Universal anomalous diffusion of weakly damped particles

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We show that anomalous diffusion arises in two different models for the motion of randomly forced and weakly damped particles: one is a generalization of the Ornstein-Uhlenbeck process with a random force, which depends on position as well as time, the other is a generalization of the Chandrasekhar-Rosenbluth model of stellar dynamics, encompassing non-Coulombic potentials. We show that both models exhibit anomalous diffusion of position x and momentum p with the same exponents: $\langle x^2\rangle \sim C_x t^2$ and $\langle p^2\rangle \sim C_p t^{2/5}$. We are able to determine the prefactors C_x, C_p analytically.

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I. INTRODUCTION

In many systems the growth of a dynamical variable X with time t satisfies $\langle X^2\rangle \sim Ct^\alpha$ where the angular brackets denote averaging. The process is said to be anomalous diffusion if $\alpha \neq 1$. Anomalous diffusion may be a consequence of a power law built into the dynamical process, such as in Lévy flight models [1], or it may be an emergent property, where the anomalous exponent α is not a direct consequence of power laws, which are built into the model. The latter case is more interesting, because noninteger exponents do not feature in the fundamental laws of physics, but there are relatively few models where emergent anomalous diffusion can be analyzed exactly. In this paper we describe two physically natural models for the diffusion of a particle that is accelerated by random forces. If the damping is sufficiently weak the particle can exhibit anomalous diffusion, having universal exponents, the same for each model.

Our two models are generalizations of two classic models for diffusion processes. The first is an extension of the Ornstein-Uhlenbeck process [2], in which a particle is subjected to a rapidly fluctuating random force, and is damped by viscous drag. In the generalized Ornstein-Uhlenbeck process the random force depends upon the position of the particle and time and is derived from a potential $\Phi(\boldsymbol{x},t)$. Earlier works analyzed this model in detail for one spatial dimension [3,4]. This model exhibits anomalous diffusion in one dimension. Here we discuss higher spatial dimensions, where the mechanism for anomalous diffusion is significantly different, as was suggested (for a closely related model) in Refs. [5,6]. Here we obtain exact formulas for the momentum distribution of the generalized Ornstein-Uhlenbeck process in two and three dimensions (for a particle that is initially at rest). We use these to obtain precise asymptotic formulas for the growth of the second moment of the coordinate: the second moments scale as $\langle p^2\rangle \sim t^{2/5}$ and $\langle x^2\rangle \sim t^2$ in the anomalous diffusion regime (the exponents are the same as those obtained in Ref. [6]; we obtain the prefactor exactly).

We also discuss an extension of the Chandrasekhar-Rosenbluth model for diffusion [7,8], in which a test particle interacts with a gas of point masses via a pair potential. The interaction should cause small changes of momentum, which can be modeled as a diffusion process. Usually the interaction is gravitational, and the application is to the motion of stars in galaxies, but here we simplify the problem by considering a nonsingular weak interaction potential. We show that, surprisingly, the diffusion tensor has the same form as for the generalized Ornstein-Uhlenbeck process, and that consequently there is anomalous diffusion with the same universal exponents. For this model too, we derive diffusion coefficients for this model precisely in terms of the microscopic parameters.

The anomalous diffusion effect that we describe is analyzed by introducing a diffusion process describing the fluctuations of the momentum p of a particle in response to a spatially and temporally fluctuating random potential. The diffusion coefficient of this process, $D(p)$, is a function of the momentum of the particle. We remark that this approach to formulating the equation of motion was first introduced by Sturrock [9], and that similar developments appeared later in the mathematical literature (see Ref. [10] and references cited therein). Our paper gives a solution to the generalized Ornstein-Uhlenbeck process in two or three dimensions. The results on anomalous diffusion are expected to find applications in studies of optical models for random potentials (see Refs. [11,12]).

II. GENERALIZED ORNSTEIN-UHLENBECK MODEL

We cosnider a particle of mass m with momentum $\boldsymbol{p}$ subjected to the generalized Ornstein-Uhlenbeck process [4]. This is described by equations of motion

$$\dot{\boldsymbol{x}} = \frac{\boldsymbol{p}}{m}, \quad \dot{\boldsymbol{p}} = -\gamma\boldsymbol{p} + \boldsymbol{f}(\boldsymbol{x},t), \tag{1}$$

where $\boldsymbol{x}$ is the particle's position. The particle experiences two types of forces: a drag force $-\gamma\boldsymbol{p}$ with γ being a damping rate and a random force $\boldsymbol{f}(\boldsymbol{x},t)$. Unlike the classic Ornstein-Uhlenbeck process, where the force depends only upon time, in our generalized model the random force depends upon the position as well. We assume that $\boldsymbol{f}(\boldsymbol{x},t)$ is a force derived from a random potential varying in time and space [i.e., $\boldsymbol{f}(\boldsymbol{x},t) = -\nabla\Phi(\boldsymbol{x},t)$] where $\Phi(\boldsymbol{x},t)$ has statistics

$$\langle\Phi(\boldsymbol{x},t)\rangle = 0, \quad \langle\Phi(\boldsymbol{x},t)\Phi(\boldsymbol{x}',t')\rangle = C(|\boldsymbol{x}-\boldsymbol{x}'|,|t-t'|). \tag{2}$$

