

A scaling law for the energy levels of a nonlinear Schrödinger equation

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Abstract

It is shown that the energy levels of the one-dimensional nonlinear Schrödinger, or Gross–Pitaevskii, equation with the homogeneous trap potential x^{2p} , $p \geq 1$, obey an approximate scaling law and as a consequence the energy increases approximately linearly with the quantum number. Moreover, for a quadratic trap, $p = 1$, the rate of increase of energy with the quantum number is independent of the nonlinearity: this prediction is confirmed with numerical calculations. It is also shown that the energy levels computed using a variational approximation do not satisfy this scaling law.

1. Introduction

The Bose–Einstein condensate is described, approximately, by a mean-field approximation, see for example Friedrich (1998), that gives the Gross–Pitaevskii equation. In the one-dimensional problem considered here this equation takes the form

$$-\frac{\hbar^2}{2\mu} \frac{d^2 y}{dx^2} + \frac{1}{2} \mu \omega^2 x^2 y + A |y|^2 y = E y, \quad (1)$$

where x is the spatial coordinate, μ the atomic mass of the atoms comprising the condensate and ω the classical frequency of a single atom in the trap potential. The nonlinear parameter A results from the use of a mean-field approximation to describe the particle interactions and is defined in terms of fundamental constants, $A = 4\pi\hbar^2\alpha_0 N/\mu$, where α_0 is the scattering length and N the effective density of atoms along the condensate axis. In most experimental circumstances the nonlinear constant A is large so perturbation methods are of little value. For the ground state, because the wavefunction varies relatively slowly and because the nonlinearity is large the Thomas–Fermi approximations, equation (11) below, provide a reasonable approximation to both the energy level and the wavefunction. For excited states no such simple approximation seems to be available. Yukalov *et al* (1997) have derived a re-normalized perturbation theory that gives approximate energy levels and wavefunctions, but we show in section 5 that this method seems to provide a poor estimate of the excited energy levels.

In this paper we show that the energy levels satisfy a simple approximate scaling law and consequently that they are given approximately by the simple formula

$$E_n(A) = \frac{1}{2} \left(\frac{3}{2} A \omega \sqrt{\mu} \right)^{2/3} + \frac{7\pi}{32} \omega \hbar n. \quad (2)$$

The first term is just the Thomas–Fermi estimate of the ground-state energy, obtained by neglecting the kinetic energy term. The second term is the dominant correction and is linear in n independent of A . We show also that the latter behaviour is a consequence of the particular form of the trap potential.

2. Theory

The eigenvalues of equation (1), $E_n(A)$, $n = 0, 1, 2, \dots$, are those values of E for which $y(x)$ satisfy the boundary conditions $|y| \rightarrow 0$ as $|x| \rightarrow \infty$ and the normalization condition

$$\int_{-\infty}^{\infty} dx |y(x)|^2 = 1. \quad (3)$$

For real eigenvalues we may assume $y(x)$ to be real.

Two of the four independent parameters in this equation may be removed by rescaling x and y and ensuring that the normalization conditions are invariant,

$$x = \alpha x', \quad y = \frac{y'}{\sqrt{\alpha}}, \quad \omega = \frac{\omega'}{\hbar}, \quad A = \alpha A', \quad \alpha = \frac{\hbar}{\sqrt{\mu}},$$

which replaces μ and \hbar by unity. In the following we drop all primes.

By re-writing equation (1) in the form

$$\frac{d^2 y}{dx^2} + \frac{\partial V}{\partial y} = 0, \quad V(y, x) = \bar{E}(x)y^2 - \frac{1}{2}Ay^4, \quad \bar{E}(x) = E - \frac{1}{2}\omega^2 x^2 \quad (4)$$

and treating x as the ‘time’ we may interpret equation (1) as that of a classical particle of unit mass moving in a time-dependent potential, $V(y, x)$. Conventional methods of classical dynamics provide a means of estimating the eigenvalues.

