

PAPER

## Uniformity transition for ray intensities in random media

To cite this article: Marc Pradas *et al* 2018 *J. Phys. A: Math. Theor.* **51** 155002

View the [article online](#) for updates and enhancements.

# Uniformity transition for ray intensities in random media

Marc Pradas<sup>1</sup>, Alain Pumir<sup>2</sup> and Michael Wilkinson<sup>1</sup>

<sup>1</sup> School of Mathematics and Statistics, The Open University, Walton Hall, Milton Keynes, MK7 6AA, England

<sup>2</sup> Université de Lyon, ENS de Lyon, Université Claude Bernard, CNRS, Laboratoire de Physique, 69342 Lyon, France

E-mail: [marc.pradas@open.ac.uk](mailto:marc.pradas@open.ac.uk), [alain.pumir@ens-lyon.fr](mailto:alain.pumir@ens-lyon.fr) and [m.wilkinson@open.ac.uk](mailto:m.wilkinson@open.ac.uk)

Received 8 November 2017, revised 19 February 2018

Accepted for publication 22 February 2018

Published 15 March 2018



CrossMark

## Abstract

This paper analyses a model for the intensity of distribution for rays propagating without absorption in a random medium. The random medium is modelled as a dynamical map. After  $N$  iterations, the intensity is modelled as a sum  $S$  of  $\mathcal{N}$  contributions from different trajectories, each of which is a product of  $N$  independent identically distributed random variables  $x_k$ , representing successive focussing or de-focussing events. The number of ray trajectories reaching a given point is assumed to proliferate exponentially:  $\mathcal{N} = \Lambda^N$ , for some  $\Lambda > 1$ . We investigate the probability distribution of  $S$ . We find a phase transition as parameters of the model are varied. There is a phase where the fluctuations of  $S$  are suppressed as  $N \rightarrow \infty$ , and a phase where the  $S$  has large fluctuations, for which we provide a large deviation analysis.

Keywords: random media, intensity fluctuations, phase transition, large deviations

(Some figures may appear in colour only in the online journal)

## 1. Introduction

We consider a model for light rays propagating through a random medium with negligible absorption. Random fluctuations of the refractive index cause rays to diverge or to focus, leading to fluctuations of the light intensity, depending on the path of the light ray to reach the point where the intensity is observed [1–6]. In addition to applications to optics, the model studied here is relevant in a dynamical systems context, where extremely large fluctuations of the density of trajectories can be demonstrated [7]. The results may also have applications in electron transport in low-temperature conduction, where very pronounced fluctuations of

current density have been observed [8]. We consider cases where interference effects, leading to ‘speckle’ phenomena [9] are not relevant, either because the light source is not phase coherent, or because the spatial resolution of observations is greater than the coherence length.

Because the effects of each successive focussing or de-focussing events are to multiply the light intensity by a random factor, the effects of focussing are expected to increase exponentially with the path length. On the other hand, the intensity at a given point is the sum of the intensities from all of the rays reaching that point. The number of rays reaching a point increases exponentially with the path length, and we expect that the proliferation of rays will tend to average out the fluctuations of the intensity. There are, therefore, two effects on the distribution of intensity fluctuations which compete as we increase the path length. Does the effect of focussing along individual rays dominate, so that the light intensity shows increasingly pronounced fluctuations? Or does the proliferation of paths become dominant, so that intensity fluctuations are averaged out and the medium behaves as a diffuser which produces a uniform intensity? In this paper we introduce and analyse a very simplified, but physically well-motivated model, which is analytically solvable. We show that this model has a phase transition between a fluctuation-dominated phase and a uniform phase.

Our model is a reasonable description of paraxial propagation, where the angular dispersion of the rays remains small, and it may, therefore, find applications in situations where light rays are scattered through small angles. Propagation of light through an atmosphere which is turbulent due to convection is an important problem where paraxial approximations are usually valid [2, 3]. However, the principle underlying our phase transition, which is a competition between increasing intensity fluctuations along a ray and the averaging effect of a proliferating number of rays, is applicable outside the paraxial context.

## 2. A model for intensity statistics

Several approaches have been proposed to compute the distribution of the intensity of waves travelling through a random medium. Many authors have treated the solution of the wave equation directly, see e.g. [4, 5]. Others have simplified the problem by considering a short-wavelength limit and concentrating on the ray trajectories [3, 6, 10]. This approach relates the high-intensity events to the effects of focussing, and makes elegant connections with catastrophe theory [11, 12]. The use of catastrophe theory is appropriate when only a few rays reach each observation point. As we move deeper into a random medium, however, the number of trajectories which can reach a given point proliferates, essentially exponentially. It is this case which is addressed in our work: we consider propagation with negligible absorption in a short-wavelength limit, so that the intensities are determined by focussing of rays, but the number of rays which could contribute is extremely large. Our objective in this paper is to analyse a solvable model which can serve as a benchmark for future studies of specific systems.

