Colliding particles in highly turbulent flows

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We discuss relative velocities and the collision rate of small particles suspended in a highly turbulent fluid. In the limit where the viscous damping is very weak, we estimate the relative velocities using the Kolmogorov cascade principle. © *2007 American Institute of Physics.*
[DOI: 10.1063/1.2768931]

This Brief Communication considers collisions of small particles suspended in a highly turbulent gas. The collisions of these particles can facilitate aggregation of the suspended particles. This process may be relevant to the precipitation of rain from turbulent cumulus clouds,[1] and to the formation of planets by aggregation of dust particles suspended in the gas surrounding a growing star.[2]

The suspended particles are characterized by a dimensionless measure of the importance of inertia, termed the Stokes number, $\text{St} = 1/\gamma\tau$, where τ is a correlation time of the flow and γ is the damping rate of the suspended particles (both quantities are defined more precisely below). In Ref. 3 we showed how the collision rate increases very rapidly when St exceeds a threshold value, due to fold caustics making the velocity field of the suspended particles multivalued. In Ref. 3, which discusses the initiation of rainfall from turbulent clouds, it was sufficient to use a single-scale flow model of the turbulent motion (described by a correlation length η and correlation time τ) because the Stokes number is never very large for particles suspended in terrestrial atmospheric clouds. However, in astrophysical contexts it is necessary to consider flows with large values of St, where the multiscale aspect of turbulent flow[4] becomes important. (It is hard to study cases where St is large in terrestrial contexts because heavy particles fall out of the fluid.)

In the following we derive an expression for the collision rate in a highly turbulent flow with large Stokes number. We employ the Kolmogorov cascade principle to deduce an expression for the variance of the relative velocities of colliding particles, which in turn determines the collision rate. In the case of particles with very unequal damping rates, we calculate the variance of the relative velocity in terms of statistics of the turbulent velocity field. The results are consistent with those surmised from the Kolmogorov scaling principle.

In the formulation of the problem, we assume that the drag force on a particle is proportional to the difference in velocity between the particle and the surrounding gas, so that the equation of motion is

$$\ddot{\boldsymbol{r}} = \gamma[\boldsymbol{u}(\boldsymbol{r},t) - \dot{\boldsymbol{r}}] \tag{1}$$

where $\boldsymbol{r}$ is the position of the particle and $\boldsymbol{u}(\boldsymbol{x},t)$ is the fluid velocity field (until the particles come into contact). This equation is familiar in the context of Stokes's law for the drag on a sphere, where the damping rate γ is proportional to the kinematic viscosity ν.

In astrophysical applications, the mean free path of the gas is typically very large compared to the size of the particles,[2] but Eq. (1) remains applicable.[5] In this "Epstein regime" the damping rate given by $\gamma = \bar{c}\rho_\text{g}/\rho_\text{p}a$, where a is the radius, $\bar{c}$ is the mean molecular speed of the gas and ρ_g, ρ_p are the densities of the gas and the particles, respectively.[5] Particles produced by random aggregation processes tend to have a fractal structure, such that the number of particles N in a cluster of characteristic size a is $N \sim a^D$, where D is a "fractal dimension." In this case the Epstein drag formula becomes $\gamma = \kappa\bar{c}\rho_\text{g}\bar{A}/m$, where $\bar{A}$ is the average projected area of the particle, m is its mass, and κ is a dimensionless factor. In the case where the clusters are fractal, we therefore expect that $\gamma \propto N^{(2-D)/D}$ if $D > 2$, whereas the γ does not decrease with cluster size if $D < 2$. The large mean free path relative to cluster size suggests that ballistic aggregation models[6] are important. Experiments have shown evidence for clusters with $D < 2$,[7] however in the astrophysical context there are also long intervals between collisions, in which clusters of particles can relax to more compact and energetically favored shapes. The correct value of the dimension D for use in astrophysical applications is uncertain and probably not universal.

In Ref. 3, we demonstrated that the rate of collision R for a single suspended particle may be well approximated by

$$R = R_\text{diff} + R_\text{adv} + \exp(-A/\text{St})R_\text{gas}. \tag{2}$$

Here R_diff is a rate of collision due to Brownian diffusion (and therefore independent of the intensity of the turbulence), R_adv is the rate of collision due to the shearing effect of the flow, described by Saffman and Turner[8] and R_gas is the collision rate predicted by a "gas-kinetic" model, introduced by Abrahamson,[9] in which the suspended particles move with velocities which become uncorrelated with each other and with the gas flow. The exponential term describes the

