Non-universality of chaotic classical dynamics: implications for quantum chaos

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Non-Universality of Chaotic Classical Dynamics: Implications for Quantum Chaos

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Abstract

It might be anticipated that there is statistical universality in the long-time classical dynamics of chaotic systems, corresponding to the universal agreement between their quantum spectral statistics and random matrix theory. It is argued that no such universality exists. Two statistical properties of long period orbits are considered. The distribution of the phase-space density of periodic orbits of fixed length is shown to have a log-normal distribution. Also, a correlation function of periodic-orbit actions is discussed, which has a semiclassical correspondence to the quantum spectral two-point correlation function. It is shown that bifurcations are a mechanism for creating correlations of periodic-orbit actions. They lead to a result which is non-universal, and which in general may not be an analytic function of the action difference.

1. Introduction

It has been appreciated for many years that sufficiently complex quantum systems exhibit a high degree of universality [1], in that many statistical properties of their spectra usually fall into one of three classes, exemplified by the three Gaussian random matrix ensembles introduced by Dyson [2]. It is natural to ask whether an analogous degree of universality exists in classical dynamics, and if it exists whether it underlies the universality observed in the behaviour of quantum systems. This paper suggests what the appropriate classical analog of quantum spectral universality should be, and gives arguments supporting the view that there is, in general, no classical universality underlying that of quantum systems.

The universality exhibited by spectral properties is confined to statistics which are sufficiently short ranged in energy. This implies that the universal features are associated with dynamics over long time scales: they may be associated with universal properties of the long-time classical dynamics, or they may be purely quantum. It is natural to anticipate that universal behaviour, if present at all, will only be manifest in properties which characterise small regions of phase space. The large scale structure of phase space can clearly be non-universal (for example, the volume of the energy shell is a non-universal function of energy).

An example of a property which could show universality is the statistical characterisation of the distribution of points periodic under the Nth iterate of a chaotic area preserving maps of the form $\alpha_{n+1} = \mathcal{M}(\alpha_n)$, with $\alpha = (x, p)$. These points are illustrated in Fig. 1 for the case of the 7th iterate of the standard map [3]

$$x_{n+1} = x_n + p_n\,, \qquad p_{n+1} = p_n + K\sin(x_{n+1}) \tag{1}$$

with $K = 6$ and the 11th iterate of a modified cat map

$$\begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ p_n \end{pmatrix} + K \begin{pmatrix} u_n \\ 0 \end{pmatrix} \tag{2}$$

with $u_n = \sin(x_n + 2p_n)$ and $K = 2$. The patterns displayed in Fig. 1 are complex and show very wide fluctuations in the density of points. They are clearly different, but local statistical properties of their fine-scale structure might be equivalent after scaling the coordinates to give the same mean density of points.

Gutzwiller's trace formula [4] gives a relation between classical periodic orbits and quantum spectra. The formula is exact for a small number of special systems, but it cannot be exact in general because it contains no information about the choice of quantisation procedure [5]. In [6,7] Gutzwiller's formula is combined with the observation that spectral correlations are described by random matrix theory to infer that the actions of long period orbits are correlated. The correlations are a function of the action difference which was predicted to be universal within each of Dyson's symmetry classes, and an analytic form was quoted for the GUE ensemble. In systems where Gutzwiller's trace formula is exact (for a discussion of the three known examples see [4,8]) such classcial correlations must certainly exist. However, in general the trace formula is not exact and it is thus necessary to find an entirely classical mechanism for such correlations. It has been suggested that action correlations are related to the symbolic dynamics of systems [7], but this has not led to quantitative predictions, and does not indicate why there should be universality. Some numerical studies have reported that action correlations exist, and are in agreement with the universal predictions [6,9].

The paper is organized as follows. In Section 2 it is demonstrated that the long-time, local structure of phase-space of chaotic systems is non-universal, using theoretical arguments and numerical experiments. Section 3 discusses the fluctuations in the density of periodic points, such as those shown in Fig. 1. It is demonstrated that the local density has a log-normal distribution. Section 4 proposes an entirely classical mechanism for periodic-orbit correlations, based on the observation that pairs of orbits have correlated actions when they are formed as a result of a bifurcation. Combined with strong, but reasonable, assumptions about the statistics of bifurcations, this mechanism gives rise to

