Large deviation analysis of rapid onset of rain showers – Supplementary materials

Michael Wilkinson

December 8, 2015

# 1 Introduction

These notes present some details of the calculation of the explicit asymptotic approximation to to the probability density $P(\tau)$ in the limit as $\tau \to 0$. They also describe an approximation which gives improved convergence when $\gamma - 1$ is small.

Equations in the Letter are referred to using square brackets, and equations in these Supplementary Materials by parentheses. To simplify expressions, these notes assume that the rate for the first collision is $R_1 = 1$.

# 2 Asymptotics of summations

The cumulant is expressed in dimensionless form by the following summation:

$$S(\kappa) = \sum_{n=1}^{\infty} \ln(1 + \kappa n^{-\gamma}) \ . \tag{1}$$

The small $\tau$ asymptotics are expressed in terms of the large $\kappa$ expansion of this sum: write

$$S = S_0 + \sum_{n=1}^{\infty} \Delta S_n \tag{2}$$

where

$$S_0 = \int_0^\infty \mathrm{d}n \ \ln(1 + \kappa n^{-\gamma}) \tag{3}$$

and

$$\Delta S_n = \int_0^1 \mathrm{d}x \ \left[\ln(1 + \kappa n^{-\gamma}) - \ln(1 + \kappa (n-x)^{-\gamma})\right] \ . \tag{4}$$

When $n \gg 1$, the $\Delta S_n$ are approximated by

$$\Delta S_n \sim -\frac{\partial}{\partial n} \ln(1 + \kappa n^{-\gamma}) \int_0^1 \mathrm{d}x \ x = \frac{-\gamma \kappa}{2n(\kappa + n^\gamma)} \ . \tag{5}$$

When $\kappa n^{-\gamma} \gg 1$, but $n$ is not necessarily large, however

$$\Delta S_n \ \sim \ \int_0^1 \mathrm{d}x \ \ln\left[\kappa n^{-\gamma}(1 + n^\gamma/\kappa)\right] - \ln\left[\kappa (n-x)^{-\gamma}(1 + (n-x)^\gamma/\kappa)\right]$$

$$
\begin{aligned}
&= -\gamma \int_0^1 \mathrm{d}x\ \ \ln(n) - \ln(n-x) + \frac{1}{\kappa}\int_0^1 \mathrm{d}x\ [n^\gamma - (n-x)^\gamma] + O(\kappa^{-2}) \\
&= -\gamma \int_{n-1}^n \mathrm{d}y\ \ \ln(n) - \ln(y) + \frac{1}{\kappa}\int_{n-1}^n \mathrm{d}y\ n^\gamma - y^\gamma + O(\kappa^{-2}) \\
&= -\gamma \left[(n-1)\ln\left(\frac{n-1}{n}\right) + 1\right] + \frac{n^\gamma}{\kappa}\left[1 - \frac{n}{\gamma+1} + \frac{n-1}{\gamma+1}\left(\frac{n-1}{n}\right)^{\gamma+1}\right] + O(\kappa^{-2})\ . \quad (6)
\end{aligned}
$$

In order to evaluate $S$ using (2), split the summation over $\Delta S_n$ into two parts: choose an integer $M$ such that $\kappa M^{-\gamma} \gg 1$. As $\kappa \to \infty$, let $M \to \infty$. Use (5) for $\Delta S_n$ with $n > M$ and (6) when $n \leq M$:

$$
S \sim S_0 - \gamma \sum_{n=1}^{M} \left[(n-1)\ln\left(\frac{n-1}{n}\right) + 1\right] - \gamma\kappa \sum_{n=M+1}^{\infty} \frac{1}{2n(\kappa + n^\gamma)}\ . \quad (7)
$$

Approximating the second summation by an integral using $y = n^\gamma/\kappa$:

$$
\begin{aligned}
-\gamma\kappa \sum_{n=M+1}^{\infty} \frac{1}{2n(\kappa+n^\gamma)} &\sim \frac{-\gamma\kappa}{2}\int_M^\infty \mathrm{d}n\ \frac{1}{n(\kappa+n^\gamma)} \\
&= -\frac{\gamma}{2}\int_{M^\gamma/\kappa}^\infty \mathrm{d}y\ \frac{1}{y(1+y)} \\
&= -\frac{\gamma}{2}\int_{M^\gamma/\kappa}^\infty \mathrm{d}y\ \frac{1}{y} - \frac{1}{1+y} \\
&= \frac{1}{2}\ln\left[\frac{M^\gamma}{\kappa + M^\gamma}\right]\ . \quad (8)
\end{aligned}
$$

