Eur. Phys. J. B (2012) 85: 18
DOI: 10.1140/epjb/e2011-20325-5

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

# Clustering of exponentially separating trajectories

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Received 26 April 2011 / Received in final form 18 October 2011
Published online 18 January 2012 – 

**Abstract.** It might be expected that trajectories of a dynamical system which has no negative Lyapunov exponent (implying exponential growth of small separations) will not cluster together. However, clustering can occur such that the density $\rho(\Delta x)$ of trajectories within distance $|\Delta x|$ of a reference trajectory has a power-law divergence, so that $\rho(\Delta x) \sim |\Delta x|^{-\beta}$ when $|\Delta x|$ is sufficiently small, for some $0 < \beta < 1$. We demonstrate this effect using a random map in one dimension. We find no evidence for this effect in the chaotic logistic map, and argue that the effect is harder to observe in deterministic maps.

## 1 Introduction

It is well known that the trajectories of simple dynamical systems can approach fractal sets known as strange attractors [1]. These are usually illustrated using systems, such as the Hénon map, for which the dynamics combines stretching and folding [2]. Stretching implies that trajectories diverge exponentially, so that the largest Lyapunov exponent is positive. In order to have an attractor there must be contraction (characterised by negative Lyapunov exponents) in other directions, so that volume elements contract. Here we demonstrate an alternative mechanism for trajectories of a dynamical system to exhibit fractal clustering effects, which is counter-intuitive, but which can be analysed quite thoroughly in one-dimensional maps. We shall demonstrate, explain and analyse how fractal clustering of trajectories may occur in a system which has no negative Lyapunov exponents, so that there is no attractor. Although we must make reference to some technical papers, this article is self-contained. Beyond the definitions of the Lyapunov exponent and of the Renyi dimension (Eqs. (1) and (3) below), we only use quite elementary results from probability theory, random walks and diffusion processes.

First we consider whether this claim is compatible with established results. Fractal measures which arise in the study of dynamical systems are typically multifractal, in the sense that their Renyi dimensions $D_q$ (defined by Eq. (3) below and discussed in [2,3]) are not all equal. Exponential separation in all directions implies that the invariant density must cover a set of finite measure, so that for systems with only positive Lyapunov exponents the box-counting dimension $D_0$ is equal to the space dimension, $d$. Furthermore, the Kaplan-Yorke formula [4] for the Lyapunov dimension $D_{\rm L}$ gives $D_{\rm L} = d$ if the sum of the Lyapunov exponents is positive. It is believed that $D_{\rm L} = D_1$ [5], so that we conclude that $D_1 = d$ for the systems which we consider. There does not appear to be any constraint which implies $D_2 = d$ when all of the Lyapunov exponents are positive. It is, however, hard to conceive of how fractal clustering can occur in a situation where trajectories are separating exponentially in each direction. If a cluster with arbitrarily high density is to form, the trajectories which participate have to 'beat the odds' by coming closer together throughout a long sequence of iterations of the map, despite the fact that positive Lyapunov exponents imply that separations are more likely to increase than to decrease. The mathematical literature has discussed special cases where the attractor has dimension lower than $d$ despite the sum of the Lyapunov exponents being positive (see [6] for an example). Here, however, we show that clustering can occur under quite general circumstances even when there is no negative Lyapunov exponent.

## 2 The Lyapunov exponent and correlation dimension

We shall discuss the simplest case which illustrates our conclusion, that of chaotic maps in one dimension. The extensions to flows and to higher dimensions are straightforward. We start by reviewing the definitions of the Lyapunov exponent $\lambda$ and of the correlation dimension $D_2$. It is possible to describe the fractal attractor of a deterministic map by considering a measure on the set of points visited by a single trajectory. In this paper, however, we shall consider random as well as deterministic maps, so that the fractal measures upon which trajectories might congregate are not necessarily fixed in the

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coordinate space. We will, therefore, describe clustering in terms of the behaviour of trajectories with different initial conditions, examined at the same 'time' (that is, iteration number). The initial distribution of the trajectories is a random scatter across the coordinate space, with uniform density.

