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Power-law distributions in noisy dynamical systems

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Abstract – We consider a dynamical system which is non-autonomous, has a stable attractor and which is perturbed by an additive noise. We establish that under some quite typical conditions, the intermittent fluctuations from the attractor have a probability distribution with power-law tails. We show that this results from a stochastic cascade of amplification of fluctuations due to transient periods of instability. The exponent of the power-law is interpreted as a negative fractal dimension, and is explicitly determined, using numerics or perturbation expansion, in the case of a model of colloidal particles in one-dimension.

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Introduction. – Chaotic dynamical systems are characterized by sensitive dependence to initial conditions. In more precise terms, the separation between two very close trajectories, $\Delta\mathbf{x}$, grows essentially exponentially, $|\Delta\mathbf{x}(t)| \sim |\Delta\mathbf{x}(0)| \times \exp(\lambda t)$, where λ is the largest Lyapunov exponent [1]. Chaotic dynamical systems are associated with a *positive* value of λ . In a vast class of dynamical systems, the trajectories sample in the long-time limit a fractal measure in phase space [2], which can be characterized by its correlation dimension, D_2 [3]. The expectation value $\langle \mathcal{N} \rangle$ of the number \mathcal{N} of points in a ball centered around a given trajectory of radius $|\Delta x|$ behaves as $\langle \mathcal{N} \rangle (|\Delta x|) \sim |\Delta x|^{D_2}$ in the small $|\Delta x|$ limit. The probability density function (PDF) for the separation of trajectories is obtained by differentiating this expression, so that in the limit as $|\Delta x| \rightarrow 0$,

$$P_{\Delta x} \sim |\Delta x|^{D_2-1}. \quad (1)$$

(We denote throughout the PDF of a quantity X by P_X .) A positive fractal dimension, $D_2 \geq 0$, implies that $P_{\Delta x}$ is normalizable.

In comparison, in deterministic stable dynamical systems with a negative value of the Lyapunov exponent λ , trajectories approach simple attractors, whose correlation dimension is, in the absence of any noise, $D_2 = 0$. In this case, $P_{\Delta x}$ reduces to a singular δ -function at $\Delta x = 0$.

Even when the Lyapunov exponent is negative, there may be transient episodes when trajectories are divergent, and it has been recognised that this may be associated with various types of intermittency [4–8]. In this work we demonstrate a *universal* consequence of transient divergent behaviour which amplifies the separation between trajectories.

Here, we consider the case of a *stable* dynamical system ($\lambda < 0$) and take into account in the equation of motion the effect of an additive noise, such as thermal noise [9]. We show, using arguments independent of any particular model, that the PDFs of the fluctuations in the separation between trajectories are described by power-law behaviours of the form

$$P_{\Delta x} \sim |\Delta x|^{-(1+\alpha)}. \quad (2)$$

The slow decay of this distribution for large values of Δx implies that there must be relatively frequent large excursions, which are a signature of intermittency, a phenomenon observed in many different contexts, in the physical sciences [10–12], or in physiology [13].

The value of the exponent α characterizes the dynamical system itself, and is independent of the amplitude of the noise. We explicitly show how the existence of this exponent can be determined from theoretical considerations, without reference to any specific model, and explain

that there are universal aspects to these fluctuations which arise because the mechanism for producing the large fluctuations is independent of the mechanism which seeds them. Note that the main difference with the case of a chaotic dynamical system is that the value of α is now *positive* which implies that eq. (2) is not normalizable because of the divergence at small Δx . The amplitude of the noise controls the small-scale cutoff of the distribution of $|\Delta x|$.

Our results are generic and the phenomena we are discussing here will arise in dynamical systems which are i) *non-autonomous*, and ii) *stable*, in the sense that in the absence of noise, solutions converge to a point attractor. It is the interaction between the noise and the fluctuating environment which generates intermittency, via a stochastic mechanism of amplification. As an example, we investigate these properties by using a simple physical model of colloidal particles in a turbulent fluid flow [14].

We conclude our introduction by commenting on the relation between our work and some earlier studies. It has been noticed that the fluctuations in the local rate of separation between trajectories may lead to strong, intermittent bursts in the separation between trajectories [9,15], but none of these earlier works addressed the power-law distribution of separations. Solutions containing power-law were obtained by analysing specific models for stochastic dynamics with *multiplicative* noise [16,17]. In these models, the exponent does depend upon the amplitude of the noise. Our own approach provides a general framework to explain the occurrence of power-law tails in a wide class of models with additive noise. The mechanism presented here is *not* included in the recent review [18] of the known mechanisms for producing power laws in statistical physics. We also show how to determine quantitatively the exponents describing the power-law tails, for a model of separations of particles in a colloidal suspension.