The correlation function $C(x,t)$ has temporal and spatial scales τ and ξ respectively [that is, $C(x,t) = c(x/\xi,t/\tau)$, where c is a function of two dimensionless variables]. We consider the case where $\gamma\tau \ll 1$, implying that the momentum satisfies a diffusion equation. We define a momentum scale $p_0 = m\xi/\tau$, such that if $|\boldsymbol{p}| \gg p_0$ the force experienced by the particle decorrelates much more rapidly than the force experienced by a stationary particle. We consider the limit where $|\boldsymbol{p}| \gg p_0$, which is realized for weak damping. The dynamics of Eq. (1) can be described by a diffusion equation for the probability density of the momentum, $P(\boldsymbol{p},t)$; we now consider how to derive this diffusion equation. We assume that the correlation function decays sufficiently rapidly that the integrals in Eq. (6) exist.

The dynamics of the momentum can be approximated by a Langevin process. Small increments of components p_i of the momentum vector $\boldsymbol{p}$ may be written as

$$\delta p_i = -\gamma p_i \delta t + \delta w_i, \tag{3}$$

where δw_i is the impulse exerted by the ith component of the force $\boldsymbol{f}$ on the particle in time δt,

$$\begin{aligned}\delta w_i(t_0) &= \int_{t_0}^{t_0+\delta t} dt_1\, f_i(\boldsymbol{x}(t_1),t_1)\\ &= \int_{t_0}^{t_0+\delta t} dt_1\, f_i(\boldsymbol{p}t_1/m,t_1) + O(\delta t^2).\end{aligned} \tag{4}$$

The Langevin process (3) is equivalent to the Fokker-Planck equation describing time evolution of the probability density of momentum $P(\boldsymbol{p},t)$. In order to construct the equation we need to know drift and diffusion coefficients, $v_i = \langle \delta p_i\rangle/\delta t$ and $D_{ij} = \langle \delta p_i \delta p_j\rangle/2\delta t$ respectively. Using the definition of the increment δw_i we find

$$\begin{aligned}\langle \delta w_i \delta w_j\rangle &\sim \int_0^{\delta t} dt_1 \int_0^{\delta t} dt_2 \langle f_i(\boldsymbol{p}t_1/m,t_1) f_j(\boldsymbol{p}t_2/m,t_2)\rangle\\ &\sim \delta t \int_{-\infty}^{\infty} dt_1\, \langle f_i(\boldsymbol{0},0) f_j(\boldsymbol{p}t_1/m,t_1)\rangle\\ &= 2D_{ij}\delta t\end{aligned} \tag{5}$$

(the second step is justified when δt is large compared to τ but small compared to γ^{-1}).

The components D_{ij} of the momentum diffusion tensor $\mathbf{D}$ depend upon the direction of the momentum. For the case where the momentum is aligned with the x axis (that is, where $\boldsymbol{p} = p\mathbf{e}_1$, the coefficients of the diffusion matrix are expressed in terms of the correlation function of the potential as follows:

$$\begin{aligned}D_{xx} &= -\frac{m}{2p}\int_{-\infty}^{\infty} dR\, \frac{\partial^2 C}{\partial R^2}(R,mR/p)\\ D_{yy} &= -\frac{m}{2p}\int_{-\infty}^{\infty} dR\, \frac{1}{|R|}\frac{\partial C}{\partial R}(R,mR/p)\\ D_{xy} &= 0.\end{aligned} \tag{6}$$

(In three dimensions $D_{zz} = D_{yy}$ and $D_{xz} = D_{yz} = 0$.) For other directions of the momentum, the elements of the momentum diffusion tensor can be obtained by applying a rotation matrix. If $\mathbf{O}$ is a rotation matrix that rotates the momentum vector $\boldsymbol{p}$ into $\boldsymbol{p}' = \mathbf{O}\boldsymbol{p} = p\mathbf{e}_1$, which is aligned with the x axis, then the elements D'_{ij} of the diffusion matrix in the transformed coordinate system are given by Eq. (6). The diffusion matrix in the original coordinate system is $\mathbf{D} = \mathbf{O}\,\mathbf{D}'\,\mathbf{O}^{\mathrm{T}}$.