The potential $V(y, x)$ is stationary at $y = 0$ and this is a minimum for times $x < x_0 = \sqrt{2E}/\omega$; for these times there are also maxima at

$$y^2 = y_m(x)^2 = \bar{E}(x)/A.$$

For larger times, when $\bar{E}(x) < 0$, there is only a maximum at $y = 0$. Hence quasi-periodic motion is possible for small times but for larger times almost all orbits diverge as $|x| \rightarrow \infty$: for every $E > 0$, however, there are initial conditions for which $y(x) \rightarrow 0$ as $x \rightarrow \infty$.

To be specific consider the even solution with initial conditions $y(0) = a > 0$ and $y'(0) = 0$. For small a and large enough E this orbit will oscillate in the potential well until the barrier at $y = y_m(x)$ is low enough for the orbit to either escape or ride on the barrier top and eventually to zero: most orbits escape to infinity. Examples of these types of orbit are shown in figure 1. Here $E = 15.0810$, $A = 100$, $\omega = 1$, $a = a_1 = 0.23975967$ and $a = a_1 \pm 0.0000001$; the converged solution is not normalized.

This figure shows that the required solutions with $y(x) \rightarrow 0$ as $|x| \rightarrow \infty$ comprise a quasi-periodic part, for $|x| < x_t$ where x_t is defined in equation (9) below, and a monotonically decreasing segment for $|x| > x_t$. It also shows that the distance between nodes is almost constant: reasons for this are discussed later.

Consider the oscillatory region. When $\bar{E} = \text{constant}$ it follows from the definition of the Jacobi elliptic function that the odd and even solutions are, respectively

$$y = a \operatorname{sn}(\tau, k), \quad (y(0) = 0), \quad y = a \operatorname{sn}(K - \tau, k), \quad (y(0) = a), \quad (5)$$

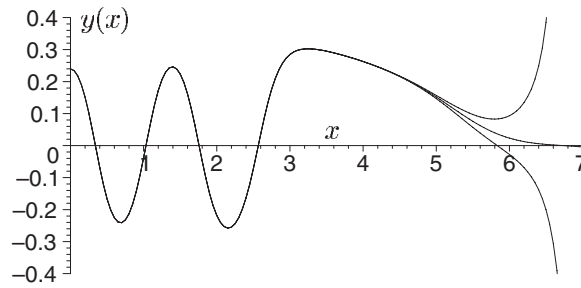


Figure 1. Some examples of even solutions of equation (1), with $E = 15.0810$ and $y(0) = a$, given in the text.

where $K = K(k)$ is the complete elliptic integral of the first kind and

$$\tau = x\sqrt{2\bar{E} - a^2A}, \quad k^2 = \frac{Aa^2}{2\bar{E} - Aa^2}.$$

The period of these oscillations is

$$T = \frac{4}{\sqrt{2\bar{E} - Aa^2}}K(k). \tag{6}$$

When \bar{E} is constant the action of the above oscillatory solution may be written in the form

$$I = \frac{1}{2}a^2\sqrt{2\bar{E}}F(k), \quad F(k) = \frac{4}{3\pi k^2\sqrt{1+k^2}}((1+k^2)E(k) - (1-k^2)K(k)), \tag{7}$$

where $E(k)$ is the complete elliptic integral of the second kind, and is not to be confused with the energy. For each \bar{E} there is bound motion if $0 < Aa^2 < \bar{E}$ and as k increases from zero to unity $F(k)$ decreases from 1 to $4\sqrt{2}/(3\pi) \simeq 0.6$. The action is bounded by $0 \leq I \leq I_s$, where I_s is the action of the bound, non-periodic motion on the separatrix, where $Aa^2 = \bar{E}$ ($k = 1$),

$$I_s = \frac{4}{3\pi A}\bar{E}^{3/2}. \tag{8}$$

Now consider the effect of \bar{E} decreasing, but changing little during one period of the unperturbed motion. The principle of adiabatic invariance (Percival and Richards 1982, chapter 9) shows that the action is almost invariant. The separatrix action, however, is not constant and decreases to zero at $x = x_0$ where $\omega x_0 = \sqrt{2\bar{E}}$. All orbits cease to oscillate before this time and if the change in \bar{E} is sufficiently slow this change occurs when the action is equal to the separatrix action. If x_t is this time it is given by the solution of