Investigation of a one-dimensional model. – In one spatial dimension we consider equations of motion in the form

$$\dot{x} = v(x, t) + \sqrt{2D}\eta(t), \quad (3)$$

where $\eta(t)$ is a white-noise signal with statistics

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = \delta(t - t') \quad (4)$$

and D is the diffusion coefficient of the corresponding Brownian motion, which we assume to be small ($\langle X \rangle$ denotes the expectation value of X). In (3), the velocity field, $v(x, t)$, satisfies its own equation of motion, and fluctuates in space and time (a specific example of such a model is described by eqs. (12), which are analysed below).

In the deterministic case ($D = 0$) the system is characterised by its Lyapunov exponent λ . If $\lambda < 0$, the system is stable, so all the trajectories of the system converge to an attractor [14] when $D = 0$. In the presence of noise ($D \neq 0$), however, the trajectories do not remain on the attractor of a stable system. As a result, the separation of

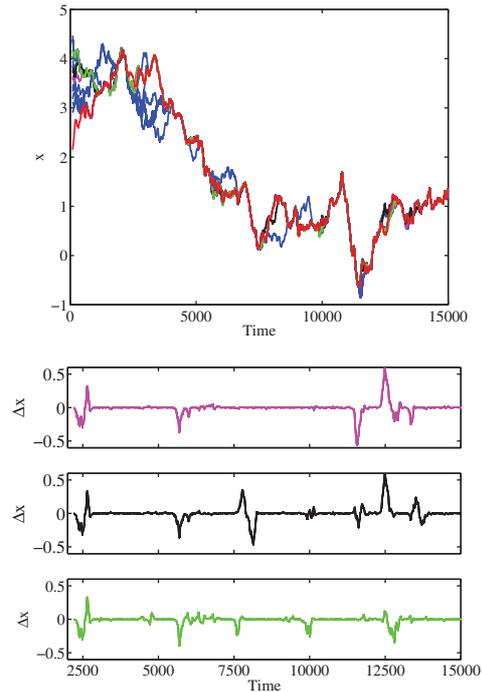


Fig. 1: (Color online) (a) Set of trajectories of a model of colloidal particles in suspension, eqs. (12). Different trajectories separate and recombine. (b) Intermittent separation of pairs of trajectories $\Delta x(t)$. The parameters are $\gamma = 0.05$, $\ell_c = 0.08$, $A = 0.02$, $D = 10^{-8}$.

two trajectories, denoted as Δx , can have large excursions away from zero [15], leading to fat tails of the PDF $P_{\Delta x}$.

One may have expected the PDF $P_{\Delta x}$ to be well approximated by a Gaussian distribution. This is because, in the case of a stable autonomous system, the motion in the vicinity of the attractor satisfies $\dot{x} = \lambda x + \sqrt{2D}\eta(t)$ with $\lambda < 0$, which is an Ornstein-Uhlenbeck (OU) process [19]. In this case the deviations from the attractors do have a Gaussian distribution, with a variance $D/|\lambda|$, so the deviations Δx between two trajectories also have a Gaussian PDF, with a variance $2D/|\lambda|$. We demonstrate here the existence of a generic class of models for which the distribution $P_{\Delta x}$ is *non-Gaussian* and has power-law tails of Δx described, when $\Delta x \rightarrow 0$, by (2).

The large excursions of $\Delta x(t)$ described by the power-law tails are a manifestation of the phenomenon of intermittency, whereby close-by trajectories are separated by the combined effect of the noise and fluctuating environment. This is illustrated in fig. 1, which shows individual trajectories and their differences for a simple model discussed in detail below (see eqs. (12) and (13)). Figure 2 shows the corresponding power-law distribution of Δx .