The infinitesimal separation of two trajectories is characterised by the Lyapunov exponent, $\lambda$ [2]. For a one-dimensional map generating a sequence of points $x_0$, $x_1$, $\ldots, x_i, \ldots$ this is defined by writing

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \left\langle \ln \left| \frac{\delta x_N}{\delta x_0} \right| \right\rangle \quad (1)$$

where the $\delta x_i$ are infinitesimal separations of two trajectories at the $i$th iteration (throughout this paper $\langle X \rangle$ denotes the expectation value of $X$). If $\lambda < 0$, the separation of two very close orbits approaches zero with a probability tending to unity as the initial separation approaches zero. When the Lyapunov exponent is negative, the trajectories coalesce, rather than cluster. This path-coalescence effect was noted by Deutsch [7,8] and its relation to the sign of the Lyapunov exponent is considered in [9]. If $\lambda > 0$, the separation of two initially very close orbits increases (with a probability approaching unity as the initial separation decreases). Given this observation, it would be surprising to see trajectories clustering together if $\lambda > 0$.

In the following we show that clustering may occur, such that the probability density $\rho$ for the separation of trajectories, $\Delta x$ has a power-law dependence for small values of $|\Delta x|$, i.e.

$$\rho(\Delta x) \sim |\Delta x|^{-\beta} \quad (2)$$

with $0 < \beta < 1$. Pikovsky [10] has also demonstrated a power-law distribution for the separation of trajectories of random maps in one dimension, and although his calculation is more technical than ours the methods do overlap. He did not, however, discuss the surprising fact that clustering can occur despite an exponential growth of particle separations.

The Renyi dimensions $D_q$ are defined by dividing the configuration space into cells, labelled by an index $i$, with size $\epsilon$, which are occupied with probability $p_i$:

$$D_q = \frac{1}{q-1} \lim_{\epsilon\to 0} \frac{\ln \sum_i p_i^q}{\ln \epsilon}. \quad (3)$$

The dimension $D_2$ is known as the correlation dimension and is related to the exponent $\beta$ in (2). It follows from (3) that the number $\mathcal{N}$ of trajectories in a ball of radius $\epsilon$ about a randomly selected test trajectory satisfies $\langle \mathcal{N}(\epsilon) \rangle \sim \epsilon^{D_2}$ [2]. The probability density for trajectory separations in one dimension is $\rho(\epsilon) = d\langle \mathcal{N} \rangle / d\epsilon$, so that $D_2 = 1 - \beta$.

## 3 Analysis of clustering

We consider the dynamics of a one-dimensional map, of the form

$$x_{n+1} = f_n(x_n). \quad (4)$$

The functions $f_n(x)$ may be independent of $n$ (the map is deterministic), or they may be selected at random from an ensemble at each iteration. We allow for the possibility of a random map because the analysis can be taken further in that case and because random maps play a role in modelling many different physical processes. We concentrate upon chaotic maps, that is maps with positive Lyapunov exponent. We assume that the iterations do not escape from some finite region, so that there is an upper bound on the separation of two trajectories.

Both the Lyapunov exponent $\lambda$ and the clustering exponent $\beta$ characterise small separations of trajectories, $\Delta x$. It is convenient to consider the variable $Y_n = \ln|\Delta x_n|$, which we express in terms of its incremental changes at each iteration, $Z_i$:

$$Y_n = \ln|\Delta x_n| = \ln|\Delta x_0| + \sum_{i=0}^{n-1} Z_i. \quad (5)$$

As $\Delta x \to 0$, the incremental changes $Z_i$ are given by

$$Z_i = \ln|f_i'(x_i)|. \quad (6)$$

This follows from the fact that in the limit $\Delta x \to 0$, the evolution of infinitesimal separations $\delta x_i$ is described by the linearised equation of motion, $\delta x_{i+1} = f_i'(x_i)\delta x_i$. Equations (5) and (6) describe how $Y_n$ changes as a result of many iterations of points initially close to each other.

We note that even for a deterministic map, the quantity $Z_i$ differs from one iteration to the next, and if the map is chaotic, this variation will appear to be random.

In many-to-one maps (which we consider here) it may occur that two initially distant points become very close in just one iteration. Such events are expected to give rise to a distribution of separations which approaches a constant in the limit as $\Delta x \to 0$. These events do not, therefore, contribute to the clustering effect. For this reason they are not given further consideration in the discussion below.