$$\frac{4}{3\pi} \left(E - \frac{1}{2}\omega^2 x_t^2 \right)^{3/2} = AI(E) \tag{9}$$

where the action is evaluated at E , the initial value of \bar{E} . Adiabatic invariance shows that the solution oscillates with a local period, T , given by equation (6), which depends upon x . However, the period, although singular at $\bar{E}(x) = Aa^2$, does not change significantly until $\bar{E}(x)$ is close to Aa^2 , so the nodes of the wavefunction are almost equally spaced.

The quantum number, n , that labels the state is the number of zeros in the eigenfunction. The ground state, $n = 0$, has no zeros: the first excited state is odd and has one zero at the origin and the second excited state is even and has two zeros. Thus the oscillatory parts of the solution are represented by orbits that encircle the phase-space origin $(n + 1)/4$ times before

approaching the origin almost parallel to the y' -axis. There are $n/4$ oscillations in the interval $0 \leq x \leq x_t$ so we have the approximate relation $x_t = nT/4$. For later use it is convenient to introduce the scaled variables

$$N = \frac{\pi}{2}\omega n, \quad \mathcal{E} = \frac{E}{N}, \quad \text{and} \quad z = \frac{2E}{Aa^2} \geq 2,$$

in terms of which $k^2 = 1/(z - 1)$ and the quantization condition becomes

$$\omega x_t(\mathcal{E}, z) = \sqrt{\frac{N}{2\mathcal{E}}}g(z), \quad g(z) = \frac{2K(k)}{\pi\sqrt{1-1/z}}. \tag{10}$$

For large z , $g(z) = 1 + \frac{3}{4z} + O(z^{-2})$.

Finally, we need an approximation to the motion for $x > x_t$. The value of $y(x_t)$ must be close to the barrier height, $y(x_t) \simeq y_m(x_t)$: if $y(x_t) \ll y_m(x_t)$ the orbit would complete another $\frac{1}{2}$ period and if $y(x_t) > y_m(x_t)$ it would escape. But if $y(x_t) \simeq y_m(x_t)$ the required subsequent orbit is approximated by expanding about the point in phase space that follows the potential maximum, by making the canonical transformation

$$y = Q + y_m(x), \quad \frac{dy}{dx} = P + \frac{dy_m}{dx}$$

and expanding the equations of motion to second order. Then if $x_0 > 0$ is the time $\bar{E}(x_0) = 0$ for $x_t < x < x_0$ the equations of motion are

$$\frac{dQ}{dx} = P, \quad \frac{dP}{dx} = \bar{E}(x)Q - \frac{d^2y_m}{dx^2}, \quad \frac{d^2y_m}{dx^2} = -\frac{E\omega^2}{2\bar{E}(x)\sqrt{A\bar{E}(x)}}.$$

These equations may be solved numerically and it is seen that $Q(x)$ remains small provided both $|P(x_t)|$ and $|\bar{E}(x_t)Q(x_t) - y_m''(x_t)|$ are small or zero. As $x \rightarrow x_0$ the solution diverges. However, over the interval of interest this expansion shows that an approximate solution is

$$y(x) \simeq y_m(x) = \sqrt{\frac{\bar{E}(x)}{A}}, \quad x_t \leq x < x_0, \quad \bar{E}(x_0) = 0. \tag{11}$$

This is, of course, the standard Thomas–Fermi approximation, obtained from equation (1) by ignoring the kinetic energy term.

Some idea of the accuracy of the approximations (5) and (11) is given in figure 2, comparing these with an exact solution. In this case $E = 15$ and $A = 100$, which gives $a = 0.23976$ and $x_t = 2.7272$.

In the next section we use equations (3), (8) and (5) to approximate the eigenvalues of equation (1) and to obtain an approximate scaling law.