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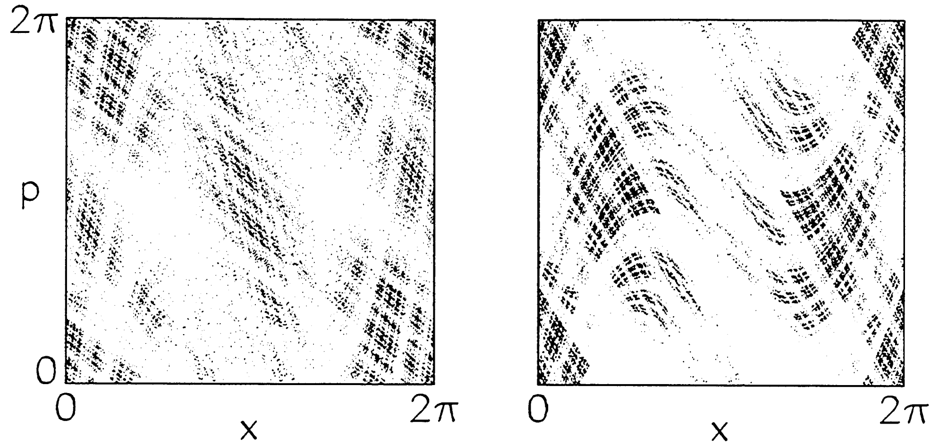

Fig. 1. Shows periodic points of the standard map with $K = 6$ and $N = 7$ (left) and for the modified cap map with $K = 2$ and $N = 11$ (right).

a non-universal and possibly non-analytic contribution to the correlation function.

2. Non-universality of long-time dynamics

The argument against universality is based upon considering a particular statistic, namely the proportion $P(N)$ of trajectories which are elliptic upon N iterations of area-preserving maps such as (1), (2). The results generalise directly to continuous time systems. The stability of a point periodic under N iterations of $\mathcal{M}$, $\alpha = \mathcal{M}^N(\alpha)$, is described by the monodromy matrix $M_N(\alpha) = \partial\mathcal{M}^N(\alpha)/\partial\alpha$. The periodic point $\alpha = \mathcal{M}^N(\alpha)$ is elliptic (hyperbolic) if $|\mathrm{tr}\, M_N| < 2$ ($|\mathrm{tr}\, M_N| > 2$). The terms elliptic and hyperbolic will be used in the same way to describe any trajectory, regardless of whether it is closed.

For chaotic maps, in the large-N limit the elliptic and hyperbolic trajectories are finely intermingled, implying that $P(N)$ is independent of any smooth scaling of the phase-space coordinates, and $P(N)$ will decrease rapidly with increasing N. Consider the distribution of values of $\mathrm{tr}\, M_N(\alpha)$ for large values of N. The monodromy matrix M_N is a product of elementary monodromy matrices $m(\alpha) = M_1(\alpha)$ for individual applications of the mapping: $M_N(\alpha) = \prod_{n=1}^{N} m(\alpha_n)$ where α_n is the phase-space point reached from α after n applications of the mapping. The typical values of the elements of the matrix M_N are expected to grow exponentially, in the sense that the mean of the logarithm of some norm of the matrix M_N should grow linearly with time: $\lambda = \frac{1}{2}\lim_{N\to\infty} N^{-1}\langle \log \mathrm{tr}\, [M_N^T(\alpha)M_N(\alpha)]\rangle_\alpha$ is termed the Lyapunov exponent. This suggests that elliptic trajectories are rare in the long-time limit, and that if a universal form exists for the function $P(N)$, it might be expected to be exponential,

$$P(N) \sim A \exp(-a\,\lambda\,N) \tag{3}$$

for $N \gg 1$, where A and a are universal constants.

Determing the fraction of elliptic trajectories is equivalent to determining the probability that $\mathrm{tr}\, M_N$ lies in the interval $[-2, 2]$. Because the typical value of $\mathrm{tr}\, M_N$ is exponentially large, this question relates to the tail of the distribution. In the case of a chaotic map the succesive positions α_n have the characteristics of random numbers, and the monodromy matrix M_N may be modelled as a product of random matrices. A product of a large number of random scalars has a log-normal distribution, and it is therefore natural to expect that $\mathrm{tr}\, M_N$ will have a log-normal distribution. The central limit theorem is only applicable sufficiently close to the maximum of the probability distribution, and the tails of the distribution of a sum of random variables depends upon the distribution of the variables. Because the distribution of the matrices $m(\alpha)$ is non-universal, the tail of the distribution of $\mathrm{tr}\, M_N$ is expected to be non-universal.