We now consider how both the Lyapunov exponent $\lambda$ and the clustering exponent $\beta$ are related to the statistics of $Z$ for small values of $|\Delta x|$. When the dynamics is ergodic, it follows from the definitions (1) and (5) that

$$\lambda = \langle Z_i \rangle. \quad (7)$$

Let us consider the dynamics of $Y_n$. Because the displacements $Z_i$ can be considered as random variables, the quantity $Y_n$ executes a random walk with drift when $|\Delta x_n|$ is sufficiently small. The variable $Y_n$ has diffusive fluctuations, with the variance of $\Delta Y_n = Y_n - Y_0$ increasing linearly as a function of the iteration number, $n$:

$$\lim_{n\to\infty} \frac{\langle (\Delta Y_n - n\lambda)^2 \rangle}{n} = 2\mathcal{D}. \quad (8)$$

The diffusion coefficient $\mathcal{D}$ is determined from the correlation function of the fluctuations of $Z_i$

$$\mathcal{D} = \frac{1}{2} \sum_{i=-\infty}^{\infty} \left[ \langle Z_i Z_0 \rangle - \langle Z_i \rangle^2 \right]. \quad (9)$$

We characterise the behaviour of $Y_n$ in terms of its probability density, $\rho_n(Y)$. If the probability density $\rho_n(Y)$ varies sufficiently slowly as a function of $Y$, we may approximate its evolution by a discrete-time version of the Fokker-Planck equation [11]:

$$\rho_{n+1} - \rho_n = -\frac{\partial}{\partial Y}(\lambda \rho_n) + \frac{\partial^2}{\partial Y^2}(\mathcal{D}\rho_n) \qquad (10)$$

where the drift velocity is, by (7), equal to the Lyapunov exponent $\lambda$. The steady-state solution of equation (10), $\rho$, is an exponential function:

$$\rho(Y) = K \exp(\lambda Y/\mathcal{D}). \qquad (11)$$

This solution is not normalisable and we should therefore consider the conditions under which it is applicable. The linearised mapping ceases to be applicable when $|\Delta x|$ is too large. Because the growth of $|\Delta x|$ is assumed to be bounded, the probability density $\rho(Y)$ should have a normalisable steady state. We can therefore meaningfully consider a solution of the form (11) with positive values of $\lambda/\mathcal{D}$, because the solution is matched to another function at larger values of $Y$ where (6) is no longer valid. There is no lower cutoff however, so that solutions with negative values of $\lambda/\mathcal{D}$ are untenable. This implies that there is no steady-state solution $\rho$ when $\lambda < 0$.

Now consider the implications of (11) for the probability density of trajectory separations. Transforming (11) to a density $\rho(\Delta x)$ by writing $dP = K \exp(\lambda Y/\mathcal{D})dY = \rho(\Delta x)d\Delta x$, we find $\rho(\Delta x) = K|\Delta x|^{\lambda/\mathcal{D}-1}$. We conclude that clustering should occur when $\lambda/\mathcal{D} < 1$, and that the correlation dimension is given by

$$D_2 = \frac{\lambda}{\mathcal{D}} = 1 - \beta. \qquad (12)$$

The use of the Fokker-Planck equation is only justified when the gradient of $\rho_n(Y)$ is sufficiently small. The condition is that $\partial\rho_n/\partial Y$ should be small compared to the inverse scale over which $Y$ varies during its correlation time. This condition for the validity of (11) is equivalent to $D_2 \ll 1$. We remark that Grassberger and Procaccia [12] also used a Fokker-Planck equation to determine the logarithm of trajectory separations in a discussion of the correlation dimension. They considered a strange attractor, rather than a system with only positive Lyapunov exponents.

The results of this section can be summarized as follows. For any chaotic map, it is possible to calculate the mean value and time correlation of $Z_i = \ln|f_i'(x_i)|$. The mean value is equal to the Lyapunov exponent, $\lambda = \langle Z\rangle$, and for a chaotic map this is a positive number. The fluctuations of $Y = \ln|\Delta x|$ execute a random walk, with diffusion coefficient $\mathcal{D}$. When the probability density of $Y$ varies sufficiently slowly, this obeys a Fokker-Planck equation (10). The stationary solution of this equation is an exponential (11) which in turn implies that the distribution of $\Delta x$ is a power-law. The assumption that the probability density is slowly varying is justified when $D_2 \ll 1$, a limit in which there is strong clustering. In this limit we have seen that the correlation dimension is given by the asymptotic expression (12).