Thus:

$$
S \sim S_0 - \gamma \sum_{n=1}^{M} \left[(n-1)\ln\left(\frac{n-1}{n}\right) + 1\right] - \frac{1}{2}\ln\left[\frac{M^\gamma}{\kappa+M^\gamma}\right]\ . \quad (9)
$$

Taking the limit as $M \to \infty$,

$$
S \sim S_0 - \frac{1}{2}\ln(\kappa) - \gamma C + O(\kappa^{-1}) \quad (10)
$$

where

$$
C = \lim_{n\to\infty} \left[(n-1)\ln\left(\frac{n-1}{n}\right) + 1\right] - \frac{\ln(n)}{2}\ . \quad (11)
$$

The numerical value of $C$ is

$$
C \approx 0.91896611\ . \quad (12)
$$

The time $T$ is obtained from the sum $S$ by differentiating with respect to $\kappa$, giving

$$
T(\kappa) = \frac{\partial S}{\partial \kappa} = \sum_{n=1}^{\infty} \frac{1}{\kappa + n^\gamma}\ . \quad (13)
$$

The asymptotic expression for $T(\kappa)$ is obtained by differentiating (10), yielding the following expression for the leading orders of the relation between $\kappa$ and $T$:

$$
T(\kappa) \sim A(\gamma)\kappa^{-\frac{\gamma-1}{\gamma}} - \frac{1}{2\kappa} \quad (14)
$$

where

$$A(\gamma) = \frac{1}{\gamma}\int_0^\infty \mathrm{d}x\, \frac{x^{-\frac{\gamma-1}{\gamma}}}{1+x} = \frac{1}{\gamma}\mathrm{B}\left(1-\frac{1}{\gamma},\frac{1}{\gamma}\right) \ . \tag{15}$$

Two values of interest are $A(2) = \pi/2$ and $A(4/3) = 3\pi/2\sqrt{2}$. Combining equations (10) and (14) gives equation [21] of the Letter.

# 3 Asymptotic expressions as a function of time

The expressions of the cumulant which are derived above enable the probability density to be expressed in terms of the saddle-point parameter $\kappa$. To obtain an explicit expression, equation (14) must be inverted to express the parameter $\kappa$ in terms of the time $T$, or in terms of the dimensionless time $\tau$:

$$\tau = \frac{T}{\langle T\rangle} = \frac{T}{\zeta(\gamma)} \ . \tag{16}$$

Inverting the relationship (14) to express $\kappa$ in terms of $T$, and hence $\tau$, the terms which do not approach zero in the limit as $\tau \to 0$ are

$$\kappa = b\tau^{-\frac{\gamma}{\gamma-1}}\left[1 - c\tau^{\frac{1}{\gamma-1}}\right] \tag{17}$$

where

$$b = \left[\frac{A(\gamma)}{\zeta(\gamma)}\right]^{\frac{\gamma}{\gamma-1}} \quad , \quad c = \frac{\gamma}{2(\gamma-1)A(\gamma)}b^{-\frac{1}{\gamma}} \ . \tag{18}$$

The entropy is

$$J(\tau) = S(\kappa^*) - T(\kappa^*)\kappa^* = (\gamma-1)A\kappa^{\frac{1}{\gamma}} - \frac{1}{2}\ln\kappa - (\gamma C - \frac{1}{2}) \ . \tag{19}$$

In terms of $\tau$, the leading order terms of the entropy are

$$J(\tau) = (\gamma-1)\, A\, b^{\frac{1}{\gamma}}\tau^{-\frac{\gamma}{\gamma-1}} + \frac{\gamma}{2(\gamma-1)}\ln\tau + D \tag{20}$$

where $D$ is another constant. Furthermore, in terms of $\tau$ the second derivative of the entropy with respect to $\kappa$ is

$$J''(\tau) = \frac{\gamma-1}{\gamma}Ab^{-\frac{2\gamma-1}{\gamma-1}}\tau^{\frac{2\gamma-1}{\gamma-1}} \ . \tag{21}$$

The asymptotic expression for the probability density

$$P(\tau) = \frac{1}{\sqrt{2\pi J''(\tau)}}\exp[-J(\tau)] \tag{22}$$

is then evaluated using (20) and (21). The result is

$$P(\tau) = K\tau^{-\frac{3\gamma-1}{2(\gamma-1)}}\exp(-\mathcal{C}/\tau^{\frac{1}{\gamma-1}}) \tag{23}$$

where $\mathcal{C} = (\gamma-1)Ab^{\frac{1}{\gamma}}$ and $K$ is another constant which can be explicitly constructed.

