AIP Physics of Fluids

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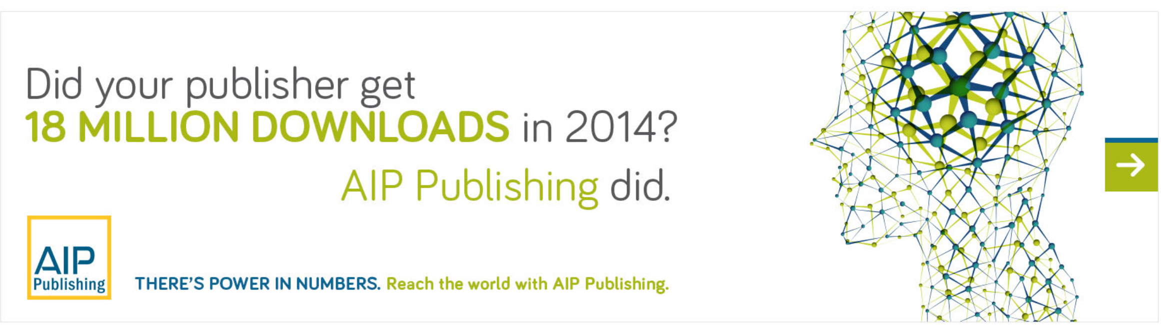
Citation: Physics of Fluids **27**, 073305 (2015); doi: 10.1063/1.4927220
View online: http://dx.doi.org/10.1063/1.4927220
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof2/27/7?ver=pdfcov
Published by the AIP Publishing





PHYSICS OF FLUIDS **27**, 073305 (2015)

# Analysis of the correlation dimension for inertial particles

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(Received 19 February 2015; accepted 9 July 2015; published online 28 July 2015)

We obtain an implicit equation for the correlation dimension which describes clustering of inertial particles in a complex flow onto a fractal measure. Our general equation involves a propagator of a nonlinear stochastic process in which the velocity gradient of the fluid appears as additive noise. When the long-time limit of the propagator is considered our equation reduces to an existing large-deviation formalism from which it is difficult to extract concrete results. In the short-time limit, however, our equation reduces to a solvability condition on a partial differential equation. In the case where the inertial particles are much denser than the fluid, we show how this approach leads to a perturbative expansion of the correlation dimension, for which the coefficients can be obtained exactly and in principle to any order. We derive the perturbation series for the correlation dimension of inertial particles suspended in three-dimensional spatially smooth random flows with white-noise time correlations, obtaining the first 33 non-zero coefficients exactly. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4927220]

## I. INTRODUCTION

In aerosols and other suspensions of microscopic bodies, it may be satisfactory to neglect hydrodynamic interactions and to assume that the particles move independently. It is known that small particles moving independently in an incompressible turbulent or complex flow may show a pronounced tendency to cluster. This occurs if the time scale for viscous damping, $\tau_{\rm p}$, is comparable to the smallest characteristic time scale for fluctuations in the flow, $\tau$. Maxey[1] proposed that these "inertial particles" cluster because they are expelled from vortices by the centrifugal effect (if they are denser than the fluid in which they are suspended, bubbles are expected to congregate in vortices). Later, Sommerer and Ott[2] showed that, in common with other chaotic dynamical processes, the trajectories of particles advected on compressible surface flows approach a fractal attractor (Ott[3] gives a good introduction to the role of fractals in dynamical systems). Numerical experiments by Bec[4] confirmed that fractal clustering is observed for inertial particles in incompressible flows, just as in the compressible surface flows considered in Ref. 2.

This clustering is of fundamental importance to understanding the effect of turbulence on aerosols, because of its potential relevance to the coalescence of cloud droplets into rain,[5] or of dust grains into planetary precursors.[6]

The clustering process and its fractal dimension have been investigated numerically in many works: Refs. 7 and 8 report state-of-the-art contributions. The present theoretical understanding of this effect is reviewed in Ref. 9.

