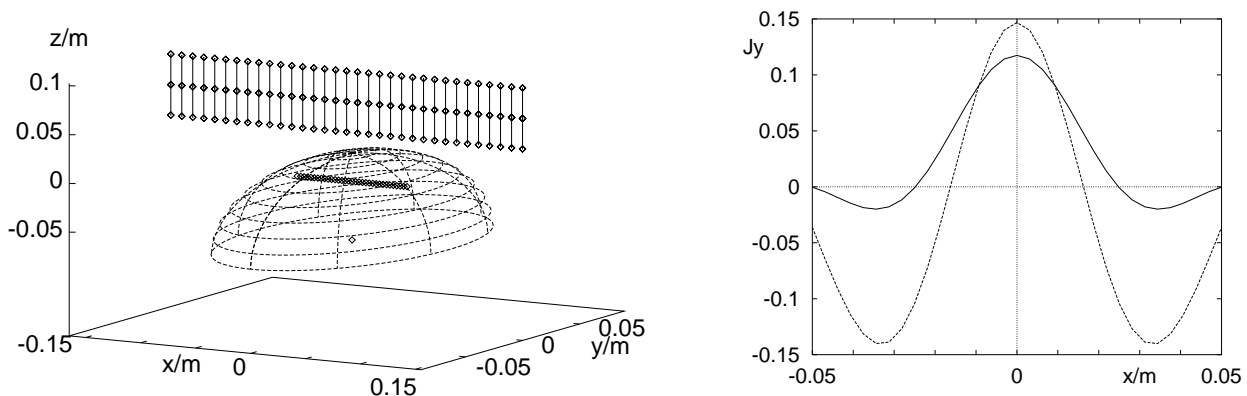


Fig. 1: (left) Geometry of apparatus which shows the positions of; the gradiometers (drawn as lines with dots at the position of the centre of each coil); the source space (a line); the conducting sphere. (right) One dimensional reconstruction. Solid line $q = 0$, dashed line $q = 2$.

The same localization effect can be seen in two dimension using data from a visually evoked response experiment. The data is the peak response (96ms) of the BT experiment described in [6]. The stimulus for the experiment was the pattern reversal of the bottom right quarter field. The left hand diagram in Fig. 2 shows the source space plane ($z = -7$ cm) below a flat 6×7 grid of second order gradiometers, baseline 3.125 cm, center coil in the plane $z = 0$ m. The head is modelled as a conducting sphere of radius 10 cm, with the centre at (0,4 cm,-14 cm). A localised source is thought to lie below (-0.02 m,-0.003 m).

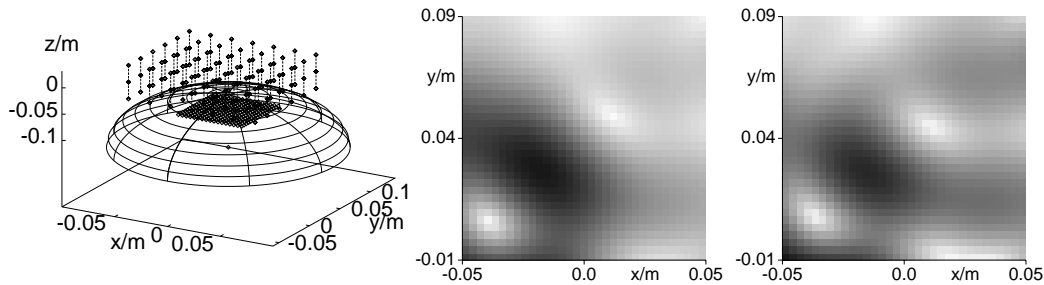


Fig. 2: (left) The arrangement for a visually evoked response experiment [6]. (middle) A real source reconstructed with a minimum norm algorithm ($\zeta = 1$). (right) The same source reconstructed using an iterated distributed current algorithm ($\zeta = 1, q = 2$).

The iterated reconstructions become more and more localized for higher q , but as q increases the algorithm develops more and more instability with respect to multiple sources. Even if there is only one source, there can be competition between the true source and spurious sources (reconstructed by the aliasing of distant sources). This is the situation in Fig. 3 where the test source is a dipole located at $(0 \text{ m}, 0.02 \text{ m}, 0)$ oriented along x . A large amount of gaussian random noise (total noise power is twice the total signal power) has been added but the minimum norm algorithm reconstructs the source accurately (Fig. 3 (left)), as does the iterated algorithm with $q = 1$ (Fig. 3 (middle)). However, the reconstruction with $q = 3$ has magnified the spurious source and this source will dominate for all higher values of q . The point at which this phenomenon occurs depends on the measurement geometry, the measurement data and the value of the smoothing parameter ζ . In conclusion we should be wary of sources that occur in the iterated reconstructions that do not appear in the minimum norm reconstruction.

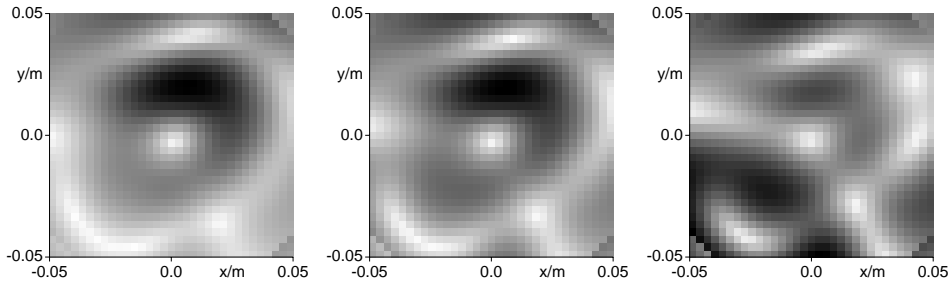


Fig. 3: (left) Minimum norm representation (i.e. $q = 0$) of a single dipole at $(0 \text{ m}, 0.02 \text{ m}, 0)$ with gaussian noise added ($\zeta = 0.5$). (middle) The same but with $q=1$. (right) The same but with $q=3$.

References

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