Further support for this prediction comes from considering moments of $M_N(\alpha)$. Modelling the α_n as random variables,

$$\langle M_N(\alpha)\rangle = \prod_{n=1}^{N}\langle m(\alpha_n)\rangle = \left[\langle m(\alpha)\rangle\right]^N \tag{4}$$

where the averages are taken over the invariant measure of the dynamics, which in the Hamiltonian case is just the uniform distribution on the phase-space area. Higher moments are obtained by averaging outer products of monodromy matrices: $\langle \mathrm{tr}\, M_N^T(\alpha)M_N(\alpha)\rangle$ for example may be expressed in terms of $\left[\langle m(\alpha)\otimes m(\alpha)\rangle\right]^N$ (see also [10]). The value of $\langle \mathrm{tr}\, M_N^T(\alpha)M_N(\alpha)\rangle$ is therefore expected to grow exponentially with N

$$\langle \mathrm{tr}\, M_N^T(\alpha)M_N(\alpha)\rangle \sim \exp(\lambda_2 N) \tag{5}$$

where λ_2 is the largest eigenvalue of $\langle m(\alpha)\otimes m(\alpha)\rangle$. A similar approach can be used to estimate the growth of $\langle(\mathrm{tr}\, M_N^T M_N)^k\rangle$: the result is of the form (5) with λ_2 replaced by λ_{2k}, the largest eigenvalue of the k-fold outer product $\langle m\otimes m\otimes\cdots\otimes m\rangle$. The values of these eigenvalues depend upon the structure of the elementary monodromy matrices m, indicating that the ratios of the eigenvalues λ_k are non-universal. It is very difficult to reconcile this with the hypothesis that $P(N)$ is universal.

To test these predictions, the fraction of elliptic trajectories and the Lyapunov exponent were evaluated for the mappings defined by Eqs. (1) and (2). Fig. 2 is a plot of the fraction $P(N)$ of elliptic trajectories as a function of λN. It clearly shows the non-universality of $P(N)$.

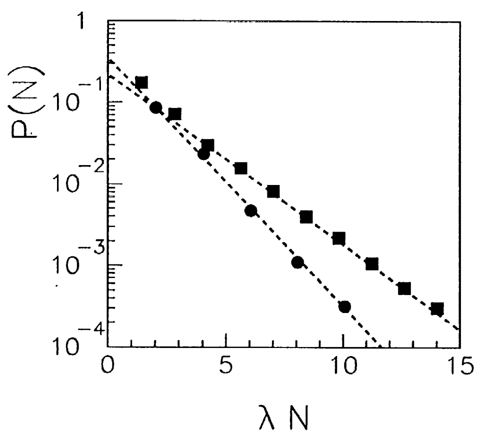

Fig. 2. Shows $P(N)$ as a function of λN for the standard map ($K = 15, \lambda \simeq \log K/2 = 2.015$. circles) and for the modified cap map ($K = 8, \lambda = 1.404$, squares).

3. Fluctuations of periodic orbit density

Figure 1 indicates that the density of periodic points is highly non-uniform, however an exact result due to Hannay and Ozorio de Almeida [11] suggests that the periodic points might have a uniform distribution in phase space. This section discusses how this apparent contradiction is resolved, presents an argument indicating that the density of periodic points has a log-normal distribution.

The discussion is based upon the Kac-Rice [12,13] approach to calculating the densities of point singularities of random functions. The density $\mathcal{D}$ of zeros of a vector-valued random function $\boldsymbol{f}(\boldsymbol{r})$ can be obtained from the joint probability distrbution function $P[\mathbf{f}, \tilde{F}]$ of $\mathbf{f}$ and of its Hessian matrix $\tilde{F}$, with elements $F_{ij} = \partial f_i/\partial r_j$. Adapting results from [13] to vector-valued functions, the density is

$$\mathcal{D} = \int d\tilde{F}|\det\tilde{F}|\, P[\mathbf{0}, \tilde{F}]. \tag{6}$$

In the following the iterated mapping is treated as if it were a random function. The problem is to find the set of points where $\alpha - \mathcal{M}^N(\alpha) = 0$ for large N. For long orbits, it may be assumed that the joint distribution function $P[\mathcal{M}, M]$ factorizes, and

$$\mathcal{D} = P[\mathcal{M}]\,\Big|_{\mathcal{M}=\alpha} \int \mathrm{d}M\, P[M]\,|\det(M - I)| = \mathcal{A}^{-1} \langle|\det(M - I)|\rangle \tag{7}$$

where $\mathcal{A}$ is the phase-space area. Similarly, the density of periodic points weighted with a smooth function $W(\alpha)$ is

$$\mathcal{D}_W = \mathcal{A}^{-1}\,\langle W(\alpha)\,|\det(M - I)|\rangle\,. \tag{8}$$