The present paper is concerned with the analysis of the correlation dimension, $D_2$, which is the most important dimension in physical applications but which is still quite poorly understood. The importance of the correlation dimension arises from its direct relation to the two-point correlation function of particles (given as Equation (5) below), which enters in theories for collision processes[10,11] and light scattering.[12]

Our approach gives an implicit equation for the correlation dimension, in terms of a propagator for a nonlinear stochastic process in which components of the velocity gradient tensor appear as additive noise. In the limit as the propagation time approaches zero, our equation becomes a solvability condition for a linear partial differential equation. We analyse this system using perturbation theory. We obtain a series expansion for the correlation dimension of the particle distribution in powers of a dimensionless parameter which measures the importance of inertial effects.

We briefly review the state of the theoretical knowledge concerning the clustering of particles. Maxey's original work[1] proposed that the particles (which we assume are much denser than the fluid) are expelled by centrifugal forces from vortices in the fluid, but that this effect can only be effective when the motion of the particles relative to the fluid is neither too lightly damped nor too heavily damped. The damping is characterised by a dimensionless number termed the Stokes number, defined by

$$\mathrm{St} = \frac{1}{\gamma\tau}, \tag{1}$$

where $\tau$ is a characteristic time scale of the fluid flow, and where $\gamma = 1/\tau_{\mathrm{p}}$ is the rate constant for damping the motion of the particles relative to the fluid. He also showed that, when inertial effects are weak, the particle velocity may be approximated by an effective velocity field which has a compressible component. Simulations do show that particles have lower density in regions of high vorticity, see Ref. 13.

The particle distribution has clustering properties which are much more significant than the instantaneous negative correlation between density and vorticity. The particles approach a fractal measure. This can be characterised in a variety of ways, but the approach which is most easily understood and most fundamental to physical applications is to consider the number of particles $\mathcal{N}$ inside a ball of radius $\delta r$ centred on a randomly selected test particle. For sufficiently small values of $\delta r$, the average of this quantity has a power-law dependence upon $\delta r$ with exponent denoted by $D_2$,

$$\langle \mathcal{N}(\delta r)\rangle \sim \delta r^{D_2}. \tag{2}$$

Throughout this paper the expectation value of $X$ is denoted by $\langle X\rangle$. The exponent $D_2$ is termed the correlation dimension of the particle distribution.[3] The fractal dimension of particle clusters has been investigated numerically, and it has been confirmed that the fractal dimension in turbulent flows is significantly less than the space dimension only when the Stokes number is of order unity (see, for example, Fig. 2 in Ref. 14, and Fig. 1 in Ref. 8).

However, a theoretical analysis leading to quantitative results concerning the dependence of the dimension $D_2$ upon the Stokes number is lacking. There are a few works in which analytical results on the correlation dimension have been obtained. Most of the literature has discussed the relation between the Renyi dimensions and the statistics of the finite-time Lyapunov exponent: these relationships were established by Grassberger and Procaccia[15] (see also Ref. 16) and are reviewed in the book by Ott.[3] Usually the finite-time Lyapunov exponent can only be investigated numerically, but its statistics can be obtained for the Kraichnan model[17] in which a particle is advected in a velocity field with white-noise temporal correlations: Falkovich *et al.*[18] discussed the calculation of the Renyi dimensions for the Kraichnan model. The first analytical studies on the correlation dimension for inertial particles were made by Bec *et al.*,[19] who considered a velocity field which has white-noise temporal correlations. Their method yields the first two terms of the series expansion of $D_2$, but it seems to be very difficult to extend to higher orders (and the second-order coefficient in Ref. 19 appears to be incorrect). In Ref. 20, we described a new method which related the correlation dimension to the solution of a partial differential equation. It was shown that the series expansion of the solution to this equation can be automated, so that coefficients of arbitrary order are obtained by repeated application of a system of annihilation and creation operators. In this way, the coefficients in a series expansion of the correlation dimension of inertial particles in two-dimensional random flows were obtained. Gustavsson and Mehlig[21] used a different technique to compute the correlation dimension for a random-flow model in one dimension where the correlation dimension could be treated as a small parameter. In a series of papers, Zaichik and Alipchenkov[22–24] developed an approach to calculating the clustering and collision rates of particles in a turbulent flow which combines empirical data on turbulence, with a stability analysis of the dispersion of particles.