Thus clustering is expected to occur for a chaotic one-dimensional map upon varying a parameter so that the Lyapunov exponent approaches zero (from above). The only way to avoid this conclusion is if the diffusion constant $\mathcal{D}$ approaches zero at the same time as the Lyapunov exponent $\lambda$.

## 4 Correlated random walk model

We now illustrate the clustering phenomenon using a simple random dynamical system, namely a random walk for which the random displacement is a smoothly varying function of the current position. The map is

$$x_{n+1} = x_n + g_n(x_n) \qquad (13)$$

and the function $g_n(x)$ is a realisation of a random process with statistics specified by a function $C(x - x')$:

$$\langle g_n(x)\rangle = 0\ , \quad \langle g_n(x) g_m(x')\rangle = \delta_{nm} C(x - x'). \qquad (14)$$

Furthermore we consider the coordinate $x$ to be cyclic, so that the positions $x$ and $x + L$ are equivalent. (This implies that the functions $g_n(x)$ are periodic with period $L$). This model has been discussed in earlier work [9] where it was pointed out that the Lyapunov exponent is negative when the functions $g_n(x)$ are sufficiently small. The model is of some physical interest as a model for the motion of particles advected by a spatially smooth but chaotic velocity field [9]. Its attraction in the present context is that it allows us to write down and analyse an exact (although implicit) equation for the correlation dimension, $D_2$.

Because there is no correlation between the random displacements at successive iterations, the probability density $\rho_{n+1}$ for $Y_{n+1}$ may be expressed exactly in terms of density $\rho_n$ of $Y_n$. For sufficiently small $|\Delta x|$ we have:

$$\rho_{n+1}(Y) = \int_{-\infty}^{\infty} dZ\ P(Z)\rho_n(Y - Z) \qquad (15)$$

where $P(Z)$ is the probability density of $Z_i = \ln|1+g'(x_i)|$ (this equation is equivalent to equation (12) of [10]). This equation has a steady-state solution of the form (11), that is $\rho(Y) = K \exp(D_2 Y)$ for some constant $K$. By substituting (11) into (15), we find that

$$\langle \exp(-D_2 Z)\rangle = \int_{-\infty}^{\infty} dZ\ \exp(-D_2 Z)P(Z) = 1. \qquad (16)$$

It is a simple exercise to reproduce equation (12) from this equation by writing $\langle \exp(-D_2 Z)\rangle = 1 - D_2\langle Z\rangle + \frac{1}{2}D_2^2\langle Z^2\rangle + O(D_2^3)$. If equation (16) has a solution with $0 < D_2 < 1$, this demonstrates conclusively that fractal clustering exists for a model with only positive Lyapunov exponents.

We made a numerical investigation of the model described by (13) and (14), in the case where the noise has a

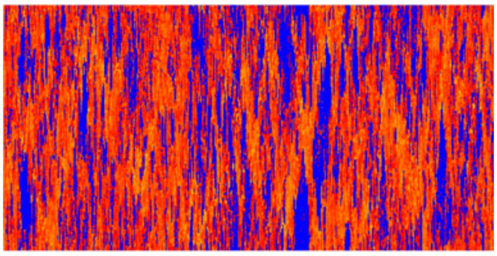

**Fig. 1.** (Color online) Illustrating clustering of the trajectories of the correlated random walk model defined by equations (13) and (14). The parameters were $\sigma^2 = 5.6$, $\xi = 0.032$. We plot position (value of $x_n$ from 0 to 1) on the vertical axis and iteration number (from 0 to 1000) on the horizontal axis). We iterated 10 000 trajectories with different initial conditions. The density is colour-coded on a logarithmic scale: orange is the highest, and blue the lowest density. The density shows very marked fluctuations, indicating that the trajectories tend to cluster without coalescing.

Gaussian distribution and where the correlation function is $C(x) = \sigma^2\xi^2 \exp(-x^2/2\xi^2)$ (for some constants $\sigma$ and $\xi$). The density of trajectories in this model is illustrated in Figure 1, where we show a scatter plot of the positions for a realisation of the dynamics described by (13) and (14) as a function of iteration number, for a case where the Lyapunov exponent is positive ($\sigma^2 = 5.6$, $\xi = 0.032$). For this choice of correlation function a numerical evaluation of $\langle Z\rangle$ indicates that the Lyapunov exponent is positive when $\sigma^2 > 2.421\ldots$, so that for the parameters used in Figure 1 the trajectories do not coalesce. It might be expected that exponentially diverging trajectories would result in a uniform density distribution, but Figure 1 shows that the trajectories cluster together, consistent with the arguments above. The probability density $\rho(\Delta x)$ for a trajectory to occur at a distance $|\Delta x|$ from a randomly chosen test trajectory was found to have the power-law form described by equation (2): see Figure 2a. This implies that the distribution can be regarded as a fractal set, with correlation dimension $D_2$.