Beyond the point at which the underlying dynamical system becomes unstable (λ becomes positive), the system has a strange attractor, where phase points cluster on a fractal measure [1], and $P_{\Delta x}$ is described by (1), with $D_2 \geq 0$. As λ approaches zero from below, the exponent

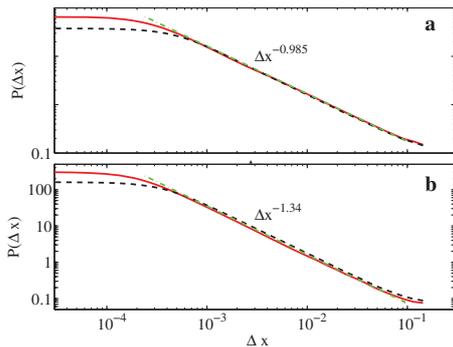


Fig. 2: (Color online) The probability distribution $P_{\Delta x}$ for the model (12), for two values of D ($D = 3 \times 10^{-10}$, full curve, and $D = 10^{-9}$, dashed curves), showing fits by a power law $|\Delta x|^{-(\alpha+1)}$. The parameters are (a) $\gamma = 0.0375$, $\ell_c = 0.08$, $A = 0.02$ ($\lambda > 0$), and (b) $\gamma = 0.075$, $\ell_c = 0.08$, $A = 0.02$ ($\lambda < 0$).

α in eq. (2) approaches zero from above. When $\lambda > 0$, the two expressions (1) and (2) are identical, so

$$\alpha = -D_2. \quad (5)$$

As a consequence, a normalisable distribution of fluctuations correspond to positive values of D_2 . Equation (2) therefore gives a physical meaning to a negative fractal dimension, but we should emphasise the difference between the interpretation of the two exponents. Equation (1) when $D_2 \geq 0$ corresponds to a system with no added noise, whereas eq. (2) describes the tail of the PDF for separations when a small noise signal is added. The formal relation between our exponent α and the correlation dimension D_2 allows us to use recent advances in computing the correlation dimension [20–22] to quantify our exponent α . An alternative definition of negative fractal dimension has been offered in [23].

General theory. – We now explain the power-law tails by analysing the separation between two nearby trajectories, namely x_1 and $x_2 = x_1 + \delta x$. To this end, we linearize eq. (3), and considering two independent noise terms, we arrive at

$$\delta \dot{x} = Z(t)\delta x + 2\sqrt{D}\eta(t), \quad (6)$$

where

$$Z(t) = \frac{\partial v}{\partial x}(x(t), t). \quad (7)$$

Note that when $D = 0$, eq. (6) implies that $Z(t)$ is the logarithmic derivative of the separation $\delta x(t)$ with respect to time, and that its expectation value is the Lyapunov exponent:

$$Z(t) = \frac{\delta \dot{x}}{\delta x}, \quad \lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t dt' Z(t') = \langle Z(t) \rangle. \quad (8)$$

The instantaneous rate of separation between trajectories, $Z(t)$, can be thought of as the *instantaneous Lyapunov exponent*. In the case of autonomous systems with an attractor, the attractor must be a fixed point in phase

space, and $Z(t)$ approaches a constant $\lambda < 0$ as $t \rightarrow \infty$. In this case the fluctuations are described by an OU process and the distribution $P_{\Delta x}$ is Gaussian. In cases where the dynamical system is non-autonomous, $Z(t)$ need not approach a constant value. If the external driving is a stationary stochastic process, $Z(t)$ is a fluctuating quantity with stationary statistics. The origin of the power-law tails described by (2) is that the fluctuations are amplified during periods when $Z(t) > 0$. This noise amplification occurs via a process of stochastic amplification. The fluctuating quantity $Z(t)$ acts multiplicatively in eq. (6). The amplification during each period where $Z(t) > 0$ is independent of the initial amplitude, thus enabling large amplitude fluctuations to build up by a succession of periods where $Z(t) > 0$. The large values of $|\Delta x|$, responsible for the power-law distribution (2) arise whenever $Z(t)$ is positive for some intervals of time, however short.

In order to quantify this picture, let us consider the dynamics of the fluctuations in a logarithmic variable

$$Y = \ln(\Delta x). \quad (9)$$

Consider the tail of $P_{\Delta x}$, where the fluctuations are much larger than the driving noise, so that the term $2\sqrt{D}\eta(t)$ in (6) can be neglected. In this limit the equation of motion for $Y(t)$ is simply

$$\dot{Y} = Z(t). \quad (10)$$

Because the fluctuations of $Z(t)$ are independent of Y , the PDF of Y is expected to be invariant under a translation of the form: $Y \rightarrow Y + \delta Y$. Thus, the only possible functional form for P_Y , the PDF of Y , is