Having considered the diffusive fluctuations of the momentum, we now consider its drift, $\langle \delta p_i\rangle = -\gamma p_i \delta t + \langle \delta w_i\rangle$. Expanding $f_i(\boldsymbol{x},t)$ we obtain (in d dimensions)

$$f_i(\boldsymbol{x},t) = f_i(\boldsymbol{0},t) + \sum_{j=1}^{d} \frac{\partial f_i(\boldsymbol{0},t)}{\partial x_j} x_j(t), \tag{7}$$

where $x_j(t)$ can be written as a solution of Eq. (1),

$$x_j(t) = \frac{1}{m}\int_0^t dt_1 \int_0^{t_1} dt_2 \exp[-\gamma(t_1-t_2)] f_j(\boldsymbol{p}t_2/m,t_2). \tag{8}$$

Combining Eqs. (4), (7), and (8) we obtain

$$\begin{aligned}\langle \delta w_i\rangle \sim \frac{1}{m}\sum_{j=1}^{d}\int_0^{\delta t} dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3\ \exp[-\gamma(t_2-t_3)]\\ \times\left\langle \frac{\partial f_i(\boldsymbol{0},t_1)}{\partial x_j} f_j(\boldsymbol{p}t_3/m,t_3)\right\rangle.\end{aligned} \tag{9}$$

We approximate $\exp[-\gamma(t_2-t_3)]$ by unity for $\gamma\tau \ll 1$, and use the assumption that $\delta t \gg \tau$ to obtain

$$\begin{aligned}\langle \delta w_i\rangle &\sim \frac{\delta t}{2m}\sum_{j=1}^{d}\int_{-\infty}^{\infty} dt\ t\left\langle \frac{\partial f_i(\boldsymbol{0},0)}{\partial x_j} f_j(\boldsymbol{p}t/m,t)\right\rangle\\ &= \delta t \sum_{j=1}^{d} \frac{\partial}{\partial p_j} D_{ij}(\boldsymbol{p}).\end{aligned} \tag{10}$$

Taking account of the fact that the drift coefficients contain derivatives of the diffusion coefficients, we obtain the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \boldsymbol{p}}\left(\gamma \boldsymbol{p} + \mathbf{D}(\boldsymbol{p})\frac{\partial}{\partial \boldsymbol{p}}\right) P. \tag{11}$$

We are primarily interested in the case of weak damping, where the typical momentum satisfies $|\boldsymbol{p}| \gg p_0$. In this limit, the diffusion coefficients have an algebraic dependence upon $p = |\boldsymbol{p}|$. We follow a similar approach to Ref. [4]: for diffusion of the momentum vector parallel to its direction, when $p \gg p_0$ the leading term is of order p^{-3},

$$D_{xx} \sim \frac{D_3 p_0^3}{p^3}, \quad D_3 = -\frac{m^3}{4p_0^3}\int_{-\infty}^{\infty} dR\, R^2 \frac{\partial^4 C}{\partial R^2 \partial t^2}(R,0). \tag{12}$$

For diffusion of $\boldsymbol{p}$ in a direction perpendicular to its direction, we have

$$D_{yy} \sim \frac{D_1 p_0}{p}, \quad D_1 = -\frac{m}{2p_0}\int_{-\infty}^{\infty} dR\, \frac{1}{|R|}\frac{\partial C}{\partial R}(R,0). \tag{13}$$

Note that when $p \gg p_0$, the diffusion of the direction of the momentum vector is much more rapid than diffusion of its magnitude. This makes the behavior of the generalized Ornstein-Uhlenbeck process in two or more dimensions very different from its behavior in one dimension.

Results equivalent to Eqs. (12) and (13) were obtained in Refs. [5,6] in an analysis of a closely related model.

III. GENERALIZED CHANDRASEKHAR-ROSENBLUTH MODEL

In this section we consider the motion of a particle traveling through an infinite homogeneous population of background particles. We assume that the test particle interacts with the background particles, and the background particles do not interact with each other. As the test particle moves, its interaction with each of the background particles causes small changes of its velocity. When the number of the background particles is very large, the velocity of the test particle changes rapidly and in an unpredictable way, so that its motion can be described by a diffusion process. The test particle could be a star moving in a galaxy interacting with the background stars (the interaction between the background stars is not considered), so that the force of interaction is gravitational and thus proportional to the inverse square of the distance r between stars. This problem was originally studied by Chandrasekhar [7], who found that the test particle experiences a gradual decrease of the velocity in the direction of motion. This phenomenon is called dynamical friction. For slow particles (with velocities much smaller than some representative velocity scale) this deceleration is proportional to the velocity of the particle v (analogous to the Stokes's law for a drag force for a particle is a viscous medium). For sufficiently fast particles the deceleration is proportional to v^{-2}.

Subsequently, Rosenbluth *et al.* [8] studied the diffusion of momentum of the test particle in this model in greater depth. They found that in the fast regime the diffusion coefficient of the momentum in the direction parallel to the direction of motion is proportional to v^{-3}, while the diffusion coefficient in the plane perpendicular to the direction of motion is proportional to v^{-1}. These dependencies are equivalent to the momentum dependencies of the radial and transverse fluctuations of the momentum in the generalized Ornstein-Uhlenbeck model, obtained in Eqs. (12) and (13). The expressions for these diffusion coefficients obtained in Ref. [8] contain logarithmic terms due to the long-ranged nature of the Coulombic potential, which makes it difficult to write down precise formulas. In order to illuminate the relation between the generalized Ornstein-Uhlenbeck model and the Chandrasekhar-Rosenbluth model in the simplest context, in this section we consider the latter model for the case when the interaction is described by a short-range potential $U(r)$ of some rather general radially symmetric form. We obtain precise expressions for these diffusion coefficients and show that the results for the scaling of the diffusion coefficients are the same as in the generalized Ornstein-Uhlenbeck process. This leads to the same anomalous diffusion behavior in both models.