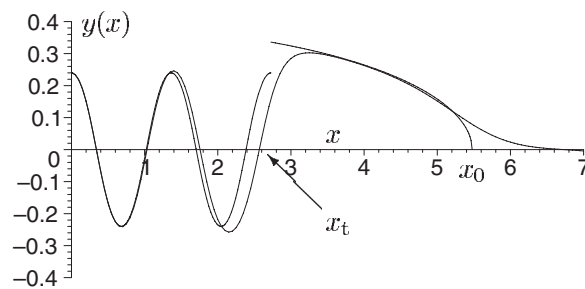


Figure 2. Graphs of the exact solution, the Thomas–Fermi solution (11) for $x_t \leq x \leq x_0$ and the adiabatic solution (5) for $0 \leq x \leq x_t$ in the case $E = 15$ and $A = 100$, for which $a = 0.23976$ and $x_t = 2.7272$.

3. An approximate scaling law

Here we show that the approximations described above may be used to derive an approximate scaling law relating the energy, E , quantum number n and nonlinearity parameter A by the single equation

$$\mathcal{E} = H(\omega AN^{-3/2}), \quad \mathcal{E} = \frac{2E}{\pi n \omega} = \frac{E}{N} \tag{12}$$

for some function H . A consequence of this is that the energy levels behave like those of the linear oscillator in that the difference $E_{n+1}(A) - E_n(A)$ is almost independent of n and also of A .

In order to derive this relation we first express z in terms of \mathcal{E} using the adiabatic and the quantization conditions, equations (9) and (10) respectively. These equations may be combined to give

$$\frac{2\sqrt{2}}{3\pi} \left(1 - \frac{g(z)^2}{4\mathcal{E}^2}\right)^{3/2} = \frac{F(k)}{z}, \quad k^2 = \frac{1}{z-1} \tag{13}$$

which, in principle, gives $z(\mathcal{E})$. The behaviour of this function is shown in figure 3, where $1/z$ is plotted as a function of $\mathcal{E} = E/N$.

As $z \rightarrow \infty, k \rightarrow 0, g \rightarrow 1$ and $F \rightarrow 1$, and so $2\mathcal{E} \rightarrow 1$: in this limit,

$$\frac{1}{z} = \frac{2\sqrt{2}}{3\pi} \frac{(4\mathcal{E}^2 - 1)^{3/2}}{(2\mathcal{E})^3}, \quad 2\mathcal{E} \sim 1.$$

As \mathcal{E} increases $1/z(\mathcal{E})$ increases monotonically to $1/2$.

The normalization condition, equation (3), can be written in the form

$$1 = 2na^2 \int_0^{T/4} dx \operatorname{sn}(\tau, k)^2 + 2 \int_{x_1}^{x_0} dx \frac{\bar{E}(x)}{A}. \tag{14}$$

The first of these integrals may be evaluated using relations given by Abramowitz and Stegun (1965, section 16.25), so we have

$$1 = \frac{2na^2}{\sqrt{2E - Aa^2}} \frac{K(k) - E(k)}{k^2} + \frac{2}{3\omega A} (2E)^{3/2} - \frac{x_1}{3A} (6E - \omega^2 x_1^2). \tag{15}$$

In terms of the scaled variables introduced in equation (10) this becomes

$$\frac{3A\omega}{2N^{3/2}} = (2\mathcal{E})^{3/2} \left\{ 1 + \frac{3}{4\mathcal{E}} \left(\frac{4}{\pi} \frac{K(k) - E(k)}{k^2 \sqrt{z(z-1)}} - g(z) \right) + \frac{g(z)^3}{16\mathcal{E}^3} \right\}. \tag{16}$$

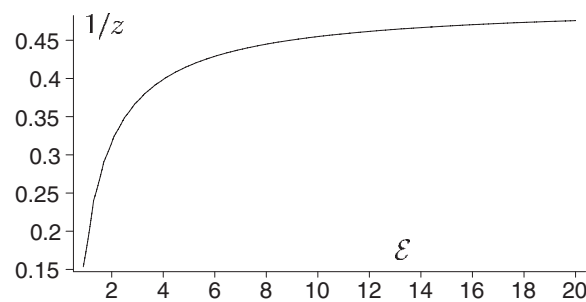


Figure 3. Graph of $1/z(\mathcal{E})$.