We motivate our model by considering a simplified one-dimensional problem of ray propagation along the  $z$  axis. The point at which a ray crosses the perpendicular axis after propagation for a distance of  $z = n\Delta z$  (where  $\Delta z$  is some fixed increment) is  $x_n$ . The evolution of the ray position  $x_n$  is described by a sequence of random one-dimensional maps,  $f_n$ :

$$x_{n+1} = f_n(x_n) . \quad (1)$$

We assume that this random dynamical system has ‘chaotic’ properties with a positive Lyapunov exponent [13]. The density of initial conditions is  $\rho_0$ , and the density of trajectories after  $N$  iterations of the map is denoted as  $\rho_N(x)$ . If the map were invertible, the density would be  $\rho_0(x_N)/F'_N(x_N)$ , where  $F_N(x)$  is the mapping for  $N$  iterations so that

$$F'_N(x) = \left( \frac{\partial x_N}{\partial x_0} \right) \tag{2}$$

is the stability factor of the trajectory,  $x_N(x)$  is the  $N$  step pre-image of  $x$ , and  $\rho_0(x)$  is the initial density at  $x$ . Usually, however, a point will have multiple pre-images, so that

$$\rho_N(x) = \sum_{j=1}^{\mathcal{N}} \frac{\rho_0(x_j)}{|F'_N(x_j)|} \tag{3}$$

where the  $x_j$  are the  $\mathcal{N}$  pre-images of  $x$ . The number of pre-images of a point is expected to proliferate exponentially (with exponent equal to the topological entropy [14]), and after  $N$  iterations we have

$$\mathcal{N} \sim \Lambda^N \tag{4}$$

for some constant  $\Lambda > 1$ . The stability factor of the trajectory is a product of terms for each time step, where the sum runs over all of the pre-images of  $x$  at  $n = 0$  and the sensitivity of each trajectory is a product of independent terms:

$$F'_N(x) = \prod_{k=1}^N \left| \frac{\partial x_k}{\partial x_{k-1}} \right|_{x_{k-1}} = \prod_{k=1}^N f'_k(x_k) \tag{5}$$

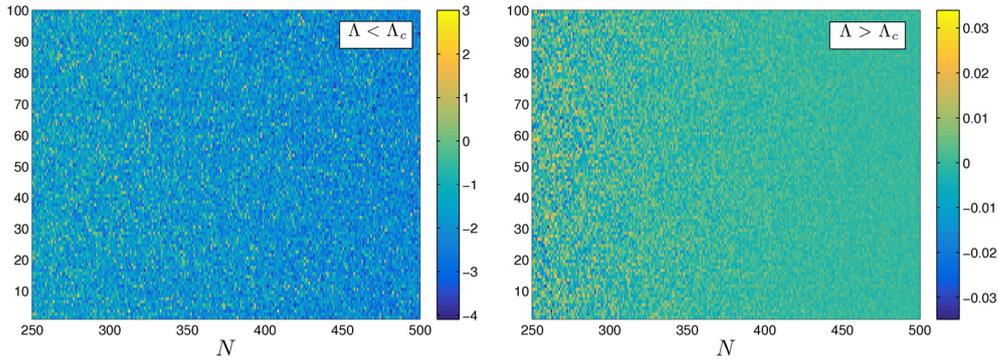
where the  $x_k$  are the successive pre-images after  $k$  iterations. When  $N$  is large, the density of trajectories is therefore constructed as a sum of a large number of terms (as implied by equations (3) and (4)), each of which is the product of a large number of factors (implied by equation (5)).

The analysis of how  $\rho_N(x)$  varies as a function of  $x$  for a specific system is clearly a difficult and usually intractable problem. However, the large number of proliferating pre-images implies that a statistical approach may yield valuable insights. In this paper we consider a statistical model for the density, represented by a sum  $S$ , which is constructed using a set of independent, identical distributed variables,  $X_k$ . The model is defined by the equations

$$\begin{aligned} \mathcal{N} &= \text{int}(\Lambda^N) \\ S &= \sum_{j=1}^{\mathcal{N}} Y_j \\ Y_j &= \prod_{k=1}^N X_k . \end{aligned} \tag{6}$$

Because intensity is a positive quantity, we assume that all of the factors  $X_k$  are positive. The problem is to characterise the probability distribution of  $S$  in the limit  $N \gg 1$ , given the value of  $\Lambda$  and the probability density function (PDF) of  $X_k$ . If  $S$  approaches a limit with small fluctuations relative to its magnitude, the density at large times is uniform. Alternatively, if the fluctuations of  $S$  relative to its size grow, then the density becomes highly inhomogeneous.

We note that in the model described by equation (6) there are competing effects. The fact that the  $Y_j$  are a product of many factors implies that they have very wide fluctuations in magnitude. On the other hand,  $S$  is a sum of an exponentially large number of independent quantities, so that fluctuations may be averaged away. We must consider which effect dominates, and whether, in the limit as  $N \rightarrow \infty$ , the dominant effect can change as the parameter  $\Lambda$  is varied. In the following we show that there is a phase transition: when  $\Lambda$  is relatively small,  $S$  shows



**Figure 1.** Representation of a set of values of the random variable  $S$  allocated in the vertical axis over different values of  $N$  (horizontal axis). Left and right panels correspond to a value of the parameter  $\Lambda$  below ( $\Lambda = 0.99\Lambda_c$ ) and above ( $\Lambda = 1.01\Lambda_c$ ) the critical point  $\Lambda_c$ , respectively. The colour coding illustrates from low (blue) to high (yellow) values of  $S$ , and the colour bar is in decimal logarithmic scale. At  $N = 500$ , the largest (smallest) fluctuations of  $S$  are larger (smaller) than the mean by a factor  $10^3$  ( $10^{-3}$ ) for the left panel, which shows the extreme nature of the fluctuations. On the contrary, the fluctuations do not exceed the mean by more than  $\approx 10\%$  for the right panel.

very large fluctuations, but as  $\Lambda$  is increased beyond a critical value  $\Lambda_c$ , the fluctuations of  $S$  in the limit as  $N \rightarrow \infty$  are suddenly suppressed. A numerical illustration of this effect is shown in figure 1, where we can see how a set  $\{S_1, \dots, S_m\}$ , with  $m = 100$ , evolves as we increase  $N$  for two different values of  $\Lambda$ . When  $\Lambda < \Lambda_c$  the random variable  $S$  exhibits inhomogeneous fluctuations spanning several decades in magnitude, while these fluctuations are largely suppressed when  $\Lambda > \Lambda_c$ . For this numerical example the random variables  $X_k$  in equation (6) are drawn from a log-normal distribution (see section 6 for more details).