In this paper, we describe a general principle (Section II) for calculating the correlation dimension, based on the invariance of the distribution of small separations under dilations, corresponding to translations in logarithmic variables. In Section III, we show how this principle can be expressed in terms of a time-propagator. We show that a large-time expansion of the propagator gives a set of equations closely related to equations derived from a large-deviation principle — discussed in Refs. 15, 16, and 3. We also show how an approximate expression for $D_2$ can be recovered from the large deviation formalism, but it is difficult to extend this because of the intractability of determining the entropy function of the large deviations of the Lyapunov exponent. A short-time expansion of the propagator, by contrast, yields a partial differential equation involving $D_2$ which is more amenable to analysis. This approach was previously outlined in Ref. 20. Here, in Section IV, we apply the method to a white-noise random-flow model in three spatial dimensions, developing a perturbation theory for $D_2$ in Section V. Because the correlation time of the flow is $\tau = 0$ for our model flow, the Stokes number is not defined for our model. However, our perturbation parameter, $\epsilon$, plays a role which is analogous to St. The relation between $\epsilon$ and St is discussed carefully in Ref. 25, where it is argued that $\epsilon^2 \propto \mathrm{St}$. The perturbation series is divergent and the methods used to extract finite results are discussed in Section VI. Section VII contains our conclusions and discusses possible extensions of this work.

## II. THE CORRELATION DIMENSION

In Secs. II and III, we define the correlation dimension and discuss several distinct but interconnected approaches to calculating it. What these approaches have in common is that they use a dynamical variable, $Z_1(t)$, which is derived from the linearised equation of motion. The statistics of $Z_1(t)$ are also closely related to the leading Lyapunov exponent. Several different probability density functions (PDFs) must be introduced. We denote the PDF of a quantity $X$ by a function $\rho_X$, so that the probability element for $X$ to lie in the interval $[X, X + \mathrm{d}X]$ is $\mathrm{d}P = \rho_X(X)\mathrm{d}X$. The expectation value of $X$ is denoted by $\langle X \rangle$.

The correlation dimension $D_2$ is defined in terms of the expected number $\langle \mathcal{N}(\delta r)\rangle$ of particles inside a ball of radius $\delta r$ surrounding a test particle,

$$D_2 = \lim_{\delta r \to 0} \frac{\ln\langle \mathcal{N}(\delta r)\rangle}{\ln(\delta r)}, \tag{3}$$

so that

$$\langle \mathcal{N}(\delta r)\rangle \sim \delta r^{D_2}, \tag{4}$$

which is the volume element of a ball in $D_2$ dimensions. If $D_2 = d$ (where $d$ is the dimensionality of space), there is no clustering. The probability density $\rho(\delta r)$ for a particle to have another particle at small distance $\delta r$ is

$$\rho(\delta r) = \frac{\mathrm{d}\langle \mathcal{N}(\delta r)\rangle}{\mathrm{d}\delta r} \sim \delta r^{D_2 - 1}. \tag{5}$$

Note that this quantity is the "two-point correlation function" which plays an important role in physical kinetics[10,11] and scattering theory.[12]