fraction of the coordinate space for which the velocity field is multivalued and A is a "universal" dimensionless constant. The exponential dependence of the rate of caustic production on St was noted in Ref. 10 and recent simulations of Navier-Stokes turbulence suggest that $A \approx 2$.[11] The rate R_{gas} greatly exceeds R_{adv} and R_{diff}, but the gas-kinetic theory is only applicable when the velocity field of the suspended particles is multivalued. The mechanism for the particle velocity field becoming multivalued is the formation of fold caustics, described in Ref. 3. The formation of caustics can be modelled as a process of diffusion-driven escape from a basin of attraction, similar to the Kramers model for a chemical reaction.[10] The exponential term is therefore analogous to the Arrhenius term $\exp(-E/kT)$ in the expression for the rate of an activated chemical reaction. The abrupt increase of the collision rate as the Stokes number exceeds a threshold value was first noted in numerical experiments by Sundaram and Collins.[12] A theory proposed by Falkovich *et al.*[13] emphasized the significance of caustics, but does not allow accurate quantitative comparisons with numerically calculated collision rates.

Accounting for the multiscale nature of the flow will change the expression for the gas-kinetic collision rate R_{gas} in (2) but will leave the others unchanged. For a gas with particle density n_0, this contribution to the rate of collision between particles of radius a is

$$R_{gas} = 4\pi a^2 n_0 \langle \Delta v \rangle \tag{3}$$

where Δv is the relative velocity of two suspended particles at the same position in space (and angular brackets denote averages throughout). For collisions in a conventional fluid a "collision efficiency" factor is included to account for deflection of particles by the fluid trapped between them (see, e.g., Ref. 8), but in the Epstein regime this factor is not required. Suspended particles exhibit a tendency to cluster when St ≈ 1,[14,15] and in some circumstances the density n_0 might have to be modified to take account of this effect.[13]

Now we consider the relative velocities in a multiscale flow. In the following, η and τ are taken to be the dissipative length and timescales of the flow. According to the Kolmogorov theory of turbulence, these quantities are determined by the rate of dissipation per unit mass, $\mathcal{E}$, and the kinematic viscosity ν: we define

$$\eta = \left(\frac{\nu^3}{\mathcal{E}}\right)^{1/4}, \quad \tau = \left(\frac{\nu}{\mathcal{E}}\right)^{1/2}. \tag{4}$$

If the turbulent motion is driven by forces acting on a length scale $L \gg \eta$, the velocity fluctuations of the fluid have a power-law spectrum for wavenumbers between $1/L$ and $1/\eta$.[4]

In a multiscale turbulent flow, when $\gamma\tau \ll 1$ the motion of the suspended particles is underdamped relative to the motion on the dissipative scale, but (unless γ is smaller than frequency scale of the largest eddies) it is overdamped relative to slower long-wavelength motions in the fluid. The relative velocity of two nearby particles is a result of the different histories of the particles. If we follow the particles far back in time to when they had a large separation, their ve-

locities were very different, but these velocity differences are damped out when the particles approach each other. A stochastic model of this process was discussed in Ref. 16 which gave an expression for the relative velocity of colliding particles in terms of the velocity and length scales of the largest eddies. Here we show how to surmise the variance of the relative velocities by using the Kolmogorov cascade principle. Our result is universally applicable, in that it is expressed in terms of the rate of dissipation per unit mass, $\mathcal{E}$, rather than in terms of the particular nature of the driving process.

The motion of two suspended particles is determined by their damping rates γ_1, γ_2 and by the properties of the velocity field. When the particles are underdamped relative to the smallest dissipative scale, but overdamped relative to the "integral" (driving) scale, we can apply the Kolmogorov principle,[4] that motion in the inertial range is independent of the mechanism of dissipation (i.e., it is determined by the rate of dissipation $\mathcal{E}$ but it does not depend on ν). The moments therefore depend only upon $\mathcal{E}$ and γ_1 and γ_2. For the second moment of the relative velocity, dimensional consideration then imply

$$\langle \Delta v^2 \rangle \propto \mathcal{E}/\bar{\gamma} \tag{5}$$

where $\bar{\gamma}$ must be some weighted average of γ_1 and γ_2, given by a formula which is symmetric under interchange of labels.