The model given by equation (6) appears to be quite realistic as a model for fluctuations of ray intensity: if rays reach the point of observation via chaotic trajectories, then it is plausible that these rays will sample different regions of the random medium and that the intensity factors will be independent. The most significant weakness of our model is that it does not represent the effects of propagation: the intensity predicted by the model after  $N + 1$  steps is un-related to the realisation of the model for  $N$  steps. A more realistic model may take account of the cumulative effect of focusing along paths. However, if we are interested in the distribution at a single point, there are no obvious reasons why the predictions of our model should be suspect. In addition, our model has the advantage of being highly amenable to analytical investigations.

We note that the model presented here is somewhat analogous to models for the partition function of the Ising model and other interacting spin systems on disordered Bethe lattices [15, 16]. We remark, however, that we are aiming at a different type of result: our quantity  $S$  is analogous to the partition function, and we are concerned with its probability density. This would be analogous to studying the probability distribution of the partition function under different realisations of the lattice disorder. The model is also somewhat reminiscent of Derrida's random energy model for a spin glass [17]. Our model is also quite closely related to a model used in studies of hopping conductivity [18, 19]. That model differs by having random elements with different signs (or, more generally, different complex phases), but it also exhibits a phase transition, associated with a transition of the sign of the sum.

### 3. Explicit analytical calculations

In the following we simplify the discussion by making a specific choice of the PDF of the  $X_k$ . We give these variables a log-normal distribution by writing

$$X_k = \exp(y_k), \quad P_y = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] \quad (7)$$

where  $\mu$  and  $\sigma$  are constant (throughout,  $P_s$  will denote the probability density function of a random variable  $s$ , and  $\langle s \rangle$  represents its expectation value). With this choice of PDF, the moments of  $X_k$  are obtained explicitly as:

$$\langle X^n \rangle \equiv \langle (X_k)^n \rangle = \exp(n\mu + n^2\sigma^2/2) \quad (8)$$

which, as we shall see below, enable us to make explicit calculations. Later, we shall show that qualitative results obtained from this distribution are true for a very general choice of the probability distribution of the factors  $X_k$ .

In the following section we describe three calculations that can be done with the model (6), giving explicit results for the special case where the  $X_k$  have a log-normal distribution (as defined by (7)).

#### 3.1. Mean value

The mean value of  $S$  is

$$\langle S \rangle \equiv \exp[N\Sigma_0] = [\Lambda \langle X \rangle]^N \quad (9)$$

where the first equality defines the growth exponent  $\Sigma_0$ . Using equation (8), we find for the log-normal model

$$\Sigma_0 = \mu + \ln \Lambda + \frac{1}{2}\sigma^2. \quad (10)$$

The parameter  $\mu$  can be adjusted to make the mean value of  $S$  independent of  $N$  (which is a physical constraint on the intensity distribution for a non-absorbing medium), but this is irrelevant to the condition for the phase transition.

#### 3.2. Normalised central moments

A central moment of  $S$  is  $\langle \Delta S^k \rangle$  where  $\Delta S = S - \langle S \rangle$ . We consider the *normalised central moments*

$$M_k \equiv \frac{\langle \Delta S^k \rangle}{\langle S \rangle^k} \sim \xi_k^N \quad (11)$$

where the second equality defines the growth factor  $\xi_k$ . We find that, in the limit as  $N \rightarrow \infty$ ,  $M_2 \sim \xi_2^N$  with

$$\xi_2 = \frac{\langle X^2 \rangle}{\Lambda \langle X \rangle^2}. \quad (12)$$

This implies that the dispersion of the distribution of  $S$  approaches zero as  $N \rightarrow \infty$  if  $\xi_2 < 1$ , suggesting that the distribution will condense onto a delta function. This can be generalised. Consider the third moment

$$\langle \Delta S^3 \rangle = \mathcal{N} [\langle Y^3 \rangle - 3\langle Y^2 \rangle \langle Y \rangle - 2\langle Y \rangle^3] . \quad (13)$$

Noting that

$$\langle Y^3 \rangle = [\langle X^3 \rangle]^N, \quad \langle Y^2 \rangle \langle Y \rangle = [\langle X^2 \rangle \langle X \rangle]^N, \quad \langle Y \rangle^3 = [\langle X \rangle^3]^N \quad (14)$$

we see that

$$M_3 \equiv \frac{\langle \Delta S^3 \rangle}{\langle S \rangle^3} \sim \xi_3^N \quad (15)$$

where

$$\xi_3 = \frac{\max_{l=0,1,2} \langle X^{3-l} \rangle \langle X \rangle^l}{(\ln \Lambda)^2 \langle X \rangle^3} . \quad (16)$$

In general,  $\langle \Delta S^k \rangle$  is (for integer  $k > 1$ ) a linear combination of  $\mathcal{N}$  times  $\langle Y^{k-l} \rangle \langle Y \rangle^l$ , with  $l = 0, \dots, k-1$ . The integer coefficients are related to Pascal's triangle, but their values are irrelevant to determining the growth factors  $\xi_k$ . The value of  $\langle \Delta S^k \rangle$  is determined by the largest (in magnitude) of the values of  $\langle X^{k-l} \rangle \langle X \rangle^l$ . We have

$$M_k \equiv \frac{\langle \Delta S^k \rangle}{\langle S \rangle^k} \sim \xi_k^N \quad (17)$$

where

$$\xi_k = \frac{\max_{l=0, \dots, k-1} \langle X^{k-l} \rangle \langle X \rangle^l}{\ln \Lambda^{k-1} \langle X \rangle^k} . \quad (18)$$

In general, we cannot conclude that  $l = 0$  is the largest term, but for the log-normal model we have an explicit expression (8) for the expectation values, and we find

$$\xi_k = \exp [(k-1) (k\sigma^2/2 - \ln \Lambda)] . \quad (19)$$

This expression has been derived for positive integer values of  $k$ . It is however an analytic function and we can consider the consequences of assuming that it is valid for arbitrary values of  $k$ .

