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Lyapunov exponent for small particles in smooth one-dimensional flows

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Abstract

This paper discusses the Lyapunov exponent λ for small particles in a spatially and temporally smooth flow in one dimension. The Lyapunov exponent is obtained as a series expansion in the Stokes number, St, which is a dimensionless measure of the importance of inertial effects. The approach described here can be extended to calculations of the Lyapunov exponents and of the correlation dimension for inertial particles in higher dimensions. It is shown that there is a correction to this theory which arises because the particles do not sample the velocity field ergodically. Using this non-ergodic correction, it is found that (contrary to expectations) the first-order term in the expansion of λ does not vanish.

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1. Introduction

It is known that small particles suspended in complex flows, such as fully developed turbulence, can cluster together [1], so that they accumulate on a fractal attractor [2]. In cases where the flow has a compressible component, it is also possible for the trajectories of particles to coalesce [3]. These effects have been analysed by calculating the Lyapunov exponents of the particle trajectories (the role of Lyapunov exponents in characterizing dynamical systems is discussed in [4]). The leading Lyapunov exponent λ is the average of the logarithmic rate of divergence of particles with an infinitesimal separation $\delta x(t)$:

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \left\langle \ln\left[\frac{\delta x(t)}{\delta x(0)}\right]\right\rangle, \qquad (1)$$

where throughout this paper $\langle X\rangle$ denotes the expectation value of X. If the leading Lyapunov exponent is negative, the particles aggregate together (the path coalescence effect) [5]. In cases where there is clustering onto a fractal attractor rather than aggregation, the dimension of the attractor can be estimated [6] from the full set of Lyapunov exponents of the particle trajectories by means of the Lyapunov dimension, given by the Kaplan–Yorke formula [7].

The clustering effect may play an important role in explaining the growth of rain drops [8] and of planetesimals [9], and deserves to be thoroughly understood.

Substantial progress has been made in analysing these clustering effects for a model with a random velocity field, in the limit where the correlation time of the velocity field is vanishingly small. All three Lyapunov exponents for the suspended particles were obtained as power series in a dimensionless parameter [10]. The methods used in that work involve a mapping to a Fokker–Planck equation, and appear to be restricted to the case of a vanishing correlation time. This paper shows how the analysis developed in [10] may be extended to models where the velocity field has a finite correlation time, allowing us to consider physically realistic velocity fields. The paper uses results from a recent work which discusses an operator-based approach to the analysis of perturbations of a generalized Ornstein–Uhlenbeck model [11].

It will be shown that there is a subtlety in the application of the finite correlation time model, which may complicate its application to models of turbulent velocity fields. For this reason, the problem is explored here in its simplest form, involving a one-dimensional flow, so that the difficulties can be explored in the simplest context.

In order to analyse the clustering effect, it is necessary to consider the dimensionless parameters of this problem. The Stokes number, a dimensionless measure of the inertia of the particles, is defined by

$$\mathrm{St} = \frac{1}{\gamma\tau}, \tag{2}$$

where τ is the correlation time for the fluctuations of the velocity gradient of the fluid, and γ is the rate at which particle motion relative to the fluid is damped. The statistics of the velocity field are described by a dimensionless parameter known as the Kubo number:

$$\mathrm{Ku} = \frac{u_0\tau}{\xi}, \tag{3}$$

where u_0 is a typical scale size of the velocity field and ξ is a correlation length. In [10] it is argued that for fully developed turbulence the relevant length, time and velocity scales are those corresponding to the Kolmogorov microscales of the velocity field. Dimensional considerations then imply that the Kubo number is of order unity for turbulent flows. However, the Kubo number will remain as a parameter in any random flow model for turbulence. It is desirable to understand how the Lyapunov exponent varies as a function of both the Stokes number and the Kubo number, so that the analytical properties of the approximations can be thoroughly understood. In section 3, the variables St and Ku will be replaced by the variables ω and κ, which are dimensionally equivalent but which are precisely defined.

The model with vanishing correlation time which is analysed in [10] corresponds to considering the limit as $\mathrm{Ku} \to 0$. The application of models with zero correlation time has been criticized [12], because it is argued that that such models fail to give the correct leading-order terms of series expansions for the Lyapunov exponents. In particular, it has been argued [12] that the difference between the fractal dimension D of the measure upon which particles cluster and the dimension d of the coordinate space is of second order in the Stokes parameter, that is, $d - D = O(\mathrm{St}^2)$, whereas for the zero correlation time model it is found that $d - D = O(\mathrm{St})$ [10]. Numerical evidence for the prediction that $d - D = O(\mathrm{St}^2)$ is presented by Bec [13], who credits Balkovsky *et al* [14] with the theoretical basis for this result. This discrepancy is addressed by the model which is developed here, and it emerges that these issues can be quite subtle. For the one-dimensional model with zero correlation time it is known [5] that $\lambda/\gamma = O(\mathrm{St})$, and here it is shown that a simple application of the model with finite correlation time indicates that the leading-order dependence of λ/γ is $O(\mathrm{St}^2)$, in line with the expectation that the finite correlation time model should have a different leading-order behaviour. A more careful analysis, however, leads to a surprising conclusion. The model with finite correlation time has the same leading-order behaviour as the zero correlation time model, namely $\lambda/\gamma = O(\mathrm{St})$. Moreover, the prefactor is the same.