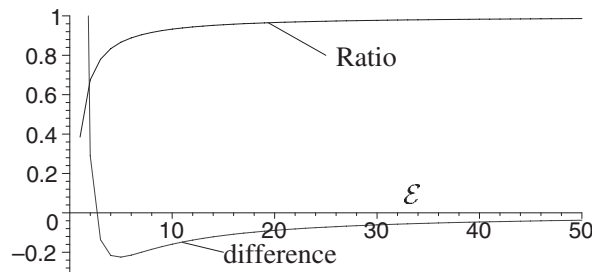


Figure 4. Graphs of the ratio $R(\mathcal{E})$, equation (17), and the difference $100(R(\mathcal{E}) - R_1(\mathcal{E}))$.

Since $k^2 = 1/(z - 1)$ and z is a function of \mathcal{E} through equation (13), the right-hand side of this equation depends only upon \mathcal{E} . Thus \mathcal{E} is a function only of the variable $\omega AN^{-3/2}$, which is the scaling law (12).

This analysis can be carried further with more approximations, but first we show in figure 4 the graph of the ratio

$$R(\mathcal{E}) = \frac{3A\omega}{2N^{3/2}} \frac{1}{(2\mathcal{E})^{3/2}}. \quad (17)$$

Figure 4 suggests that $R(\mathcal{E})$ tends to unity and this can be seen from equation (16), and the fact that $z \rightarrow 2$, as $\mathcal{E} \rightarrow \infty$.

Expanding equation (16) in powers of $1/z$ gives

$$\frac{3A\omega}{2N^{3/2}} = (2\mathcal{E})^{3/2} \left\{ 1 - \frac{3}{4\mathcal{E}} \left(1 - \frac{1}{4z} + \dots \right) + \frac{1}{16\mathcal{E}^3} \left(1 + \frac{9}{4z} + \dots \right) \right\}. \quad (18)$$

An analysis of $R(\mathcal{E})$ suggests that $1 - R(\mathcal{E}) \sim \mathcal{E}^{-1}$ for large \mathcal{E} , that in this range $z \simeq \frac{1}{2}$ and that z changes relatively slowly with \mathcal{E} . Thus a simple approximation to this ratio is given by setting z equal to its asymptotic value, $z = 2$, to give

$$R(\mathcal{E}) \simeq R_1(\mathcal{E}) = 1 - \frac{21}{32\mathcal{E}}.$$

The graph of $100(R(\mathcal{E}) - R_1(\mathcal{E}))$ is shown in figure 4 and this demonstrates the accuracy of this simple approximation.

On using R_1 to approximate $R(\mathcal{E})$ in equation (16) and rearranging the equation we obtain

$$E_n(A) = \frac{1}{2} \left(\frac{3A\omega}{2} \right)^{2/3} + \frac{7\pi}{32} \omega n + \text{higher-order terms}. \quad (19)$$

The first term in this equation is just the Thomas–Fermi approximation, which follows from the normalization condition, equation (14), by setting $x_t = 0$. The second term increases linearly with n and, because the trap potential is quadratic, is independent of A . Higher-order corrections come from the expansion about the asymptotic value of z and are complicated and not warranted because of other approximations made.

The scaling law (12) exists because the trap potential is homogeneous in x , so the adiabatic condition (9) may be expressed in terms of only two variables. For the quadratic potential these are $\mathcal{E} = \frac{2E}{\pi\omega n} = \frac{E}{N}$ and $z = \frac{2E}{Aa^2}$ and it is the form of these variables that gives the scaling law (12) and ultimately the energy level (19). If the trap potential is $(\omega x)^{2p}/2p$, $p \geq 1$, the scaled energy may be taken to be $\mathcal{E} = 2EN^{-2p/(2p+1)}$ and then the scaling law 12 becomes

$$E = N^{\frac{2p}{2p+1}} H \left(\frac{A\omega}{N^{\frac{2p+1}{p+1}}} \right)$$

and the energy levels become

$$E_n(A) = \frac{1}{2^p} \left(\left(p + \frac{1}{2} \right) A \omega \right)^{\frac{2p}{2p+1}} + \frac{7\pi \omega n}{32\sqrt{p}} \left(\left(p + \frac{1}{2} \right) A \omega \right)^{\frac{p-1}{2p+1}}. \quad (20)$$

When $p = 1$ this reduces to equation (19), but when $p \neq 1$ the coefficient of n depends upon the nonlinearity, A .