### 3.3. Largest element in sum

We can consider the PDF of  $Y_m$ , the largest element of the sum in equation (6), using a combination of large deviation [20, 21] and extreme value [22] approaches. The distribution of  $Y$  is more conveniently described in terms of a logarithmic variable

$$Z = \frac{1}{N} \ln Y = \frac{1}{N} \sum_{k=1}^N y_k \quad (20)$$

where  $y_k = \ln X_k$ . Note that  $Z$  is the mean value of  $y_k$ , so that the distribution of  $Z$  is expected to be described by a large-deviation ansatz [20, 21]:

$$P_Z \sim \exp[-NJ(Z)] \quad (21)$$

where  $J(Z)$  is termed the large deviation *entropy function* or *rate function* [21]. For the log-normal model the entropy function can be determined explicitly:

$$J(Z) = \frac{(Z - \mu)^2}{2\sigma^2}. \quad (22)$$

The precise form of the distribution of the maximal value of  $Z$ , namely  $Z_m = \ln Y_m/N$  is then determined from the Gumbel distribution [22]. However the essential features are easily explained. The peak of the distribution of  $Z_m$  is at position  $Z_0$ , determined by the condition that the product of the probability density and the number of samples is of order unity:

$$\mathcal{N}P_Z(Z_0) \sim 1. \quad (23)$$

By using  $\mathcal{N} \sim \Lambda^N$ , the above condition can be expressed in terms of the entropy function as  $J(Z_0) = \ln \Lambda$ , which has two solutions. Of these we must consider the larger solution, because we are considering the distribution of maximal values. For the log-normal model this gives

$$Z_0 = \mu + \sqrt{2\sigma^2 \ln \Lambda}. \quad (24)$$

The probability density to obtain  $Z_m < Z_0$  is extremely small: exponentials of exponentials. The probability density for  $Z_m > Z_0$  is approximately [22]

$$P_{Z_m}(Z_m) \sim \mathcal{N} \exp[-NJ(Z_m)]. \quad (25)$$

When  $Z_m - \Sigma_0$  is sufficiently small, we can approximate this using a Taylor expansion about  $\Sigma_0$ , the value corresponding to  $\langle S \rangle$ , see (9) and (10). The derivative of  $J(Z)$  at  $\Sigma_0$  is

$$J'(\Sigma_0) = \frac{\Sigma_0 - \mu}{\sigma^2} = \frac{1}{2} + \frac{\ln \Lambda}{\sigma^2} \quad (26)$$

so that  $P_{Z_m} \sim \exp[-\alpha N(Z_m - \Sigma_0)]$  with

$$\alpha = \frac{1}{2} + \frac{\ln \Lambda}{\sigma^2}. \quad (27)$$

Therefore, the corresponding PDF of  $Y_m$  is

$$P_{Y_m} \sim Y_m^{-(1+\alpha)}. \quad (28)$$

We remark that the case where  $\alpha = 1$  may be significant. If  $\alpha < 1$ , the approximation (28) suggests that the integral determining the mean value is divergent. In this case the mean value is determined by the behaviour of the tail of the distribution at values much larger than the typical value of  $Y$ . For our log-normal model, the critical point where  $\alpha = 1$  is determined by the condition

$$\ln \Lambda_c = \sigma^2/2. \quad (29)$$

#### 4. Inferences from calculations

The calculations discussed in section 3 can be used to infer properties of the distribution  $P_S$ , as follows.

##### 4.1. Existence of delta-function measures

Let us consider the consequences of finding that  $\xi_k < 1$  for some value of  $k$ . This condition may be satisfied if the distribution  $P_S$  approaches a delta function as  $N \rightarrow \infty$ . This is also consistent with the distribution  $P_S$  having a long ‘tail’, provided this tail decreases sufficiently rapidly as  $S \rightarrow \infty$  and as  $N \rightarrow \infty$ . For example, a distribution of the form

$$P_S \sim [1 - w(N)]\delta(S - S_0) + w(N)\Theta(S - S_0)(S - S_0)^{-(1+\alpha)}, \quad (30)$$

with  $w(N) \sim \exp[-\beta N]$ ,  $\alpha > k$ , and  $\beta > 0$ , is consistent with having normalised central moments that go to zero,  $M_k \rightarrow 0$ , as  $N \rightarrow \infty$ . In this sense, showing that  $\xi_k < 1$  for  $k > 1$  implies that there is a delta-function component of  $P_S$  that emerges as  $N \rightarrow \infty$ . It is hence desirable to determine the region of parameter space for which the delta-function component of  $P_S$  is present.

We have seen that  $M_k \sim \xi_k^N$ , where, in the log-normal case,  $\xi_k$  is given by expression (19), which is an analytical function of  $k$ . In the following, we assume that this expression is valid for any real positive value of  $k$ . This is very similar in spirit to the ‘replica trick’ where the free energy is obtained from the  $n^{\text{th}}$  moment of the partition function by taking the limit as  $n \rightarrow 0$  [23, 24]. Let us determine for which combination of the model parameters ( $\Lambda$ ,  $\mu$  and  $\sigma$ ) the value of  $\xi_k$  may be less than unity for some choice of  $k > 1$ . Clearly the value of  $\mu$  is irrelevant because it does not appear in (19). If  $\sigma^2/2 < \ln \Lambda$ , then, for all values of  $k$ , satisfying  $1 < k < 2 \ln \Lambda / \sigma^2$ ,  $\xi_k < 1$ . This therefore suggests that there is a delta-function component whenever  $\ln \Lambda > \sigma^2/2$ .