An alternative model for the dynamics of inertial particles which involves finite correlation times has been proposed by Falkovich *et al* [15]. Their paper assumes that the time dependence of the velocity gradient of a one-dimensional flow is described by a telegraph noise process. Their work complements the model considered here: because the velocity gradient is discontinuous in time the model is less physical, but it is more amenable to analytic treatment.

2. A model for the Lyapunov exponent of particles in a random flow

Consider a one-dimensional model for particles embedded in a fluid with velocity $u(x, t)$. It is assumed that the motion of the particle relative to the fluid experiences a viscous damping force which is proportional to the velocity of the particle relative to the fluid. The equation of motion for a particle with position x and velocity v is therefore

$$\dot{x} = v, \qquad \dot{v} = -\gamma[v - u(x, t)], \tag{4}$$

where γ is proportional to the fluid viscosity. The Lyapunov exponent characterizes the time dependence of the infinitesimal separation δx of a trajectory from a reference trajectory. To analyse this quantity, linearize the equation of motion to obtain $\delta\dot{x} = \delta v$ and $\delta\dot{v} = -\gamma[\delta v - Y(t)\delta x]$, where $Y(t) = \frac{\partial u}{\partial x}(x(t), t)$ is the velocity gradient at the current position of the particle. Now, following [5], define a variable $X = \delta v/\delta x$. This has a simple relation to the Lyapunov exponent:

$$\lambda = \langle X\rangle \tag{5}$$

and the equation of motion for X is

$$\dot{X} = -\gamma X - X^2 + \gamma Y(t). \tag{6}$$

Thus, the task of calculating λ resolves into calculating the expectation value of a variable X which satisfies equation (6).

In turbulent or other unsteady, complex flows the velocity gradient $Y(t) = \frac{\partial u}{\partial x}(x(t), t)$ can be modelled as a stochastic variable. If the fluid velocity field $u(x,t)$ is statistically homogeneous in space and time, it seems reasonable to model the fluctuations of the velocity gradient by a random variable which has mean value equal to zero. Consider the following model for $Y(t)$:

$$\dot{Y} = -\frac{1}{\tau}Y + \eta(t), \tag{7}$$

where $\eta(t)$ is a white noise signal, with statistics $\langle\eta(t)\rangle = 0$ and $\langle\eta(t_1)\eta(t_2)\rangle = 2D\delta(t_1 - t_2)$. This is an Ornstein–Uhlenbeck process [16], discussed in [17]. The diffusion coefficient for the white noise process is obtained from the correlation function of the velocity gradient:

$$D = \frac{1}{2\tau^2}\int_{-\infty}^{\infty} \mathrm{d}t\,\langle Y(t)Y(0)\rangle. \tag{8}$$

The correlation function of the Ornstein–Uhlenbeck process (7) is $\langle Y(t_1)Y(t_2)\rangle = D\tau\exp(-|t_1 - t_2|/\tau)$ [17].

Equations (6) and (7) can be transformed into a dimensionless form which is precisely the same as equations (4) and (5) of [11]. The results developed there can be used to obtain a series expansion of the Lyapunov exponent, $\lambda = \langle X\rangle$. It is convenient to transform to a new dimensionless time variable, $t' = \gamma t$. The variables X, Y are related to dimensionless variables by writing $X = \alpha X'$, $Y = \alpha Y'$, where the quantity α has dimensions of inverse time. By writing $\alpha = \tau\sqrt{D\gamma}$ the equations of motion (6) and (7) can be written in the form

$$\frac{\mathrm{d}X'}{\mathrm{d}t'} = Y' - X' - \epsilon X'^2$$
$$\frac{\mathrm{d}Y'}{\mathrm{d}t'} = -\omega Y' + \omega\zeta(t'), \tag{9}$$

where $\zeta(t')$ is a white noise with statistics

$$\langle\zeta(t')\rangle = 0, \qquad \langle\zeta(t_1')\zeta(t_2')\rangle = 2\delta(t_1' - t_2'). \tag{10}$$