4. Variational method

Yukalov *et al* (1997) have used re-normalized perturbation theory to obtain analytic approximations to the energy levels of the three-dimensional nonlinear Schrödinger equation. Here we show that this method is equivalent to a Euler–Lagrange variational method and that the resulting energy levels of the excited states do not satisfy the scaling law described in equation (19). Thus this method cannot be as accurate as implied by Yukalov *et al* (1997).

With the Lagrangian

$$L(y, y', x) = \frac{1}{2} \left(\frac{dy}{dx} \right)^2 + \frac{1}{2} \omega^2 x^2 y^2 + \frac{1}{2} A y^4 \quad (21)$$

and treating the energy as the Lagrange multiplier we see that the Euler–Lagrange equations with the functional and the constraint

$$\bar{J}[y] = \int_{-\infty}^{\infty} dx [L(y, y', x) - E y^2], \quad \int_{-\infty}^{\infty} dx y(x)^2 = 1,$$

give equation (1), with $\mu = \hbar = 1$, and that the energy is then given by

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \left(\frac{dz}{dx} \right)^2 + \frac{1}{2} \omega^2 x^2 z^2 + A z^4 \right] \quad (22)$$

where $z(x)$ is a solution of the Euler–Lagrange equation. For trial functions satisfying the normalization condition we may use the simpler functional

$$J[y] = \int_{-\infty}^{\infty} dx L(y, y', x). \quad (23)$$

A natural trial function is

$$z(x) = \sqrt{\frac{a}{h_n}} H_n(ax) \exp\left(-\frac{1}{2} a^2 x^2\right), \quad h_n^2 = 2^n n! \sqrt{\pi} \quad (24)$$

where a is the variational parameter. Then the functional (23) becomes

$$J(a) = \frac{1}{2} \left(n + \frac{1}{2} \right) \left(a^2 + \frac{\omega^2}{a^2} \right) + \frac{aA}{2h_n^2} I_n, \quad I_n = \int_{-\infty}^{\infty} dw H_n(w)^4 e^{-2w^2}. \quad (25)$$

This is stationary so the appropriate value of a is given by the positive root of

$$\frac{\omega^2}{a^3} = a + \frac{AI_n}{(2n+1)h_n^2} \quad \text{and then} \quad E_n = \frac{1}{2} \left(n + \frac{1}{2} \right) \left(a^2 + \frac{\omega^2}{a^2} \right) + \frac{aA}{h_n^2} I_n. \quad (26)$$

If $A = 0$ these equations give the unperturbed energy levels and if A is small perturbation theory may be used to obtain the equivalent of Yukalov *et al* (1997), equation (44). For $A \gg 1$ and $n = 0$ they give $E_0 = 0.677(\omega A)^{2/3}$, which is 3.4% larger than the Thomas–Fermi energy, given by the first term in equation (19). In this limit of large A perturbation theory may be used to give

$$E_n = \frac{5}{4}(2n+1)\omega^2 B^{2/3} \left(1 + \frac{\epsilon}{15} + \frac{\epsilon^2}{15} + \dots \right), \quad B = \frac{AI_n}{(2n+1)\omega^2 h_n^2}, \quad \epsilon = \frac{1}{\omega^2 B^{4/3}}. \quad (27)$$

It is also clear from equations (26) that E/N depends only upon the variable $z = AI_n/((2n+1)h_n^2\sqrt{\omega})$, which is different from the scaling law derived in the previous section.

5. Numerical results

In this section we compare the behaviour of the energy levels of equation (1), computed numerically, with the predictions of the above formula, equations (19) and (26).