In the case where  $\sigma^2/2 > \ln \Lambda$ , the values of  $\xi_k$  can be less than 1 only for  $k < 1$ . In this case, we cannot infer the existence of a delta-function component. We have seen that when  $\sigma^2/2 > \ln \Lambda$ , the exponent in (28) satisfies  $\alpha < 1$ . This implies that the integral defining the mean value is divergent in the approximation (28), and that the mean value (which is finite) is determined by the behaviour of  $P_S$  far into the tail of its distribution. If  $0 < \alpha < 1$  and  $k < 1$ , the value of  $\langle \Delta S^k \rangle$  is determined by  $P_S$  at small values of  $S$ , so that  $\langle \Delta S^k \rangle / \langle S \rangle$  is small, without the necessity for a delta-function component.

Hence we can conclude that when  $\ln \Lambda > \ln \Lambda_c = \sigma^2/2$ , we always have a  $k > 1$  such that  $\xi_k < 0$ , implying that  $P_S$  condenses onto  $\delta(S - S_0)$  as  $N \rightarrow \infty$ . When  $\Lambda < \Lambda_c$ , we can have  $\xi_k < 0$ , when  $k < 1$ . This however is just a consequence of the very long tail of the distribution, and it does not imply condensation onto a delta function.

#### 4.2. Sum is dominated by its largest term

It is possible that the tail of the distribution of  $S$  is, in fact, dominated by the largest value of  $Y$ , so that when  $S \gg S_0$ ,  $P_S$  approaches  $P_{Y_m}$ . In the following we provide evidence that this is indeed the case. First, we note that if  $\xi_k > 1$ , the divergence of  $M_k$  as  $N \rightarrow \infty$  will be determined by the tails of the distribution of  $S$ . By making the change of variables  $Y = \exp(NZ)$ , and so  $Y_m = \exp(NZ_m)$ , in the tail of the distribution we have  $P_{Z_m} \sim \mathcal{N} \exp[-NJ(Z_m)]$ , so that

$$\langle \Delta S^k \rangle \sim \int dZ \exp[N(\ln \Lambda + kZ - J(Z))]. \quad (31)$$

Using the Laplace principle and writing  $F(Z) = \ln \Lambda + kZ - J(Z)$  we estimate

$$\langle \Delta S^k \rangle \sim \exp[NF(Z^*)] \quad (32)$$

where  $F'(Z^*) = 0$ . For the log-normal model we have  $Z^* = \mu + k\sigma^2$ , so that

$$\langle \Delta S^k \rangle \sim \exp[N(\ln \Lambda + k\mu + k^2\sigma^2/2)]. \quad (33)$$

Combining this estimate with equations (9), (10) and (17) we recover equation (19), hence suggesting that the tails of the distribution of  $P_S$  are asymptotic to the distribution of the largest element of the sum. Numerical results presented in the next section show that this is indeed the case, see figure 3.

### 4.3. Nature of the phase transition

The arguments presented so far imply the existence of a phase transition, which occurs at a critical value  $\Lambda_c$ . For the log-normal model we have shown this is  $\Lambda_c = \exp(\sigma^2/2)$ . To quantify this phase transition we will analyse the asymptotic behaviour of the PDF of  $S$  as  $N \rightarrow \infty$ . Consider the predicted form of  $P_S$  for the supercritical case,  $\ln \Lambda > \sigma^2/2$ . As  $N \rightarrow \infty$ , the distribution approaches a delta function, but we have also seen that the tail is in agreement with the distribution of the maximum value  $Y_m$ . By making use of  $P_{Z_m} \sim \mathcal{N} \exp[-NJ(Z_m)]$  and changing back to  $Y_m$ , we write

$$\begin{aligned} P_S &\sim \delta(S - S_0) + \mathcal{N} \exp[-NJ(Z_m)] \\ &\sim \delta(S - S_0) + \frac{1}{N} \exp(-ND) \left(\frac{S}{S_0}\right)^{-(1+\alpha)} \end{aligned} \quad (34)$$

where  $D = \alpha\Sigma_0 + J(\Sigma_0) - \ln \Lambda$ , which is a positive quantity. On the other hand, for the subcritical case,  $\ln \Lambda < \sigma^2/2$ , we predict that

$$P_S \sim \frac{1}{N} \exp(-ND) \left(\frac{S}{S_0}\right)^{-(1+\alpha)}. \quad (35)$$

## 5. Generalisation

Thus far we have derived results using explicit formulae for the log-normal model. It is desirable to understand how to address the same issues for a general probability distribution of the positive factors  $X_k$ . We introduce the auxiliary variable  $y_k$ , defined by  $X_k = \exp(y_k)$ . It is convenient to express the results in terms of the cumulant generating function  $\lambda(k)$  for the distribution of  $Z$ , defined by

$$\langle \exp(NkZ) \rangle = \exp[N\lambda(k)]. \quad (36)$$

Because

$$\langle \exp(NkZ) \rangle = \left\langle \exp\left(k \sum_{i=1}^N y_i\right) \right\rangle = \langle \exp(ky) \rangle^N \quad (37)$$

and  $X = \exp(y)$  it follows that

$$\lambda(k) = \ln \langle X^k \rangle. \quad (38)$$

The cumulant generating function is a Legendre transform of the entropy function for the distribution of  $Z$ :

$$J(Z) = kZ - \lambda(k), \quad k = J'(Z). \quad (39)$$

We can assume that  $J(Z)$  is a convex function, so that, for any value of  $Z_1$ ,

$$J(Z) \geq J(Z_1) + J'(Z_1)(Z - Z_1) \quad (40)$$

and a similar result holds for  $\lambda(k)$ . Now consider how the results of sections 3 and 4 generalise.