Hannay and Ozorio de Almeida [11] considered the case $W(\alpha) = |w_j|^2$, where w_j is the weight of the jth periodic orbit in the Gutzwiller sum. This case is of importance when estimating quantum two-point correlation functions semiclassically. The periodic orbit weights are $w_j = C\exp(i\pi\mu_j/2)\det(M_j - I)^{-1/2}$ where μ_j are integers termed Maslov indices which account for the phase changes associated with focusing or reflection, and C is a constant. This gives $W(\alpha) = C^2|\det(M_j - I)|^{-1}$, so that in this case $\mathcal{D}_W = C^2/\mathcal{A}$, which constitutes the version of the sum rule derived in [11] which is applicable to maps. This simple, universal form is therefore a result of the cancellation of the weight $|\det(M - I)|$ against its inverse. In general, however, the weighted density of periodic points depends upon the statistics of the monodromy matrices M.

Now consider the highly non-uniform distribution of periodic points shown in Fig. 1. It is natural to attempt to characterise this by a local density $\mathcal{D}(\alpha)$ defined in terms of the number of orbits inside a ball of radius, say, ε. In order to construct a satisfactory definition using this approach, the local density would have to converge over a range of values of ε, this range becoming broader as N increases. The fluctuations in density are so wild that it appears to be impossible to define a local density in this way. Instead, *define* the local density

$$\mathcal{D}(\alpha) = |\delta\alpha|^{-d}\,, \tag{9}$$

where $\delta\alpha$ is the displacement from the point α to the nearest periodic point, $|\delta\alpha|$ is the corresponding Euclidean distance, and d is the phase-space dimension. Assume that there is a periodic point at a small displacement $\delta\alpha$ from the test point α. Considering the condition for periodicity $\alpha + \delta\alpha = \mathcal{M}(\alpha + \delta\alpha)$, and expanding the mapping to first order in $\delta\alpha$ leads to the approximation

$$\delta\alpha = (M - I)^{-1}(\alpha - \mathcal{M}) \tag{10}$$

(where the index N designating the N-th iterate has been dropped). The fluctuations of this quantity are dominated by those of M which are log-normal. The distribution of the local density evaluated according to (9) is plotted in Fig. 3. It is well described by a log-normal distribution.

4. Bifurcations as a mechanism for periodic orbit correlation

Argaman *et al.* [6] introduced a classical correlation function of periodic-orbit actions, of (essentially) the form

$$C(T, \Delta S) = \sum_{j\neq j'} w_j\, w^*_{j'}\, \delta_\eta(T - T_j)\; \delta_\varepsilon\big(\Delta S - (S_j - S_{j'})\big)\,. \tag{11}$$

Here, S_j and T_j are the action and period of the jth periodic orbit, and ε, η are the widths of broadened delta functions. In systems where Gutzwiller's trace formula is exact, and which exhibit universal spectral two-point correlations, $C(T, \Delta S)$ must be scale to a universal form for each of Dyson's symmetry classes [6,7]. In general, however, the trace formula cannot be exact. It is thus desirable to explore classical models for periodic-orbit correlations contained in (11).

A mechanism for such correlations is revealed by embedding the Hamiltonian (or mapping) into a one-parameter family. We remark that systems for which exact trace formulae exist (Riemann's zeroes, motion of a surface with constant negative curvature [4] and quantum graphs [8]) do not form one-parameter families exhibiting bifurcations. The parameter X could be a coupling constant such as K in Eq. (1). Varying the parameter X produces bifurcations of orbits: immediately after a bifurcation the two orbits have the same actions $S_j = S_{j'}$, and their weights w_j and $w_{j'}$ may be related in a simple way (for example, they may have opposite signs). This mechanism creates cor-