We shall only discuss the two-dimensional case for simplicity and proceed as follows (some of the presentation adapts the discussion of the Chandrasekhar model in Ref. [13]). We first calculate the change of the velocity of a test particle due to the encounter with a stationary background particle. We denote components of this change by $\Delta v'_\parallel$ and $\Delta v'_\perp$, in the directions parallel and perpendicular to the initial velocity of the test particle. Next, we consider the change of the velocity of the test particle for the case when the background particle propagates with velocity $\boldsymbol{v}_b$. We denote components of this change by $\Delta v_\parallel$ and $\Delta v_\perp$. Using geometrical arguments, we then express $\Delta v_\perp$ and $\Delta v_\parallel$ in terms of $\Delta v'_\perp$ and $\Delta v'_\parallel$.

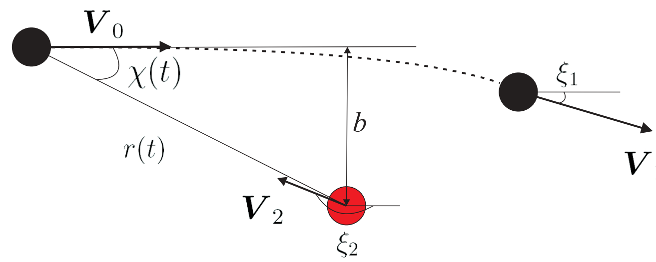

FIG. 1. (Color online) The encounter between a test particle (black) and a background particle (shaded, red online). The interaction causes a small change of the velocity of both particles.

Let us consider a test particle of mass m moving in the horizontal direction with the initial velocity $\boldsymbol{V}_0 = (V_0,0)$. The test particle interacts with a background particle of mass M, which is initially at rest. After an encounter the test particle propagates with velocity $\boldsymbol{V}_1$ described by its magnitude V_1 and the polar angle ξ_1, and the background particle moves with velocity $\boldsymbol{V}_2$ described by its magnitude V_2 and the polar angle ξ_2 (see Fig. 1).

The changes of the velocity of the test particle in the direction parallel and perpendicular to $\boldsymbol{V}_0$ are

$$\Delta v'_\parallel = V_1 \cos\xi_1 - V_0, \quad \Delta v'_\perp = V_1 \sin\xi_1. \tag{14}$$

The conservation of momentum before and after the encounter yields

$$\begin{aligned} mV_0 &= mV_1\cos\xi_1 + MV_2\cos\xi_2, \\ 0 &= mV_1\sin\xi_1 + MV_2\sin\xi_2, \end{aligned} \tag{15}$$

and from the conservation of energy we obtain

$$mV_0^2 = mV_1^2 + MV_2^2. \tag{16}$$

This enables us to write an equation for V_1,

$$V_1^2(m+M) - 2mV_0V_1\cos\xi_1 + V_0^2(m-M) = 0. \tag{17}$$

We assume that the encounter induces only a small change of the direction of motion of the test particle, so that ξ_1 can be taken as being small. In this approximation the solution of Eq. (17) is

$$V_1 = V_0\big(1 - \alpha\xi_1^2\big) + O\big(\xi_1^4\big), \tag{18}$$

where $\alpha = m/(2M)$. We substitute V_1 from Eq. (18) into Eq. (14) and obtain

$$\Delta v'_\parallel \sim -V_0\beta\xi_1^2, \quad \Delta v'_\perp \sim V_0\xi_1, \tag{19}$$

where $\beta = \alpha + 1/2 = (m+M)/2M$. In the small-angle approximation ξ_1 is defined by the change of the momentum in the perpendicular direction, so that $\xi_1 \sim \Delta p_\perp/(mV_0)$. The change of the momentum $\Delta p_\perp$ is determined by the force of interaction between particles separated by distance r with magnitude $f(r) = -dU(r)/dr$, so that $\Delta p_\perp$ can be written as

$$\Delta p_\perp = -\int_{-\infty}^{\infty} dt \frac{dU}{dr}[r(t)] \sin\chi(t), \tag{20}$$

where $r(t)$ is a distance between the particles and $\chi(t)$ is the angle between $\boldsymbol{V}_0$ and the vector connecting the particles. We define an impact parameter b to be the initial distance between the test and background particles along the axis perpendicular to $\boldsymbol{V}_0$. From Fig. 1 we find that $r(t) = \sqrt{x(t)^2 + b^2}$ and $\sin\chi(t) = b/\sqrt{x(t)^2 + b^2}$, where $x(t)$ is a coordinate of the test particle along the direction parallel to $\boldsymbol{V}_0$. Changing the variable $x(t) = V_0 t$, in the weak-scattering limit where the deflection is small we obtain

$$\Delta p_\perp \approx -\frac{1}{V_0}\int_{-\infty}^{\infty} dx\, \frac{dU}{dr}(\sqrt{x^2+b^2})\frac{b}{\sqrt{x^2+b^2}}. \tag{21}$$