### 5.1. Central moments

Using (18) and assuming that the maximum growth exponent occurs for  $l = 0$ , we find that the exponents for the central moment are given by:

$$\ln \xi_k = \lambda(k) - k\lambda(1) - (k - 1) \ln \Lambda . \quad (41)$$

When  $k < 2$ , only the  $l = 0$  case need be considered, so that (41) is certainly valid in that case. The critical point for the phase transition is where  $\xi_{1+\epsilon} = 1$  as  $\epsilon > 0$  approaches zero, that is

$$\lambda'(1) - \lambda(1) = \ln \Lambda_c . \quad (42)$$

In the non-uniform phase, we can use (42) together with the convexity of  $\lambda(k)$  and the positivity of  $\ln \Lambda$  to establish that (41) is valid for all  $k$ .

### 5.2. Distribution of maximum element of sum

The maximum value of  $Y$  has a power-law distribution  $P(Y_m) \sim Y_m^{-(1+\alpha)}$ , with the exponent given by  $\alpha = J'(\Sigma_0)$ . Using (39), we obtain an implicit equation for  $\alpha$ : we have  $\lambda(\alpha) = \alpha\Sigma_0 - J(\Sigma_0)$  with  $\lambda'(\alpha) = \Sigma_0$ . Noting that  $\Sigma_0 = \ln \Lambda + \lambda(1)$ , we arrive at

$$\lambda'(\alpha) - \lambda(1) = \ln \Lambda, \quad (43)$$

which is an implicit equation for  $\alpha$ . In the case of the log-normal model we found that the critical point, i.e. where the delta-function component for the large  $N$  limit of  $P_S$  appears, corresponds to the point at which  $\alpha = 1$ . Equation (43) implies that, in the general case, the condition  $\alpha = 1$  is satisfied at a value of  $\Lambda$  which satisfies equation (42). We conclude that, in our model, the delta function distribution occurs whenever the decay of the distribution of the largest element is sufficiently rapid that the mean value of  $S$  is close to the mode of the distribution of  $S$ .

### 5.3. A consistency check

As a consistency check, we should verify that  $\Sigma_0 \geq Z_0$ , that is, the peak of the distribution of the maximum value of  $Y$  lies below the mean value of  $S$ , consistent with equation (34). This ensures that the PDF of the tail of  $P_S$  is already exponentially small for  $S$  just slightly greater than  $\langle S \rangle$ .

This is true for the log-normal model, where, setting  $A^2 = \ln \Lambda$  and  $B^2 = \sigma^2/2$ , we write  $\Sigma_0 = \mu + A^2 + B^2$  and  $Z_0 = \mu + 2AB$ . Because  $A^2 + B^2 - 2AB = (A - B)^2 \geq 0$ , we do confirm that  $\Sigma_0 \geq Z_0$ , as expected.

It is less easy to see why this should be true in the case of a general distribution of  $S$ . Recalling that  $\langle X^k \rangle = \exp[\lambda(k)]$ , the values of  $Z_0$  and  $\Sigma_0$  are defined via the following relations

$$J(Z_0) = \ln \Lambda, \quad \Sigma_0 = \ln \Lambda + \lambda(1) . \quad (44)$$

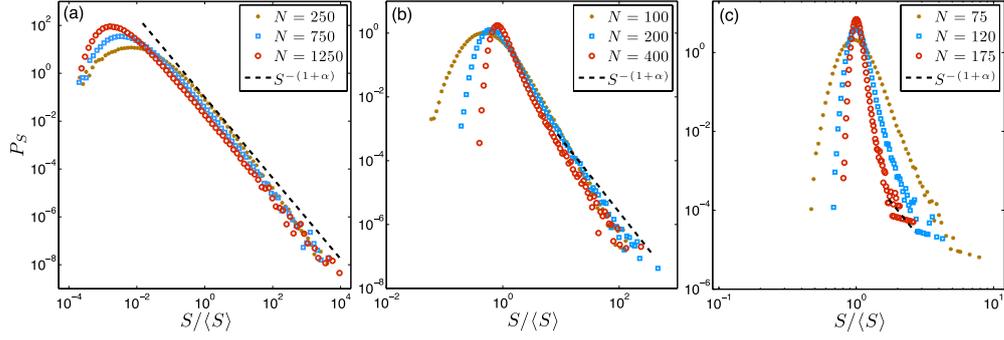
Define  $Z_1$  to be the image point of  $k = 1$  under the Legendre transformation:

$$J(Z_1) = Z_1 - \lambda(1), \quad J'(Z_1) = 1 . \quad (45)$$

Equation (44) give  $\Sigma_0 = J(Z_0) + \lambda(1)$ , and hence

$$\Sigma_0 - Z_0 = J(Z_0) + \lambda(1) - Z_0 = J(Z_0) - J(Z_1) + Z_1 - Z_0 . \quad (46)$$

Noting that  $J'(Z_1) = 1$ , the convexity relation (39) then establishes that  $\Sigma_0 - Z_0 \geq 0$ .