We can gain some information about the form of the weighted average $\bar{\gamma}$ by considering the relative velocities of particles with very different damping rates. In the case where one particle is much more heavily damped, $\gamma_2/\gamma_1 \gg 1$ say, the particles with damping rate γ_2 may be treated as if they are advected with the flow. Thus $\Delta v = |\boldsymbol{v}|$, where $\boldsymbol{v} = \dot{\boldsymbol{r}} - \boldsymbol{u}$ is the velocity of the particle with damping rate γ_1 relative to the surrounding fluid. From Eq. (1), this satisfies $\dot{\boldsymbol{v}} = \dot{\boldsymbol{u}} - \gamma_1 \boldsymbol{v}$, which has the solution

$$\boldsymbol{v}(t) = \int_{-\infty}^{t} dt' \exp[\gamma_1(t'-t)]\dot{\boldsymbol{u}}(t')$$

$$= \int_{-\infty}^{t} dt' \exp[\gamma_1(t'-t)]\frac{D\boldsymbol{u}}{Dt'}(t')$$

$$+ \int_{-\infty}^{t} dt' \exp[\gamma_1(t'-t)](\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u}(\boldsymbol{r}(t'),t') \tag{6}$$

where $D\boldsymbol{u}/Dt = \partial \boldsymbol{u}/\partial t + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ is the Lagrangian acceleration of the fluid, which fluctuates on a timescale τ. If $|\boldsymbol{v}| \gg \eta/\tau$, the integrand of the final term in (6) fluctuates on a timescale $\eta/|\boldsymbol{v}| \ll \tau$, because in this limit fluctuations of $\boldsymbol{u}(\boldsymbol{r}(t'),t')$ are determined by the rate of change of its first argument. Under this condition on the typical size of the velocity $\boldsymbol{v}$ (which is verified for $\gamma_1\tau \ll 1$ below) the final integral in Eq. (6) may be neglected because its integrand fluctuates very rapidly. In the limit as $\gamma_1\tau \to 0$, the variance of the velocity of the particle relative to the fluid approaches

$$\langle \boldsymbol{v}^2 \rangle = \frac{\mathcal{I}}{2\gamma_1}, \quad \mathcal{I} = \int_{-\infty}^{\infty} dt \left\langle \frac{D\boldsymbol{u}}{Dt}(t) \cdot \frac{D\boldsymbol{u}}{Dt}(0) \right\rangle. \tag{7}$$

Noting that $\mathcal{E}\tau = (\eta/\tau)^2$, we confirm that $\langle \boldsymbol{v}^2 \rangle \gg (\eta/\tau)^2$ when $\gamma_1\tau \ll 1$. The dimensional arguments of Kolmogorov theory imply that $\mathcal{I}$ is a function of $\mathcal{E}$ and ν; observing that $\mathcal{I}$ has the same dimensions as $\mathcal{E}$ and we conclude that $\mathcal{I} \propto \mathcal{E}$. Kolmogorov's 1941 theory of turbulence suggests that we should write $\mathcal{I} = K\mathcal{E}$ with K a universal constant, but in practice K may have a weak dependence upon Reynolds number due to intermittency effects.[4] Thus we have

$$\langle \Delta v^2 \rangle = \frac{\mathcal{I}}{2\gamma_1} = \frac{K\mathcal{E}}{2\gamma_1} \tag{8}$$

when $\gamma_2/\gamma_1 \gg 1$. Equations (5) and (8) are consistent if we write

$$\langle \Delta v^2 \rangle = \frac{\mathcal{I}\sqrt{\gamma_1^2 + \gamma_2^2}}{2\gamma_1\gamma_2} g(|\ln(\gamma_1/\gamma_2)|) \tag{9}$$

where $g(x)$ is everywhere positive and approaches a finite limit as $x \to 0$. Furthermore

$$\lim_{x \to \infty} g(x) = 1 \tag{10}$$

for consistency with (8).

In the case where the values of γ are very different, we expect that the probability density function of the relative velocity is a three-dimensional Gaussian (or Maxwell-Boltzmann) distribution, because the central-limit theorem is applicable to Eq. (6) in the limit $\mathrm{St} \to \infty$. This implies that $\langle \Delta v \rangle = \sqrt{8/3\pi}\sqrt{\langle \Delta v^2 \rangle}$. In the case where the damping rates are comparable, it would be of interest to evaluate the distribution function from a mechanistic model.

In conclusion, Eqs. (9), (2), and (3) give the collision rate for underdamped particles in a highly turbulent flow, in terms of a measure of the turbulence intensity $\mathcal{I} = K\mathcal{E}$. The constant of proportionality K could be evaluated by direct numerical simulation of turbulent Navier-Stokes flow. The collision rate R is asymptotically proportional to $\sqrt{\mathcal{E}}$ as the turbulence intensity $\mathcal{E}$ is increased, provided that the Stokes number of the suspended particles is very large. The results will be of value in producing reliable estimates of rate of collision of dust particles in accretion disks.

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