As well as confirming that clustering occurs, it is interesting to consider the quantitative predictions from (16) in the case where the random functions $g_n(x)$ have Gaussian statistics. Specifically, we assume that the quantities $G_n = g'(x_n)$ are Gaussian distributed, with variance $\sigma^2$ (the mean value is obviously zero). In this case the relation (16) becomes

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} dG\ \delta(Z - \ln|1+G|) P(G) \exp(-D_2 Z) \\ &= \frac{1}{\sqrt{\pi}2^{D_2/2}\sigma^{D_2}} \Gamma\left(\frac{1-D_2}{2}\right) {}_1F_1\left[\frac{D_2}{2}; \frac{1}{2}; -\frac{1}{2\sigma^2}\right]. \end{aligned} \tag{17}$$

where $\Gamma$ is the Euler gamma function and ${}_1F_1$ is the Kummer confluent hypergeometric function. In the limit as $\sigma \to \infty$ the exponent $D_2$ approaches unity as

$$D_2 \sim 1 - \sigma^{-1} \tag{18}$$

so that the clustering tendency is always present in (13), (14), no matter how strong the random impulses $g_n(x)$.

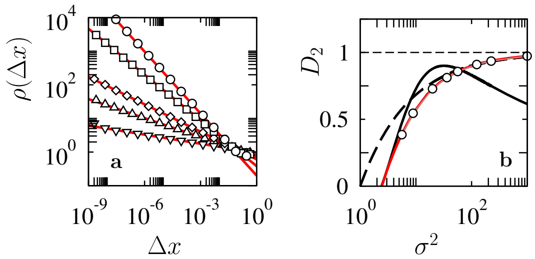


**Fig. 2.** (Color online) (a) Probability density of trajectory separations $\rho(\Delta x)$ according to the random map (13), (14). Simulation data is represented by data markers, power laws (2) with $\beta = 1 - D_2$ are fitted for small $|\Delta x|$ (solid lines, red in color) which numerically determines the correlation dimension $D_2$ for different $\sigma^2$. The parameters are $\xi = 0.032$ and $\sigma^2 = 124$ (▽), 36 (△), 20 (◇), 8.9 (□), and 5.6 (○). The data is based on $10^8$ iterations of trajectory pairs. (b) The correlation dimension of the random map (13), (14), determined by solving (17) (solid line, red in color) is compared with exponents determined as in panel a (○). The asymptotic approximations to $D_2$, (12) (solid black line) and (18) (dotted line) are also shown.

The value of $D_2$ which solves equation (17) is plotted as a function of $\sigma^2$ in Figure 2b, where it is compared with the asymptotic approximations, (12) and (18) and with values of the exponent obtained by simulation.

## 5 Extension to deterministic maps

One dimensional deterministic maps can be chaotic, with a positive Lyapunov exponent. It is natural to ask whether these can also exhibit clustering behaviour. The best studied example [2] is the logistic map,

$$x_{n+1} = Ax_n(1 - x_n) = f(x_n) \tag{19}$$

which has been studied exhaustively and which is considered to have generic properties. There are intervals of the parameter $A$ where the trajectory is attracted to stable periodic orbits, interspersed with sets where the Lyapunov exponent is positive. We have seen that when the correlation dimension is small it is well approximated by $D_2 = \lambda/\mathcal{D}$. We therefore expect that clustering of trajectories will occur close to the boundaries of the chaotic regions, where $\lambda$ is small.

A numerical investigation of the correlation dimension for the logistic map was carried out, for a large set of values of the coupling parameter $A$. This set included many values close to the edge of a stable region, where the Lyapunov exponent, although positive, was very small. This investigation produced a surprising result. With one exception, no values of $A$ were identified for which the numerically determined correlation dimension was clearly different from unity (in the chaotic zones) or zero (in the stable zones). The only exception was when $A$ was set equal to the value which corresponds to the limit point of the Feigenbaum period-doubling sequence. At that value of $A$, the results suggested a value of $D_2$ which is less than unity but greater than zero, consistent with results by Grassberger [13].