**Figure 2.** PDF of the random variable  $S$  when the variables  $x_k$  follow a log-normal distribution with  $\mu = 0$  and  $\sigma = 0.18$ , and for  $\Lambda = 0.99\Lambda_c$  (a),  $\Lambda = 1.01\Lambda_c$  (b), and  $\Lambda = 1.05\Lambda_c$  (c). Different symbols correspond to different values of  $N$ . The dashed line is a power-law function that goes as  $S^{-(1+\alpha)}$ , where  $\alpha$  is given by equation (27).

## 6. Numerical investigations

We investigated the distribution of  $S/\langle S \rangle$  for our model to verify that the phase transition exists as  $N \rightarrow \infty$ , and that it is correctly described by our theory. We used the log-normal distribution and the uniform distribution.

### 6.1. Log-normal distribution

The explicit calculations for this model have been derived in the previous section, where we have computed that the critical value of the phase transition occurs at  $\Lambda_c = \sigma^2/2$ . Figure 2 shows numerical results for  $\sigma = 0.18$  and for different values of  $\Lambda$  that are below and above the critical point. In the sub-critical case the distribution is approximated by a power-law, with an exponent which is independent of  $N$ , whereas in the super-critical case the distribution sharpens as  $N$  increases.

### 6.2. Uniform distribution

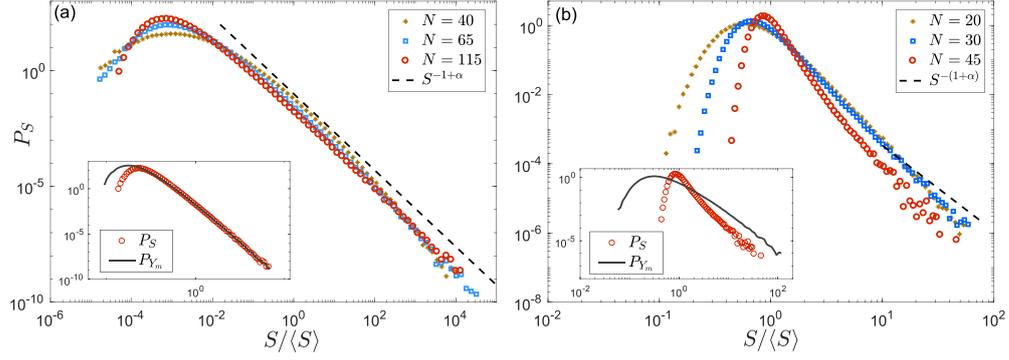
We also consider the case where the random variable  $x_k$  follows a uniform distribution in the interval  $[0, \ell]$ . We first use the results of section 5 to determine the critical point  $\Lambda_c$ , and the exponent  $\alpha$ .

To determine explicitly the entropy function  $J$ , we start with finding the moments, which is then used to determine the cumulant via equation (38):

$$\langle X^k \rangle = \int_0^\ell dX X^k = \frac{\ell^k}{k+1}, \quad \lambda(k) = k \ln \ell - \ln(1+k). \quad (47)$$

To determine the entropy function, we could adapt Example 2.3 p.6 of Ref. [21], or else use (39) to express  $J(Z)$  as the Legendre transform of  $\lambda(k)$ :  $J(Z) = kZ - \lambda(k)$ , with  $\lambda'(k) = Z$ . (Note that the way things are defined,  $Z < \ln \ell$  and  $k > -1$ ). We find  $Z = \ln \ell - 1/(1+k)$ , and eliminating  $k$  in (39) immediately gives:

$$J(Z) = -Z - 1 + \ln \ell - \ln(\ln \ell - Z) \quad (48)$$



**Figure 3.** PDF of the random variable  $S$  when the variables  $X_k$  follow a uniform distribution in  $[0, 1]$ , and for  $\Lambda = 0.95\Lambda_c$  (a), and  $\Lambda = 1.05\Lambda_c$  (b). Different symbols correspond to different values of  $N$ , and the dashed line is a power-law function that goes as  $S^{-(1+\alpha)}$ , where  $\alpha$  is given by equation (49). The inset panels show a comparison between  $P_S$  and the PDF of the largest element in the sum,  $P_{Y_m}$ , shown by the full line.

which is clearly convex, positive and has a minimum when  $Z = \log \ell - 1$ . Using now equation (43) we determine the exponent for the decay of the distribution  $P_S \sim S^{-(1+\alpha)}$ . We find  $\ln \Lambda = \ln(2) - 1/(1 + \alpha)$  that gives

$$\alpha = \frac{1}{\ln(2/\Lambda)} - 1, \quad (49)$$

which is independent of  $\ell$ . Setting  $\alpha = 1$ , or equivalently applying equation (41), we find that the critical value is given by

$$\Lambda_c = \frac{2}{\exp(1/2)}. \quad (50)$$

Figure 3 shows numerical results for  $P_S$  and  $P_{Y_m}$  for the case where the  $x_k$  have a uniform distribution, with  $\ell = 1$ .

## 7. Conclusions

We model intensity fluctuations by a sum of an exponentially increasing number of path contributions  $\mathcal{N} \sim \Lambda^N$ , each of which have a multiplicative distribution, with  $N$  random factors. Our calculations indicate that there is a phase transition, with a critical value of  $\Lambda$ :

- (i) The distribution  $P_S \sim \delta(S - S_0)$  as  $N \rightarrow \infty$  when  $\Lambda > \Lambda_c$ , apart from a power-law tail, with a coefficient which becomes exponentially small in the large  $N$  limit.
- (ii) When  $\Lambda < \Lambda_c$ ,  $P_S \sim S^{-(1+\alpha)}$  is approximately a power-law, with  $\alpha < 1$ .
- (iii) In both cases, there is a tail of  $P(S)$  which is asymptotic to the PDF of the largest element of the sum. This PDF can be obtained analytically.