If we denote an integral

$$I(b) = -\int_{-\infty}^{\infty} dx\, \frac{dU}{dr}(\sqrt{x^2+b^2})\frac{b}{\sqrt{b^2+x^2}} \tag{22}$$

we obtain

$$\xi_1(b,V_0) \sim \frac{I(b)}{mV_0^2}. \tag{23}$$

Using this relation we find

$$\Delta v'_\parallel = -\frac{\beta I^2(b)}{m^2 V_0^3}, \quad \Delta v'_\perp = \frac{I(b)}{mV_0}. \tag{24}$$

Thus, the contribution to the change of the velocity of the test particle due to a single encounter is proportional to V_0^{-3} and V_0^{-1} in the directions parallel and perpendicular to $\boldsymbol{V}_0$, respectively. Averaging over collisions with many particles is expected to add another factor of V_0, as the particle propagates with this velocity (see also the derivation below). This suggests that the second moments of the change of the velocity scale as V_0^{-5} and V_0^{-1} in the directions parallel and perpendicular to $\boldsymbol{V}_0$, respectively. While the latter result is consistent with the behavior of the diffusion coefficient in the generalized Ornstein-Uhlenbeck process, the former result is different, as in the previous model the result is $\langle\Delta v_\parallel^2\rangle \sim v^{-3}$. However, if the background particles are not stationary, these estimates must be corrected, as shown below.

We assume that the background particle moves with velocity $\boldsymbol{V}_b$, in which case the discussion above is valid if $\boldsymbol{v}_0$ is a relative velocity of the test particle in the frame of reference moving with the background particle. The velocity of the test particle in a fixed frame of reference is therefore $\boldsymbol{v}_0 = \boldsymbol{V}_0 + \boldsymbol{v}_b$. We are interested in the changes of the velocity of the test particle in the directions parallel and perpendicular to $\boldsymbol{v}_0$. We denote these $\Delta v_\parallel$ and $\Delta v_\perp$ and deduce from Fig. 2

$$\begin{aligned}\Delta v_\parallel &= \Delta v'_\parallel \cos\Omega + \Delta v'_\perp \sin\Omega,\\ \Delta v_\perp &= \Delta v'_\perp \cos\Omega - \Delta v'_\parallel \sin\Omega,\end{aligned} \tag{25}$$

where Ω is the angle between $\boldsymbol{V}_0$ and $\boldsymbol{v}_0$. These relations are equivalent to the rotation of the coordinate system by angle Ω. Assuming that $v_0 \gg v_b$, where $v_b = |\boldsymbol{v}_b|$ and $v_0 = |\boldsymbol{v}_0|$, we obtain

$$\Omega \sim \frac{\boldsymbol{v}_b \wedge \boldsymbol{V}_0}{V_0^2} \equiv \frac{v_{b\perp}}{V_0} \tag{26}$$

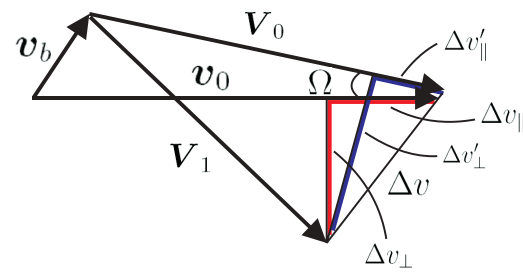

FIG. 2. (Color online) Geometrical construction illustrating the changes of the velocity of the test particle Δv parallel and perpendicular to the relative velocity $\boldsymbol{V}_0$ ($\Delta v'_\parallel$ and $\Delta v'_\perp$, solid lines, blue online) and the velocity of the test particle $\boldsymbol{v}_0$ ($\Delta v_\parallel$ and $\Delta v_\perp$, solid lines, red online).

and $v_0 \sim V_0$. In this approximation, to the leading order in v_0, we have

$$\Delta v_\parallel \approx \frac{I(b)v_{b\perp}}{mv_0^2}, \quad \Delta v_\perp \approx \frac{I(b)}{mv_0}. \tag{27}$$

We now imagine that the test particle is traveling through an infinite homogenous population of background particles with the spatial density number n (measuring a number of particles per unit area) and the probability density of the velocity is $f(\boldsymbol{v}_b)$. Let dN be the number of background particles it encounters in time Δt with velocity $\boldsymbol{v}_b$ in a volume element of velocity space $d\boldsymbol{v}_b$ and impact parameter between b and $b + db$. This is the number of particles in two thin stripes, each of width db and length equal to the distance traveled by the particle in Δt, multiply by the probability $f(\boldsymbol{v}_b)\,d\boldsymbol{v}_b$. We have

$$dN \sim 2nV_0\Delta t\, db\, f(\boldsymbol{v}_b)\,d\boldsymbol{v}_b. \tag{28}$$