### A. Logarithmic separation dynamics

It is not immediately clear why the limit in Equation (3) should exist. In this paper, we show why it does, and how to extract information about $D_2$ by considering a quantity $Z_1(t)$ defined by

$$\frac{\delta\dot{r}}{\delta r} = Z_1. \tag{6}$$

Here, $\delta\dot{r}$ denotes the time derivative of $\delta r$, and $Z_1$ is the logarithmic derivative of $\delta r$. We also consider the variable

$$Y(t) = \ln \delta r(t). \tag{7}$$

The two variables $Y$ and $Z_1$ are related by

$$Y(t) = Y(0) + \int_0^t \mathrm{d}t'\, Z_1(t'). \tag{8}$$

We will argue that, in the limit as $Y(t) \to -\infty$, the variable $Z_1$ obeys an equation of motion which is independent of $Y$. This implies translational invariance in the statistics of $Z_1$. Correspondingly, the PDF $\rho_Y(Y)$ of $Y$ exhibits translational invariance: $\rho_Y(Y)$ and $\rho_Y(Y - Y_0)$ must be the same function, up to a normalisation factor, for any choice of the displacement $Y_0$. Hence,

$$\rho_Y(Y) = C(Y_0)\rho_Y(Y - Y_0) \tag{9}$$

for some choice of $C(Y_0)$. The solution of this equation is

$$\rho_Y(Y) = A \exp(\alpha Y) \tag{10}$$

for some constant $\alpha$ and normalisation $A$. This expression is valid only for $Y \to -\infty$, so that we must require $\alpha > 0$ to give a normalisable probability density. Consider the corresponding PDF of $\delta r$, denoted by $\rho_{\delta r}(\delta r)$: the element of probability is $\mathrm{d}P = \rho_{\delta r}(\delta r)\,\mathrm{d}\delta r = \rho_Y(Y)\,\mathrm{d}Y = A\delta r^{\alpha-1}\mathrm{d}\delta r$, so that the distribution of $\delta r$ corresponding to (10) is

$$\rho_{\delta r}(\delta r) = A\,\delta r^{\alpha-1}. \tag{11}$$

By comparison with (5), it follows that the exponent of the distribution of $Y$ and the correlation dimension are equal,

$$D_2 = \alpha. \tag{12}$$

Thus, we conclude that $D_2$ can be determined by studying the statistics of the logarithmic derivative $Z_1 = \delta\dot{r}/\delta r$. Specifically, if $Z_1(t)$ is a random variable with statistics that become independent of $Y$ as $Y \to -\infty$, then the distribution of $Y$ is $\rho_Y(Y) \sim \exp(D_2 Y)$. So, to determine $D_2$, we need to study the equation of motion for $Z_1(t)$ and how the statistics of $Z_1$ determine the exponent $\alpha$.

Before going on to consider the equation of motion for $Z_1$, we remark that the variable $Z_1(t)$ also gives information about the leading Lyapunov exponent $\lambda$: provided the separations remain sufficiently small, we have

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\left\langle \ln\left(\frac{\delta r(t)}{\delta r(0)}\right)\right\rangle. \tag{13}$$

We can express this in terms of a limit of a finite-time Lyapunov exponent $\sigma(t)$,

$$\sigma(t) \equiv \frac{1}{t}\left\langle \ln\left(\frac{\delta r(t)}{\delta r(0)}\right)\right\rangle = \frac{1}{t}\int_0^t \mathrm{d}t'\, Z_1(t'). \tag{14}$$

The leading Lyapunov exponent is therefore an expectation value of $Z_1(t)$,

$$\lambda = \lim_{t\to\infty} \sigma(t) = \langle Z_1(t)\rangle. \tag{15}$$

### B. Equation of motion for the logarithmic derivative

We have shown that information about $D_2$ is contained in the dynamics of the logarithmic derivative of the separation, $Z_1(t)$. To proceed further, we need an equation of motion for this quantity. The equations of motion for a small spherical body moving in a viscous fluid are discussed in Refs. 26 and 27. We consider the case where the density of the body is much higher than that of the surrounding fluid. In this limit, the equations of motion for the particle position $\boldsymbol{r}(t)$ and velocity $\boldsymbol{v}(t)$ are

$$\dot{\boldsymbol{r}} = \boldsymbol{v}\,, \quad \dot{\boldsymbol{v}} = \gamma[\boldsymbol{u}(\boldsymbol{r}(t),t) - \boldsymbol{v}]. \tag{16}$$