$$P_Y \sim \exp(-\alpha Y). \quad (11)$$

Therefore, by using the above functional form of eq. (11) along with the change of variables $dP = P_Y dY = P_{\Delta x} d\Delta x$, we conclude that the PDF $P_{\Delta x}$ is described by eq. (2). Note that when $Y \rightarrow -\infty$, or alternatively, $\Delta x \rightarrow 0$, the noise term dominates in eq. (6), so eq. (2) does not apply. This prevents any difficulty with the divergence of P_Y . In fact, the role of the noise term reduces to providing a natural cutoff at small separations ($\Delta x \rightarrow 0$), thus preventing potential normalisation problems. We stress that the exponent α in eq. (11) and eq. (2) is independent of D , provided that $D > 0$. This is a consequence of the fact that α depends only on the statistics of $Z(t)$.

Colloidal particle model. – Next we describe a concrete and physically important example of a system that produces fluctuations described by (2). This is provided by *colloidal particles in a turbulent fluid flow*, with velocity field $\mathbf{u}(\mathbf{x}(t), t)$. The motion of small particles suspended in the flow is determined by viscous drag, which makes their velocity $\mathbf{v}(t)$ relax to that of the surrounding fluid. When the particles have a density which is much higher than that of the fluid in which they are dispersed, the viscous

drag is proportional to the difference between the particle velocity and the fluid velocity at its current location (limitations of the model are discussed in [24,25]). The problem studied in this paper is based upon a one-dimensional version of this model, which includes Brownian motion of the particles:

$$\begin{aligned}\dot{x} &= v + \sqrt{2D}\eta(t), \\ \dot{v} &= \gamma[u(x,t) - v],\end{aligned}\quad (12)$$

where γ is proportional to the viscosity of the fluid. We consider a random velocity field with a vanishingly small correlation time, with statistics given by

$$\begin{aligned}\langle u(x,t) \rangle &= 0, \\ \langle u(x,t)u(x',t') \rangle &= A^2 \exp\left[-\frac{(x-x')^2}{\ell_c^2}\right] \delta(t-t').\end{aligned}\quad (13)$$

All of the explicit work shown in this paper is based on eqs. (12) and (13). This model has two very different random elements. The random velocity field $u(x,t)$ is the same for all trajectories, whereas the Brownian noise $\eta(t)$ has a different realisation for each particle trajectory. This model has been extensively analysed for $D = 0$. The Lyapunov exponent, obtained in [14], was found to be negative for sufficiently large values of γ , with the attractor not being a fixed point, but a random walk. The separation of trajectories was analysed in [26] and the correlation dimension was investigated in [21] for the case in which $D_2 \geq 0$. When $D \neq 0$, we find that the deviations of the trajectories exhibit a power-law tail in their PDF, described by eq. (2), see fig. 2.

Consider the linearisation of eqs. (12):

$$\begin{aligned}\delta\dot{x} &= \delta v + 2\sqrt{D}\eta(t), \\ \delta\dot{v} &= \gamma[S(t)\delta x - \delta v],\end{aligned}\quad (14)$$

where $S(t) = \frac{\partial u}{\partial x}(x(t), t)$ is the velocity gradient at the position of a particle, which can be modelled by a white-noise signal with diffusion coefficient \mathcal{D} :

$$S(t) = \sqrt{2\mathcal{D}}\zeta(t), \quad \mathcal{D} = \frac{A^2}{\ell_c^2}, \quad (15)$$

where $\zeta(t)$ is independent of $\eta(t)$ but has the same statistical properties.

When $D = 0$, from eqs. (14) and (15) we obtain the following stochastic differential equation of motion for $Z(t)$ (previously obtained in [14]):

$$\dot{Z} = -\gamma Z - Z^2 + \sqrt{2\mathcal{D}}\gamma\zeta(t). \quad (16)$$

We wish to use this equation, together with (10), to determine the PDF for Y in the form (11). From eqs. (16) and (10) we can construct a Fokker-Planck equation for the joint probability density $\rho(Y, Z, t)$ of Y and Z . This is

$$\frac{\partial \rho}{\partial t} = -Z \frac{\partial \rho}{\partial Y} + \frac{\partial}{\partial Z} [(\gamma Z + Z^2)\rho] + \gamma^2 \mathcal{D} \frac{\partial^2 \rho}{\partial Z^2}. \quad (17)$$