One method of numerically solving equation (1) is to perform a two-dimensional search in the (a, E) plane, where E is the energy and for even solutions $y(0) = a > 0$ and for odd solutions $y'(0) = a > 0$. These solutions must (a) satisfy the quantization condition, (b) tend to zero as $x \rightarrow \infty$ and (c) satisfy the normalization condition. Since most solutions are unbounded this calculation is expedited by using a good first approximation, which is given by

$$\tilde{y}(x) = \begin{cases} (a + xy_m(x_t)/x_t) \cos \Omega x, & 0 \leq x \leq x_t \\ y_m(x), & x_t \leq x \leq x_0 \\ 0, & x > x_0 = \sqrt{2E} \end{cases}$$

where $y_m(x)$ is the Thomas–Fermi solution defined in equation (11) and $\Omega = 2\pi/T$ where T is the period defined in equation (6). In practice the harmonic balance approximation $\Omega^2 = 2E - 2a^2A/2$ was used for Ω . The oscillatory part of this approximation has a slowly increasing amplitude in order that $\tilde{y}(x)$ is continuous at $x = x_t$.

This approximation has two free parameters, a and E , which were varied using the Marquardt algorithm to find values that simultaneously satisfied the normalization condition (3) and the quantization condition (10). For $A = 200$ this crude approximation gives a relative error of less than 1% for the ground state and 5% for the 16th energy level.

In the second stage of the calculation we use the energy E found above and vary a to find a value at which $|y(x_f)| < \delta$, for some small δ and where $x_f = 1.25x_0$. This was achieved using a shooting algorithm that varied a according to the value of $y(x_f)$. The solution obtained in this manner is not normalized, but we find that for small changes in E , $\int_0^{x_f} dx y(x)^2$ depends approximately linearly on E so it is possible to interpolate the energy to obtain values of (a, E) that give a correctly normalized solution.

In table 1 energy levels are shown for $A = 100$ and 200. The exact numerical values are well approximated by the straight lines $E_n \simeq 14.04 + 0.66n$ and $E_n \simeq 22.40 + 0.74n$, for $A = 100$ and 200 respectively, and the gradient of these lines is close to that predicted

Table 1.

$A = 100$				
	$n = 0$	$n = 2$	$n = 4$	$n = 6$
E_n (numerical)	14.02	15.37	16.69	17.98
E_n (equation (19))	14.12	15.47	16.84	18.20
E_n (equation (26))	14.60	18.70	20.34	21.69
$A = 200$				
	$n = 0$	$n = 2$	$n = 4$	$n = 6$
E_n (numerical)	22.42	23.87	25.34	26.86
E_n (equation (19))	22.41	23.77	25.13	26.49
E_n (equation (26))	23.17	29.53	31.77	33.34

by equation (19). The energy levels of the variational method do not behave in this manner, particularly for large A , and we conclude that the excited energy levels given by the re-normalized perturbation method used by Yukalov *et al* (1997) are not accurate for the one-dimensional nonlinear Schrödinger equation.

6. Conclusions

We have shown that the energy levels E_n of the Gross–Pitaevskii equation (1) satisfy the approximate scaling law (12), which relates the variables E , n , ω , A in a single equation, which leads to the approximate energy levels (2). We have shown that other homogeneous trap potentials lead to similar scaling laws but only the energy levels of the quadratic trap have a coefficient of n that is independent of the nonlinear constant (see equation (20)). It is also shown that the energy levels of the re-normalized perturbation method of Yukalov *et al* (1997) are equivalent to a simple variational method and do not satisfy the scaling law derived here.

The method used to derive these results involves interpreting the Gross–Pitaevskii equation as a mechanical system with a slowly varying potential, so that the idea of adiabatic invariance can be used. With this equivalence the spatial coordinate becomes the time, so the generalization to the two- or three-dimensional Gross–Pitaevskii equation is not apparent. For symmetric, many-dimensional systems, however, a similar approach may be possible, though there are some problems with singularities at the origin that need to be resolved.

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