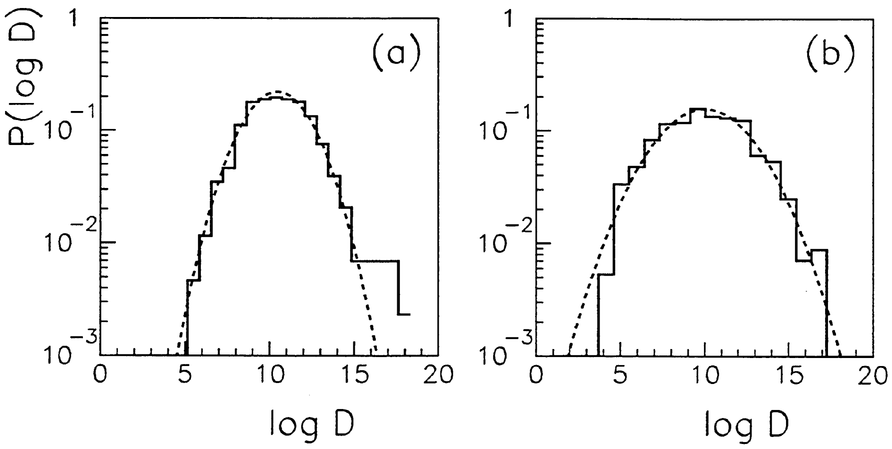

Fig. 3. Shows the distribution of the local density of periodic orbits $\mathcal{D}(\alpha)$ for the standard map with $K = 6$ and $N = 7$ (a) and for the modified cat map with $K = 2$ and $N = 11$ (b). In both cases the distribution is log-normal (— — —).

relations between actions which persist until one of the orbits undergoes a further bifurcation.

Bifurcations are not expected to exhibit universal statitical properties. In the case of systems with a smooth Hamiltonian and a mixed (but predominantly hyperbolic) phase space, bifurcations occur when a periodic orbit becomes elliptic, $|\mathrm{tr}(M)| = 2$. We have already seen that the fraction of elliptic trajectories is non-universal, and the frequency with which periodic orbits become elliptic must also be non-universal. The Gutzwiller trace formula is, strictly speaking, only applicable in fully hyperbolic systems. In such systems the condition $|\mathrm{tr}(M)| = 2$ is never realised, and the only possible bifurcations are associated with singularities of the Hamiltonian. There is even less prospect for universality here, because the form of the bifurcations is non-generic. Some parametric families do not even exhibit bifurcations. An example is a parametric version of the Baker's map, defined for (x, y) in the unit square by

$$\begin{aligned} x_{n+1} &= a x_n\ , \quad y_{n+1} = y_n/a && \text{if } y_n < a, \\ x_{n+1} &= a + (1-a)x_n\ , \quad y_{n+1} = (y_n - a)/(1-a) && \text{if } y_n \geq a. \end{aligned} \tag{12}$$

Other hyperbolic systems do exhibit bifurcations. In the Lorentz gas (scattering from a regular array of circular discs), orbits which undergo a glancing collision with a disc annihilate with orbits of nearly the same length which just miss the disc (see Fig. 4 for an illustration of this type of bifurcation). In this case the radius of scattering disc would be a suitable parameter.

A contribution to the correlation function (11) at small values of ΔS can be related directly to the form of the bifurcations. Assume that a bifurcation occurs at X_i as a parameter X is varied. The action difference in the neighbourhood of the bifurcation is of the form

$$|\Delta S_i| \sim v_i |\Delta X_i|^{\beta}\ , \tag{13}$$

where $\Delta X_i = X - X_i$. The exponent β will be determined from the type of bifurcation, and the constant v_i depends on the particular bifurcation. The weights w_j in (11) may be either singular or regular in the neighbourhood of the bifurcation: they may have algebraic singularities with exponents γ_j, which may be different for the two orbits.

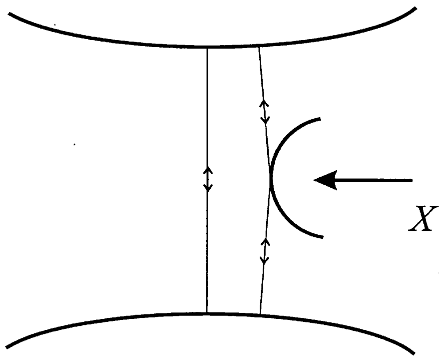

Fig. 4. The parameter X could define the shape of a billiard. In this example the circular section moves to the left as X increases, and the glancing periodic orbit annihilates the other orbit at an inverse bifurcation.