We seek a solution in the form $\rho(Y, Z) = \exp(-\alpha Y)\rho_Z(Z)$ for consistency with (11). This leads to a differential equation in the variable Z , with α as a parameter, of the form

$$\hat{F}\rho_Z(Z) + \alpha Z\rho_Z(Z) = 0, \quad (18)$$

where \hat{F} is a differential operator defined by writing

$$\hat{F}\rho(Z) = \frac{\partial}{\partial Z} \left[(\gamma Z + Z^2) + \mathcal{D}\gamma^2 \frac{\partial}{\partial Z} \right] \rho(Z). \quad (19)$$

Because $\partial/\partial Z$ is a left-factor of \hat{F} , any normalisable solution of (18) with $\alpha \neq 0$ must satisfy

$$\int_{-\infty}^{\infty} dZ Z \rho_Z(Z) = 0. \quad (20)$$

Equation (20) provides a constraint that allows us to determine α . The relation (20) must be imposed on any approximate solutions of (18) constructed by perturbation theory. We remark that the integral in (20) is distinct from the Lyapunov exponent, because $\rho_Z(Z)$ is a distribution of Z which is conditional upon the value of Y .

In order to facilitate the analysis of (18), we replace Z by a scaled variable x , and introduce a dimensionless parameter ε :

$$x(t) = \frac{1}{\gamma} \sqrt{\frac{\gamma}{\mathcal{D}}} Z(t), \quad \varepsilon = \sqrt{\frac{\mathcal{D}}{\gamma}}. \quad (21)$$

With these definitions, eq. (18) is replaced by an equation containing a single dimensionless parameter, ε :

$$\frac{\partial}{\partial x} \left[x + \varepsilon x^2 + \frac{\partial}{\partial x} \right] \rho(x) + \alpha \varepsilon x \rho(x) = 0. \quad (22)$$

From (20) it follows that this equation is to be solved subject to the condition that $\langle x \rangle = 0$. Our numerical results indicate that the exponent α , determined from $P_{\Delta x}$, obtained from direct simulation of the colloidal model (12), is a function of the scaling variable ε . The results shown in fig. 3 demonstrate that the exponent α varies smoothly as a function of ε , from positive values (representing dispersion induced by noise) to negative values (which describe fractal clustering, even in the absence of noise).

Perturbation theory. – We consider two different perturbative approaches of (22) in order to determine α as a function of ε . First we consider an expansion about $\varepsilon = 0$, where we identify the general term. Then we also consider the leading term of an expansion about the critical point, ε_c , where α changes sign.

First consider the perturbation expansion in powers of ε . We explicitly expand both $\rho(x)$ and α in powers of ε :

$$\alpha = \sum_{n=0}^{\infty} \alpha_n \varepsilon^n, \quad \rho(x) = \sum_{n=0}^{\infty} \rho_n(x) \varepsilon^n. \quad (23)$$

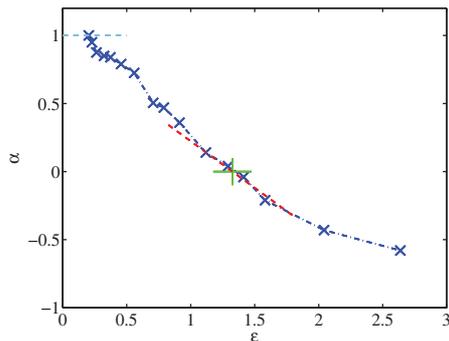


Fig. 3: (Color online) Exponent α defined by (2) as a function of the dimensionless parameter ε defined by (21). Note that $\alpha \rightarrow 1$ as $\varepsilon \rightarrow 0$ and that in the vicinity of the critical point ε_c where $\lambda = 0$, we have $\alpha \sim K(\varepsilon_c - \varepsilon)$. The dashed line at $\alpha = 1$ represents the result of the series expansion about $\varepsilon = 0$, the line where α passes through zero uses our theoretical value for the coefficient K and the cross shows the critical point location.