Equations (9) and (10) are precisely the system which is analysed in [11]. The parameters ϵ and ω in (9), and an additional parameter κ, are defined by

$$\epsilon = \tau\sqrt{\frac{D}{\gamma}}, \qquad \omega = \frac{1}{\gamma\tau}, \qquad \kappa = \sqrt{D\tau^3}. \tag{11}$$

The dimensionless parameters κ and ω are precisely defined realizations of the dimensionless parameters introduced in section 1, that is,

$$\mathrm{St} = \omega, \qquad \mathrm{Ku} = \kappa, \qquad \epsilon = \mathrm{Ku}\,\mathrm{St}^{1/2}. \tag{12}$$

Note that the equations of motion (9) are expressed most naturally in terms of the dimensionless parameters ω and ϵ, rather than in terms of the Stokes and Kubo numbers, ω and κ.

The Lyapunov exponent is $\lambda = \alpha\langle X'\rangle = \gamma\epsilon\langle X'\rangle$. Using (11) to express ϵ in terms of κ and ω and using equation (30) of [11] gives

$$\lambda = \frac{\gamma\kappa^2\omega^2}{1+\omega}\left[1 + C_1(\omega)\left(\frac{\kappa^2\omega^2}{1+\omega}\right) + C_2(\omega)\left(\frac{\kappa^2\omega^2}{1+\omega}\right)^2 + \cdots\right], \tag{13}$$

where the $C_j(\omega)$ are ratios of polynomials with the same degree, with $C_1(0) = 15$, $C_2(0) = 630\ldots$. The leading-order behaviour of the zero correlation time model is $\lambda \sim \gamma\epsilon^2 = \gamma\kappa^2\omega$ [5]. Thus, equation (13) is in agreement with the expectation that for a velocity field with finite correlation time, the leading-order behaviour is different. It also has the satisfying feature that it is based upon a model which yields all of the coefficients of the series expansion exactly by using the operator algebra developed in [11]. The technique appears to have a straightforward extension to multi-dimensional cases, and can also be adapted to provide the full series expansion of the subdominant Lyapunov exponents [10] and the correlation dimension [18]. There is, however, a deficiency in (13), which will be discussed in the next section.

3. Non-ergodic correction to the velocity statistics

The model introduced in section 2 uses the seemingly uncontroversial assumption that the mean value of the velocity gradient $Y(t)$ is zero. For a velocity field $u(x,t)$ which is statistically homogeneous as a function of x, the expectation value of $\partial u/\partial x$ must certainly vanish. It is, however, possible for $\langle Y(t)\rangle$ to be non-zero because $Y(t)$ is the velocity gradient evaluated at the current position of the particle, $x(t)$. Because $x(t)$ is itself dependent upon the history of the velocity field, $Y(t)$ may not represent an ergodic sampling of this field.

This non-ergodic correction may, in most circumstances, represent a small refinement of the theory. In the overdamped limit (as $\omega \to 0$) it could, however, be of critical importance in determining the leading-order behaviour of λ. In the limit as $\omega \to 0$, the Ornstein–Uhlenbeck noise term Y' in equation (9) varies very slowly. The stochastic variable X' must therefore also vary very slowly, and the time derivative in (9) can therefore be neglected. In the overdamped limit, the equation of motion for X' can therefore be approximated by

$$Y' - X' - \epsilon X'^2 = 0. \tag{14}$$

This is a quadratic equation determining X' as a function of Y'. In the overdamped limit (where $\omega \to 0$ with κ fixed), $\epsilon = \kappa\sqrt{\omega} \to 0$. The solution of this quadratic equation is approximated by $X' = Y' - \epsilon Y'^2 + O(\epsilon^2)$. Taking expectation values, and using (5), the Lyapunov exponent is

$$\lambda = \gamma\epsilon[\langle Y'\rangle - \epsilon\langle Y'^2\rangle] + O(\epsilon^3). \tag{15}$$

For the Ornstein–Uhlenbeck noise process in (9), $\langle Y'\rangle = 0$ and $\langle Y'^2\rangle = \omega$, so that the approximate equation (15) gives $\lambda = \gamma\epsilon^2\omega = \gamma\kappa^2\omega^2$, a result in precise agreement with the leading-order term in (13). If, however, a more refined approximation shows that $\langle Y(t)\rangle$ is non-zero, it may be discovered that the dominant contribution to λ/γ does, in fact, come from a term which is linear in the Stokes number ω.