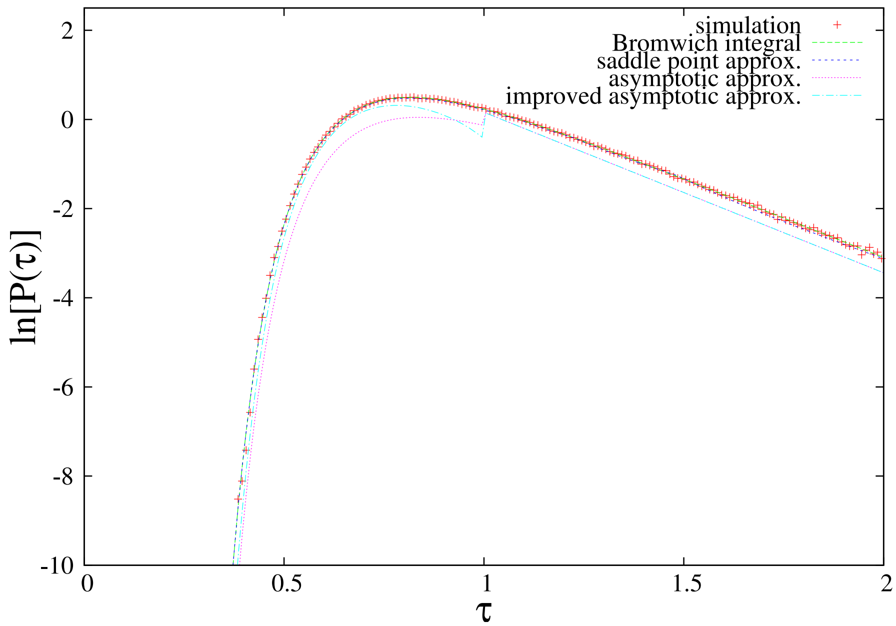


Figure 1: (Color online). Plot of $\ln[P(\tau)]$, for $\mathcal{N} = 10^5$ and $\gamma = 4/3$.

## 4 Accuracy of the asymptotic approximation

Figure 1 of the Letter shows a comparison of the asymptotic approximation, equation (23) with the numerically exact probability density for the case of $\gamma = 2$, which was argued to be most relevant to rainfall. The agreement is excellent. As $\gamma$ approaches unity from above, however, it is more difficult to see agreement of (23) with numerical results, and for the case of $\gamma = 4/3$, equation (23) only becomes accurate for very small values of $\tau$.

Equation (23) is derived by assuming that the second derivative of $J(\tau)$ is given correctly by equation (21), which used the leading order approximation to $\kappa$ as function of $\tau$. It is found that the asymptotic approximation is improved if (21) is replaced by

$$J''(\tau) = \frac{\gamma - 1}{\gamma} A(\gamma)\, \kappa^{\frac{1}{\gamma} - 2} \tag{24}$$

with $\kappa$ given by equation (17), which incorporates the sub-leading term.

Figure 1 of these notes compares different estimates of $P(\tau)$ when $\gamma = 4/3$. The simulation, evaluation of the Bromwich integral, and the saddle point estimate are all in excellent agreement. For the values of $\tau$ which can be investigated numerically, the agreement with the asymptotic estimate (23) is only fair. The improved asymptotic estimate using equation (24) shows a significant improvement. The plot switches to the large $\tau$ asymptotic estimate, equation [26] of the Letter, with a discontinuity visible in the curves at $\tau = 1$.

When $\gamma - 1$ is small, the convergence of the sum defining $T$, equation [4] is very slow, and a larger value of $\mathcal{N}$ is required to see convergence of the asymptotic estimate: that is why a very large value, $\mathcal{N} = 10^5$, was used for these simulations.