This can be done by following the method discussed in [20], expanding each of the functions $\rho_n(x)$ in terms of an orthogonal basis set:

$$\rho_n(x) = \sum_{k=0}^{\infty} A_{nk} \phi_k(x). \quad (24)$$

Here the basis functions $\phi_k(x)$, $k = 0, 1, 2, \dots$ are un-normalised harmonic-oscillator states, generated by raising and lowering operators:

$$\hat{a} = -\partial_x, \quad \hat{b} = \partial_x + x \quad (25)$$

with

$$\hat{a}\phi_k(x) = \phi_{k+1}(x), \quad \hat{b}\phi_k(x) = k\phi_{k-1}(x). \quad (26)$$

By substituting (23) into (22) and using (25), the functions $\rho_n(x)$ are found to satisfy the recursion

$$-\hat{a}\hat{b}\rho_n(x) + \alpha_{n-1}\phi_1(x) + Q_n(x) = 0, \quad (27)$$

where

$$Q_n(x) = \sum_{k=0}^{\infty} Q_{nk} \phi_k(x) \quad (28)$$

is explicitly expressed in terms of the functions $\rho_j(x)$ and coefficients α_j obtained at lower orders ($j < n$). This recursion must be solved so that (20) is satisfied for all values of ε . Because

$$\int_{-\infty}^{\infty} dx x \phi_k(x) = \delta_{k1} \int_{-\infty}^{\infty} dx \phi_0(x) \quad (29)$$

this is achieved by requiring that the coefficients in (24) satisfy $A_{n1} = 0$ for all n . This requires that we set $\alpha_{n-1} = -Q_{n1}$ at each iteration of (27).

Starting the iteration with $\rho_0(x) = \phi_0(x)$ yields $\alpha_0 = 1$, and then $\rho_1(x) = \frac{1}{3}\phi_3(x)$, $\alpha_1 = 0$. For $n > 1$, the

expression for the function $Q_n(x)$ is then given by

$$Q_n(x) = \sum_{j=1}^{n-2} \alpha_j (\hat{a} + \hat{b}) \rho_{n-j-1}(x) + \left[\hat{b} - \hat{a}\hat{b}^2 - \hat{a}^3 - 2\hat{a}^2\hat{b} \right] \rho_{n-1}(x). \quad (30)$$

Solving recursively the system of perturbation equations (27) is then found to yield $\alpha_0 = 1$, and $\alpha_j = 0$ for all $j > 0$. The fact the coefficients α_j are all zero for $j > 0$ can be justified by an elementary inductive argument. Assume that $\alpha_j = 0$ up to order $j = n - 2$. Then all of the terms in the summation on the first line of (30) are zero. The coefficient Q_{n1} results from the application of the operator $\hat{b} - \hat{a}\hat{b}^2$ to the component $A_{n-1,2}\phi_2(x)$ of $\rho_{n-1}(x)$, and this is automatically zero, which ensures that $\alpha_{n-1} = 0$.

An equivalent result was previously noted by Mehlig and Gustavsson [27], who analysed the correlation dimension of the system (12), (13) using the method in [20], without giving a meaning to the negative value of D_2 which emerges.

The implication of this analysis is not that $\alpha = 1$ exactly, but rather that α has a non-analytic dependence upon ε , such as $\alpha \sim 1 - c \exp(-S/\varepsilon^2)$ (where c and S are constants). Our numerical data (shown in fig. 3) are consistent with this type of asymptotic behaviour, but they are not yet sufficiently precise to determine the non-analytic term reliably.

An alternative approach is to make a perturbative expansion about the critical point where the Lyapunov exponent changes sign. For the model underlying (12), this occurs at $\varepsilon_c \approx 1.33 \dots$. We follow an approach used in [21] (see also [28]). To leading order in $\varepsilon - \varepsilon_c$ we obtain

$$\alpha = K(\varepsilon_c - \varepsilon). \quad (31)$$

Using the approach described in [21] and [28] the coefficient K is expressed in terms of finite-dimensional integrals, whose numerical evaluation leads to $K \approx 0.688$. This value is found to be in good agreement with the results shown in fig. 3.

Conclusions. – To conclude, we have explained and characterised a class of intermittent fluctuations in dynamical systems, leading to power-law distributions between very close trajectories. They arise in a wide class of systems which are non-autonomous, stable (*i.e.* have negative Lyapunov exponents), and in the presence of additive noise. Using symmetry arguments, and the observation that the instantaneous Lyapunov exponent has positive fluctuations, allows us to justify these power-law distributions. Our results are consistent with those in [16,17], obtained by solving a quite specific model with a multiplicative noise. We have also discussed perturbative methods for estimating the exponent α . The mechanism of stochastic amplification for producing these fluctuations

is generic, so that they should be observable in a wide class of systems.

In the case of higher-dimensional systems, the separation of two trajectories can still be represented by an equation of form similar to (6), with the variable $Z(t)$ replaced by two or more variables with coupled equations of motion: see [20] for an example. Equation (2) remains valid in higher-dimensional cases.

* * *

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