In principle it is possible to expand $\langle Y\rangle$ as a series in ω and κ. Only the leading term of this expansion is considered here. In the limit where $\omega \ll 1$ a particle is simply advected by the flow and the equation of motion may be approximated by the advection equation, $\dot{x} = u(x(t), t)$. The position of the particle is then

$$x(t) = x(0) + \int_0^t \mathrm{d}t'\, u(x(t'), t') + O(\omega). \tag{16}$$

Now use this approximation to calculate the time average of $Y(t)$, that is,

$$\langle Y(t)\rangle = \lim_{T\to\infty}\left[\frac{1}{T}\int_0^T \mathrm{d}t\,\frac{\partial u}{\partial x}(x(t), t)\right]. \tag{17}$$

Using equation (16),

$$\int_0^T \mathrm{d}t\,\frac{\partial u}{\partial x}(x(t), t) = \int_0^T \mathrm{d}t\left[\frac{\partial u}{\partial x}(x(0), t) + \frac{\partial^2 u}{\partial x^2}(x(0), t)\int_0^t \mathrm{d}t'\, u(x(0), t')\right] + \cdots. \tag{18}$$

Averaging this, using translational invariance and assuming $T \gg \tau$ gives

$$\begin{aligned}\langle Y(t)\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\int_0^t \mathrm{d}t'\big\langle\partial_x^2 u(0, t)u(0, t')\big\rangle + \cdots \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\big\langle\partial_x^2 u(0, t)u(0, 0)\big\rangle + \cdots.\end{aligned} \tag{19}$$

The correlation function in (19) is easily related to that in (8). To this end define correlation functions $C_0(t)$, $C_1(t)$ and $C_2(t)$ as follows:

$$\begin{aligned}C_0(t) &= \left\langle\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}(x, t)u(x, 0)\right]\right\rangle, \\ C_1(t) &= \left\langle\frac{\partial^2 u}{\partial x^2}(x, t)u(x, 0)\right\rangle, \\ C_2(t) &= \left\langle\frac{\partial u}{\partial x}(x, t)\frac{\partial u}{\partial x}(x, 0)\right\rangle.\end{aligned} \tag{20}$$

Note that $C_0(t) = C_1(t) + C_2(t)$. Consider the evaluation of $C_0(t)$ by averaging over the spatial coordinate: because the quantity being averaged is a spatial derivative,

$$C_0(t) = \lim_{L\to\infty}\frac{1}{L}\int_0^L \mathrm{d}x\,\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}(x, t)u(x, 0)\right] = 0 \tag{21}$$

implying that $C_1(t) = -C_2(t)$. Now return to the evaluation of $\langle Y(t)\rangle$ using (19). Recalling the definition of D in (8),

$$\langle Y(t)\rangle = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, C_1(t) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, C_2(t) = -\tau^2 D. \tag{22}$$

From this it follows that $\langle Y'\rangle = \tau\sqrt{D/\gamma} = \epsilon$, and hence, using (15), one obtains the following estimate for the Lyapunov exponent:

$$\lambda = -\gamma\epsilon^2 = -\gamma\kappa^2\omega. \tag{23}$$

This is exactly the form obtained using the zero correlation time model [5].

4. Conclusions

This paper has discussed a new approach to extending the theoretical understanding of the dynamics of particles suspended in random flows, through making series expansions of the Lyapunov exponent in powers of the Stokes number. Until now, this has only been possible for models of the fluid velocity field with zero correlation time. The technique for obtaining the series expansion has been extended to a class of models for flows with a finite correlation time, using a perturbative analysis of a multi-dimensional Ornstein–Uhlenbeck process, considered in [11].

At first sight the application of the method in [11] appears to be straightforward, but there is a complicating factor which has been explored here in the one-dimensional case. This lies in considering the statistics of the velocity gradient of the particle at its current position, $Y(t) = \partial u/\partial x(x(t), t)$. In the case where the correlation time approaches zero, statistics of $Y(t)$ are clearly the same as those of $\partial u/\partial x$ sampled at a randomly chosen point in space. When the correlation time of the velocity field is finite, this ergodic relation can be questioned, because the position $x(t)$ has been influenced by the history of the velocity field, which is correlated with its current value. In section 3 it was shown that these correlations are significant, and imply that the mean value of the velocity gradient at the particle position $Y(t)$ is non-zero, despite the fact that the mean value of $\partial u/\partial x$ is zero.

When this correction is accounted for in the one-dimensional model, the result is surprisingly simple: the leading-order behaviour of the finite correlation time model becomes exactly the same as that of the zero correlation time model, including the prefactor. A non-ergodic behaviour of the velocity gradient also occurs in the 'telegraph noise' model analysed by Falkovich *et al* [15], but its effect is not so easily described in that system.

Finally, it should be emphasised that although the result that $\lambda \sim \gamma\mathrm{St}$ is at variance with an expectation which arises from the arguments in [12–14], this conclusion is specific to the one-dimensional model. The conclusion that $d - D \sim \mathrm{St}^2$ is correct in the two or three dimensions.

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