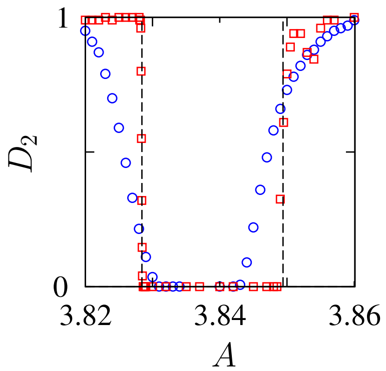


**Fig. 3.** (Color online) The correlation dimension $D_2$ of the logistic map with added noise, $x_n = Ax_n(1-x_n) + \eta_n$, with $\langle \eta_n \rangle = 0$. The noise intensity is $\sqrt{\langle \eta_n^2 \rangle} = 10^{-3}$ (blue circles) and $10^{-4}$ (red boxes). In the limit as the intensity approaches zero, the correlation dimension changes discontinuously between 0 and 1.

The observation that $D_2$ does not become small when $\lambda$ is small is consistent with (12) if the diffusion constant $\mathcal{D}$ approaches zero at the same time as $\lambda$ approaches zero. In the case of deterministic systems, this is in fact what happens, as the following heuristic argument shows. Consider what happens as the parameter $A$ of the logistic map is varied from $A_-$, where $\lambda$ is slightly negative, to the nearby value $A_+$ where $\lambda$ is slightly positive. At $A_-$ the map has an attractor which is a stable periodic orbit of period $M$ (say). The Lyapunov exponent for the stable orbit is

$$\lambda = \frac{1}{M} \sum_{j=1}^{M} \ln|f'(x_j^*)| \tag{20}$$

where $x_j^*$ are the points on the periodic orbit. In the case of the periodic orbit the shift of $Y$ with every orbit is precisely $M\lambda$, implying that $\mathcal{D} = 0$ for the stable orbit. At $A_+$ the periodic orbit has either become unstable, or else has ceased to exist. The trajectory then explores a dense subset of the line, but it must spend most of the time in the vicinity of the sequence of $M$ points visited by the nearby stable orbit. While the trajectory is trapped close to the periodic orbit, there is negligible randomness in the values of $Z$. We therefore expect that $\mathcal{D}$ approaches zero at the boundary of the unstable region, which is consistent with $D_2$ changing discontinuously from zero to unity upon going from a stable to a chaotic orbit.

If a small amount of noise is added to a deterministic map, the value of $\mathcal{D}$ remains finite at the boundary of the chaotic region, and we find that $D_2$ approaches zero smoothly at the boundary. We investigated the effect of adding noise to the logistic map: $x_n = Ax_n(1-x_n) + \eta_n$ where $\eta_n$ is uncorrelated Gaussian white noise with $\langle \eta_n \rangle = 0$. Figure 3 shows the correlation dimension $D_2$ as a function of $A$, in a small window near $A_c = 1 + \sqrt{8}$ where a stable three cycle forms. Data for zero noise amplitude are not shown: in this case the numerically found values for $D_2$ are consistent with either 0 or 1. The first dashed line is at $A_c = 1 + \sqrt{8}$. Figure 3 plots $D_2$ for two different values of the noise amplitude: $\sqrt{\langle \eta_n^2 \rangle} = 10^{-3}$ (circles) and $10^{-4}$ (boxes). At the lower noise intensity the change in $D_2$ is very abrupt, but at the higher intensity the change in $D_2$ is smooth. Noise is present in most phenomena which are described by dynamical systems. The discontinuous change in $D_2$ for deterministic dynamical systems is, therefore, a non-generic phenomenon.

## 6 Conclusions

Fractal clustering may occur even if there is no negative Lyapunov exponent. This was analysed and demonstrated in a random dynamical system, namely the correlated random walk. In the case of the logistic map, the clustering effect was not observed. We argued that it is harder to see the clustering effect in a deterministic dynamical system, although there does not appear to be any reason why it is forbidden. It would be interesting to find an example.

The work of BM and KG was supported by Vetenskapsrådet, and that of BM, KG and EW by the platform 'Nanoparticles in an interactive environment' at Göteborg university.

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