In the vicinity of the bifurcation,

$$w_j \propto |\Delta X_i|^{\gamma_j}\ . \tag{14}$$

The following discussion illustrates the effects of correlation through bifurcations by discussing a model for evaluating the contributions to (11) due to long periodic orbits. The model treats the w_j as random variables, which are independent except for those pairs of orbits which are related by a bifurcation:

$$\langle w_j \rangle = 0\ , \quad \langle w_j w_{j'} \rangle = w^2\, \delta_{jj'} \tag{18}$$

where in the latter case it is assumed that the orbits j and j' are not related by a bifurcation. At a bifurcation, one or both of the weights may be singular. In this case, it is reasonable to model the behaviour of $\langle w_j w_{j'} \rangle$ as follows

$$\langle w_j w_{j'} \rangle \sim \pm w^2\, \Lambda^{-(\gamma_j+\gamma_{j'})} |\Delta X|^{\gamma_j+\gamma_{j'}} \tag{16}$$

where Λ characterizes the frequency with which bifurcations occur. The factor ± 1 is included because for some types of bifurcation the weights may have opposite signs: note that the sign is not a random variable. The discussion will be simplified by assuming that all of the constants v_i in (13) have the same value, v.

Let $\delta P_\varepsilon(\Delta S)$ be the probability that a periodic orbit is connected to another orbit for which the action difference is in the interval $[\Delta S, \Delta S + \varepsilon]$. According to the model above, the only contributions to the correlation function come from pairs of periodic orbits related by a bifurcation. The contribution from these orbits may be estimated as follows

$$C \sim \pm \frac{1}{\varepsilon}\left(\frac{dN}{dT}\right) w^2 \left(\frac{\Delta X}{\Lambda}\right)^{(\gamma_j+\gamma_{j'})} \delta P_\varepsilon(\Delta S) \tag{17}$$

where $N(T)$ is the number of periodic orbits with period less than T, and ΔX is the distance to a bifurcation. From (13), the bifurcation occurs at a displacement in parameter space of the form $\Delta X \sim (\Delta S/v)^{1/\beta}$.

The probability that a periodic orbit does not undergo a bifurcation in a distance ΔX from an arbitrarily chosen test point is modelled by a Poisson distribution. This probability is $P_\pm(\Delta X) = \exp[-\Lambda_\pm |\Delta X|]$ for displacements to either side of the test point. For a sufficiently small separation ΔX, the probability of finding a bifurcation in a small interval of size $\delta\Delta X$ on either side of X_0 is

$$\delta P \sim \Lambda\, \delta\Delta X \tag{18}$$

with $\Lambda = \Lambda_+ + \Lambda_-$. Combining (13) and (18), one obtains $\delta P_\varepsilon(\Delta S) \sim \Lambda v^{1/\beta} |\Delta S|^{1/\beta - 1} \delta\Delta S$. Identifying $\delta\Delta S$ with ε, and substituting into (18) produces the following result, valid for small $|\Delta S|$:

$$C(T, \Delta S) \propto |\Delta S|^{(1+\gamma_j+\gamma_{j'}-\beta)/\beta}\,. \tag{19}$$

Evaluation of the exponent in (19) requires information about the nature of the bifurcations. As an example, consider a hyperbolic billiard system with no corners, such as the Lorentz gas. In this case inverse bifurcations occur when pairs of similar orbits, only one of which bounces off a surface, become tangential to that scattering surface (see Fig. 4). Geometrical considerations imply that $\beta = 2$, and $\gamma_j = 0$ for the orbit which does not bounce tangentially but $\gamma_{j'} = 1$ for that which does. In this case the exponent in (19) is zero, but for other types of bifurcation the exponent may be non-zero.

5. Conclusions concerning action correlations

In systems with a smooth Hamiltonian, the statistic $P(N)$ is very closely related to the condition for bifurcations (namely that a periodic orbit intersects the manifold where $\mathrm{tr}M = \pm 2$). The statistic $P(N)$ was shown to be non-universal, and it is implausible that bifurcations will show universal behaviour in such systems. Totally chaotic systems have no elliptic trajectories, and bifurcations are associated with singularities of the Hamiltonian: in this case the arguments against universality are no less compelling. It follows that there is a non-universal component to the periodic orbit correlation function. Equation (19) also indicates that the contributions to the correlation function coming from bifurcations has a singularity if the exponent $1 + \gamma_j + \gamma_{j'} - \beta$ is not an even integer. It is difficult to conceive of a non-universal and non-analtyic contribution to the correlation function which could combine with this one to give a universal result.

The arguments presented in this paper indicate that classical dynamics does not have the same degree of universality as quantum spectral statistics. It has been shown that bifurcations can be associated with action correlations, which are hypothesised to be in semiclassical correspondence with spectral correlations. Bifurcations, however, differ both in statistical properties and in form between different chaotic systems. In the case of generic chaotic systems, it is therefore unlikely that the Gutzwiller trace formula contains a complete description of spectral universality.

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