In order to obtain the total contribution of many background particles with different impact parameters and velocities, we integrate over b and $\boldsymbol{v}_b$,

$$\begin{aligned}\frac{\langle\Delta v_\parallel^2\rangle}{2\Delta t} &= \frac{n}{m^2 v_0^3}\int_0^\infty db\, I^2(b)\int d\boldsymbol{v}_b\, f(\boldsymbol{v}_b)|\boldsymbol{v}_{b\perp}|^2,\\ \frac{\langle\Delta v_\perp^2\rangle}{2\Delta t} &= \frac{n}{m^2 v_0}\int_0^\infty db\, I^2(b).\end{aligned} \tag{29}$$

Here we used the assumption that $U(r)$ is short ranged, allowing us to let the upper limit of the integral over b approach infinity. In the case of the original Chandrasekhar-Rosenbluth model, an upper limit to the impact parameter must be introduced because of the long-range interaction between the particles. This leads to logarithmic correction terms considered in Refs. [7,8].

Equations (29), describing the velocity increments for the Chandrasekhar-Rosenbluth model has the same scaling (as a function of v_0) as scaling of the diffusion coefficients for the generalized Ornstein-Uhlenbeck model [as a function of p; see Eqs. (12) and (13)]. This indicates that the anomalous diffusion behavior of these models is equivalent.

IV. PROBABILITY DENSITY FUNCTION AND MOMENTS OF THE MOMENTUM

Now we return to the generalized Ornstein-Uhlenbeck process and obtain the closed-form solution of the Fokker-Planck equation for a particular choice of the initial conditions. We use this solution to obtain an exact expression for the growth of the moments of the momentum.

We first consider the two-dimensional case. The probability density for the momentum satisfies Eq. (11), with the diffusion coefficients given by Eqs. (12) and (13). We transform to polar coordinates and seek a probability density $P(p,\theta,t)$, and consider the case when the particle is initially at rest, so that the initial condition is $P(p,\theta,0) = \delta(p)$. This circularly symmetric solution, $P = \rho(p,t)$, satisfies

$$\frac{\partial \rho}{\partial t} = \frac{D_3 p_0^3}{p^3}\frac{\partial^2 \rho}{\partial p^2} + \left(\gamma p - \frac{2D_3 p_0^3}{p^4}\right)\frac{\partial \rho}{\partial p} + 2\gamma\rho. \quad (30)$$

Using the MAPLE mathematical package, by analogy with the solution of the one-dimensional generalized Ornstein-Uhlenbeck model in Ref. [4] we find the following normalized closed-form solution of Eq. (30):

$$\rho(p,t) = \frac{5}{\Gamma(2/5)}\frac{\gamma^{2/5}}{\left[5D_3 p_0^3(1-e^{-5\gamma t})\right]^{2/5}} \times \exp\left[-\frac{\gamma p^5}{5D_3 p_0^3(1-e^{-5\gamma t})}\right]. \quad (31)$$

In the long-time limit the density is non-Maxwellian given by

$$\rho_0(p) = \frac{5\gamma^{2/5}}{\Gamma(2/5)\left(5D_3 p_0^3\right)^{2/5}}\exp\left[-\frac{\gamma p^5}{5D_3 p_0^3}\right]. \quad (32)$$

Using the probability density Eq. (31) we determine the lth moment of p,

$$\langle p^l(t)\rangle = \int_0^\infty dp\; p^{l+1}\rho(p,t) = \left(\frac{5D_3 p_0^3}{\gamma}\right)^{l/5}\frac{\Gamma[(2+l)/5]}{\Gamma(2/5)}(1-e^{-5\gamma t})^{l/5}. \quad (33)$$

We remark that an additional factor of p in the expression above appears as a weight in the transformation to polar coordinates.

In the three-dimensional case we find a similar solution to Eq. (10) in the case where the particles are initially stationary. We write this equation in spherical polar coordinates, and seek a spherically symmetric solution, $P(p,\theta,\phi,t) = \rho(p,t)$. The solution of the corresponding equation for $\rho(p,t)$ is obtained similarly to the two-dimensional case and we have

$$\rho(p,t) = \frac{5}{\Gamma(3/5)}\frac{\gamma^{3/5}}{\left[5D_3 p_0^3(1-e^{-5\gamma t})\right]^{3/5}} \times \exp\left[-\frac{\gamma p^5}{5D_3 p_0^3(1-e^{-5\gamma t})}\right]. \quad (34)$$

This determines moments of the momentum

$$\langle p^l(t)\rangle = \left(\frac{5D_3 p_0^3}{\gamma}\right)^{l/5}\frac{\Gamma[(3+l)/5]}{\Gamma(3/5)}(1-e^{-5\gamma t})^{l/5}. \quad (35)$$