Numerical investigations on two solvable models verify these results, showing that there is a transition between a phase where  $S$  has a delta-function distribution in the limit as  $N \rightarrow \infty$ , and a phase dominated by fluctuations, where  $S$  has a very broad distribution approximated by a power-law.

We postulate that this is a reasonable model for the distribution of intensity at a single point. It would be desirable to investigate how the parameters of our model could be estimated

for specific physical systems. Another interesting question is to consider the spatial structure of the intensity distribution, which is not addressed at all by the present model.

We remarked in the Introduction that we are neglecting interference effects, which can lead to ‘speckle’ patterns when the light source is coherent. In such a case, the delta-function intensity distribution of the uniform phase would be replaced by the intensity distribution of a homogeneous speckle pattern, and the intensity distribution of the non-uniform phase would also be broadened slightly. The statistics of speckle patterns has been studied quite extensively: see for example [9], which discusses spatial correlation functions, and [25] which considers recent work on intensity fluctuations of speckle as a potential tool for medical imaging. We remark that speckle due to coherence effects can also be observed in semiconductor systems at low-temperatures: for example Topinka *et al* [8] show images of ‘branched’ current flows, suggesting focussing effects creating large variations of current, together with small-scale fluctuations due to interference of the wavefunction.

Finally, it would be interesting to explore whether the ideas developed in this work could shed light on the transition observed in models of hopping conductivity [18, 19]. The model studied here could also conceivably shed light on the phenomenon of concentration of density in models of particles transport by a compressible flow discussed in [7].

## ORCID iDs

Marc Pradas  <https://orcid.org/0000-0002-8814-2403>

Michael Wilkinson  <https://orcid.org/0000-0002-5131-9295>

## References

- [1] Tatarski V I 1961 *Wave Propagation in a Turbulent Medium* (New York: McGraw-Hill)
- [2] Jakeman E and Tough R J A 1988 Non-Gaussian models for the statistics of scattered waves *Adv. Phys.* **37** 471–529
- [3] Uscinski B J 1968 The multiple scattering of waves in irregular media *Phil. Trans. R. Soc.* **262** 609–43
- [4] Shapiro B 1986 Large intensity fluctuations for wave propagation in random media *Phys. Rev. Lett.* **57** 2168–71
- [5] Martin J M and Flatté S M 1988 Intensity images and statistics from numerical simulation of wave propagation in 3D random media *Appl. Opt.* **27** 2111–26
- [6] Hannay J H 1982 Intensity fluctuations beyond a 1D random refracting screen in the short-wavelength limit *Opt. Acta: Int. J. Opt.* **29** 1631–49
- [7] Pradas M, Pumir A, Huber G and Wilkinson M 2017 Convergent chaos *J. Phys. A: Math. Theor.* **50** 275101
- [8] Topinka M A, LeRoy B J, Westervelt R M, Shaw S E J, Fleischmann R, Heller E J, Maranowskik K D and Gossard A C 2001 Coherent branched flow in a two-dimensional electron gas *Nature* **410** 183
- [9] Giglio M, Carpineti M and Vailati A 2000 Space intensity correlations in the near field of the scattered light: a direct measurement of the density correlation function  $g(r)$  *Phys. Rev. Lett.* **85** 1416–19
- [10] Metzger J J, Fleischmann R and Geisel T 2014 Statistics of extreme waves in random media *Phys. Rev. Lett.* **112** 203903
- [11] Berry M V 1977 Focusing and twinkling: critical exponents from catastrophes in non-Gaussian random short waves *J. Phys. A: Math. Gen.* **10** 2061–81
- [12] Berry M V 1981 *Singularities in Waves (Les Houches Lecture Series Session XXXV)* ed R Balian *et al* (Amsterdam: North Holland) pp 453–543
- [13] Ott E 2002 *Chaos in Dynamical Systems* 2nd edn (Cambridge: Cambridge University Press)
- [14] Nitecki Z and Przytycki F 1999 Preimage entropy for mappings *Int. J. Bifurcation Chaos* **9** 1815–43

- [15] Bethe H A 1935 Statistical theory of superlattices *Proc. R. Soc. A* **150** 552–75
- [16] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
- [17] Derrida B 1981 Random energy model: an exactly solvable model of disordered systems *Phys. Rev. B* **24** 2613–26
- [18] Nguen V L, Spivak B Z and Shklovskii B I 1985 Tunnel hopping in disordered systems *Sov. Phys.—JETP* **89** 1770–84
- [19] Spivak B, Feng S and Zeng F 1996 Sign phase transition and directed paths in random media *Pis'ma v ZhETF* **64** 283–8
- [20] Freidlin M I and Wentzell A D 1984 *Random Perturbations of Dynamical Systems (Grundlehren der Mathematischen Wissenschaften vol 260)* (New York: Springer)
- [21] Touchette H 2009 The large deviation approach to statistical mechanics *Phys. Rep.* **478** 1
- [22] Gumbel E J 1935 Les valeurs extremes des distributions statistiques *Ann. Inst. Henri Poincaré* **5** 115–58
- [23] Edwards S F and Anderson P W 1975 Theory of spin glasses *J. Phys. F: Met. Phys.* **5** 965–74
- [24] Parisi G 1984 An introduction to the statistical mechanics of amorphous systems *Recent Advances in Field Theory and Statistical Mechanics* ed B Zuber and R Stora (Amsterdam: North-Holland) pp 473–523
- [25] Kirillin M Y, Golnaz F, Sergeeva E A, Kolios M C and Vitkin A 2014 Speckle statistics in OCT images: Monte Carlo simulations and experimental studies *Opt. Lett.* **39** 3472–5