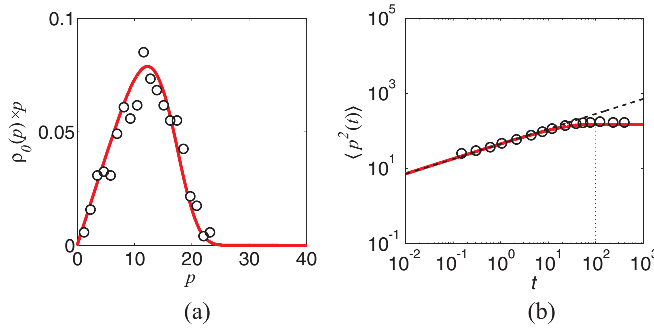

FIG. 3. (Color online) Results of the numerical simulation of Eq. (1) for the motion in the two-dimensional potential force field. Panel (a) shows stationary non-Maxwellian density function Eq. (32) (solid line) and data from the numerical simulation (circles). Panel (b) shows the second moment of the momentum, Eq. (33) with $l = 2$ (solid line) and data from the numerical simulation (circles). The dashed line shows the slope $t^{2/5}$ and dotted line indicates time γ^{-1} at which the density becomes stationary. The results are for the case of a Gaussian correlation function of the potential $C(x,t) = \sigma^2\exp[-x^2/(2\eta^2) - t^2/(2\tau^2)]$ with $\sigma = 15$, $\tau = 0.1$, and $\eta = 0.1$. The other parameters were $m = 1$ and $\gamma = 0.01$.

For both two- and three-dimensional cases we obtain that at short times the variance of the momentum grows as

$$\langle p^2(t)\rangle \sim t^{2/5}. \quad (36)$$

Thus, at short times the momentum diffuses anomalously with the same exponent as in the one-dimensional model [5,6]. The results for the stationary probability density and diffusion of the momentum in the two-dimensional case were verified by a numerical simulation, documented in Fig. 3.

V. SPATIAL DIFFUSION

In this section we find the mean-square value of the displacement of a particle which starts at the origin,

$$\langle|\boldsymbol{x}(t)|^2\rangle = \frac{1}{m^2}\int_0^t dt_1\int_0^t dt_2\,\langle \boldsymbol{p}(t_1)\cdot\boldsymbol{p}(t_2)\rangle = \frac{1}{m^2}\int_0^t dt_1\int_0^t dt_2\,\langle p(t_1)p(t_2)\cos\theta\rangle, \quad (37)$$

where θ is the angle between $\boldsymbol{p}(t_1)$ and $\boldsymbol{p}(t_2)$. We recall that when the force is the gradient of the potential, we have $D_{xx} \lll D_{yy}$ for $p \gg p_0$ implying that the correlation of the angle vanishes much more rapidly than the correlation of the magnitude of the momentum. We can, therefore, perform the averaging in Eq. (37) by first integrating over the correlation function of the angular variable, with the momentum held fixed, and then finally performing the averaging over fluctuations of the momentum.

In the two-dimensional case the probability density $P(\theta,t)$ of θ satisfies the diffusion equation on a circle with the initial condition $P(\theta,0) = \delta(\theta)$. The solution is Gaussian:

$$P(\theta,t) = \frac{1}{2\sqrt{\pi\mathcal{D}t}}\exp\left(-\frac{\theta^2}{4\mathcal{D}t}\right), \quad (38)$$

where $\mathcal{D} = D_1 p_0 / p^3(t_1)$. Using this probability density we calculate the expectation value

$$\langle \cos\theta \rangle = \exp(-\mathcal{D}|t_2 - t_1|). \tag{39}$$

We thus obtain from Eqs. (37) and (39)

$$\langle |\boldsymbol{x}(t)|^2 \rangle \sim \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \int_0^\infty dp\ \rho(p,t_1) p^3(t_1) \times \exp(-\mathcal{D}|t_2 - t_1|). \tag{40}$$

Introducing a new variable $T = t_1 - t_2$ we have

$$\begin{aligned}\langle |\boldsymbol{x}(t)|^2 \rangle &\sim \frac{2}{m^2} \int_0^t dt_1 \int_0^t dT \\ &\quad \times \int_0^\infty dp\ \rho(p,t_1) p^3(t_1) \exp\left(-\frac{D_1 p_0}{p^3(t_1)} T\right) \\ &= \frac{2}{m^2 D_1 p_0} \int_0^t dt_1 \int_0^\infty dp\ \rho(p,t_1) p^6(t_1) \\ &\quad \times \left[1 - \exp\left(-\frac{D_1 p_0}{p^3(t_1)} t\right)\right].\end{aligned} \tag{41}$$

When the forcing is strong we have $D_1 p_0 t \gg p^3$ for $t \gg \tau$, and therefore

$$\begin{aligned}\langle |\boldsymbol{x}(t)|^2 \rangle &= \frac{2}{m^2 D_1 p_0} \int_0^t dt_1 \int_0^\infty dp\ \rho(p,t_1) p^6(t_1) \\ &= \frac{2}{m^2 D_1 p_0} \int_0^t dt_1 \langle p^5(t_1) \rangle.\end{aligned} \tag{42}$$

Using Eq. (33) we obtain

$$\langle |\boldsymbol{x}(t)|^2 \rangle = \frac{4 D_3 p_0^2}{5 D_1 m^2 \gamma^2} (5\gamma t + e^{-5\gamma t} - 1). \tag{43}$$

In the three-dimensional case the probability density of $\cos\theta$ can be found by considering the diffusion equation on a spherical surface, with polar coordinates (θ,ϕ), starting from the pole, $\theta = 0$. The solution $P(\theta,\phi,t)$ of the diffusion equation may be expressed as a linear combination of spherical harmonics. Because the problem has a rotational symmetry, the solution is independent of the azimuthal angle ϕ, and it may be written as

$$P(\theta,\phi,t) = \sum_{l=0}^{\infty} A_l \exp\left[-\frac{l(l+1) p_0 D_1}{p^3} t\right] P_l(\cos\theta), \tag{44}$$

where $P_l(z)$ is a Legendre polynomial of degree l. Using the orthogonality relations for Legendre polynomials, in view of the initial condition $\cos\theta = 1$, we obtain $A_l = (2l+1)/2$. Also, the quantity that we wish to average is itself a spherical harmonic, $\cos\theta = P_1(\cos\theta)$, so that only the $l = 1$ term in Eq. (44) contributes to the correlation function. Hence we obtain

$$\langle \cos\theta(|t_2 - t_1|) \rangle = \exp(-3\mathcal{D}|t_2 - t_1|), \tag{45}$$

where $\mathcal{D} = p_0 D_1 / p^3$ is the same as in the two-dimensional case. Using this angular correlation function we calculate

$$\langle |\boldsymbol{x}(t)|^2 \rangle \sim \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \int_0^\infty dp\ \rho(p,t_1) p^4(t_1) \times \exp(-3\mathcal{D}|t_2 - t_1|). \tag{46}$$

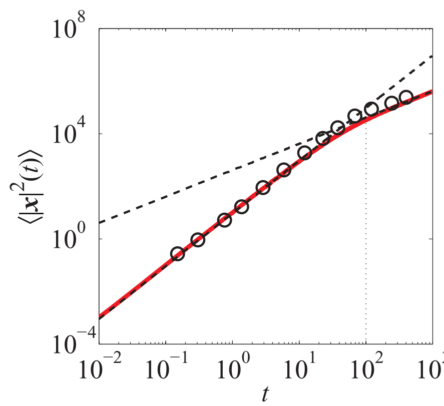

FIG. 4. (Color online) Results for the spatial diffusion in the two-dimensional potential force field. The results from the numerical simulation (circles) are compared with Eq. (43) (solid line). Dashed lines show the slopes t^2 and t and dotted line indicates the time γ^{-1}. The parameters of the simulation are the same as in Fig. 3.

The evaluation of the integral using the probability density, Eq. (35), yields

$$\langle |\boldsymbol{x}(t)|^2 \rangle = \frac{2 D_3 p_0^2}{5 D_1 m^2 \gamma^2} (5\gamma t + e^{-5\gamma t} - 1). \tag{47}$$

We find that in two- and three-dimensional cases $\langle |\boldsymbol{x}(t)|^2 \rangle \sim t^2$ at short times, so that the particle diffuses ballistically. The results are consistent with a short-time asymptotic behavior of the undamped particle obtained in Ref. [6] for $d > 1$. The long-time behavior is naturally diffusive, $\langle |\boldsymbol{x}(t)|^2 \rangle \sim t$. In Fig. 4 we show the comparison of the analytical and numerical results for $\langle |\boldsymbol{x}(t)|^2 \rangle$ for the case of motion in the two-dimensional potential force field, illustrating the short-time ballistic diffusion.

VI. SUMMARY

We have investigated generalizations of two classical models for diffusion of a particle accelerated by random forces. We discussed a generalization of the classical Ornstein-Uhlenbeck process where the force depends on the position of the particle as well as time. We also modified the Chandrasekhar-Rosenbluth model by considering motion due to a short-range interaction potential. Although both models are described by different microscopic equations of motion, surprisingly, they have the same scaling of the diffusion coefficients, leading to the same short-time asymptotic dynamics.

We solved the Fokker-Planck equation for the generalized Ornstein-Uhlenbeck process exactly in two and three dimensions, building upon our earlier analysis of the one-dimensional case in Refs. [3,4]. We have shown that this dynamics is characterized by anomalous diffusion of the momentum, with the variance, which scales as $\langle p^2 \rangle \sim t^{2/5}$. At long time, the distribution of the momentum has been found to be non-Maxwellian. The second moment of the displacement grows ballistically at short times, that is $\langle x^2 \rangle \sim t^2$, in accord with a surmise made by Rosenbluth for a closely related model [6], and at long time a simple diffusive behavior of the displacement is recovered.

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