An equation of motion for $Z_1$ is derived from the linearised equations of motion describing a pair of particles with a separations $\delta\boldsymbol{r}$ and $\delta\boldsymbol{v}$ in position and velocity,

$$\delta\dot{\boldsymbol{r}} = \delta\boldsymbol{v}\,, \quad \delta\dot{\boldsymbol{v}} = -\gamma\delta\boldsymbol{v} + \gamma\mathbb{E}\,\delta\boldsymbol{r}. \tag{17}$$

Here, $\mathbb{E}$ is the matrix of flow-velocity gradients with elements $E_{ij} = \partial u_i/\partial r_j$. From these equations, we must obtain an equation of motion for $Z_1 = \delta\dot{r}/\delta r$, where $\delta r = |\delta\boldsymbol{r}|$. To illustrate the approach in its simplest context, we show how this is done for a one-dimensional model, where $x$ is the coordinate of the particle. In one dimension, we have $\delta r = |\delta x|$, and simple manipulation of Equations (17) gives

$$\dot{Z}_1 = -\gamma Z_1 - Z_1^2 + \gamma E(t), \tag{18}$$

where

$$E(t) = \frac{\partial u}{\partial x}(x(t),t). \tag{19}$$

In two or three dimensions, the variable $Z_1(t)$ is coupled to one or more additional variables, but there are always a finite number of variables, $Z_1, Z_2, \ldots$, which are coupled in a closed system of equations analogous to (18).

The one-dimensional version of Equation (17) allows particles to exchange positions, that is, $\delta x$ passes through zero while $\delta v$ remains finite. This corresponds to a "caustic" singularity[28] where $Y(t)$ goes to $-\infty$ and returns, while $Z_1(t)$ goes to $-\infty$ and returns from $+\infty$. This divergence of $Z_1$ is a special feature of the one-dimensional version of the model and it is absent in higher dimensions. We should nevertheless consider its effect.

The finite-time singularities give rise to a "tail" of the distribution of $Y$. Consider the form of the distribution of $Y$ resulting from a fold event in a one-dimensional system, where one phase point passes another with a finite difference in their velocity. Because the relative velocity has no singularity as one particle passes the other, the PDF of the spatial separation also has no singularity. It may therefore be approximated by a uniform distribution in the vicinity of $\delta x = 0$. The corresponding distribution for $Y$ is obtained by writing the probability element as follows: $\mathrm{d}P = \rho_{\delta x}(\delta x)\mathrm{d}\delta x = \rho_Y(Y)\mathrm{d}Y$. Hence,

$$\rho_Y(Y) \sim \text{const.} \times \frac{\mathrm{d}\delta x}{\mathrm{d}Y} \sim \exp(Y). \tag{20}$$

This contribution is negligible compared to that from the analysis of the differential equation whenever the latter predicts $\alpha < 1$. The contribution from the folding events is therefore smaller than that due to fractal clustering whenever $D_2 < 1$. This condition is never violated in one dimension.[21] In higher dimensions, the equation analogous to (15) does not have finite-time singularities, although there are caustic singularities where volume elements vanish.[28,25]

## III. MARKOVIAN APPROXIMATIONS

We wish to use information about statistics of $Z_1(t)$ to determine $D_2 = \alpha$. The most practicable approach is to use a Markovian assumption, where the future development of a system can be assumed to be independent of its past history. In the present context, we assume that future evolution of $Y(t)$ is determined by its current value, and by the current value of $Z_1$. We therefore consider a joint PDF of $Y$ and of $Z_1$. Given $Y$ and $Z_1$, let $K(\Delta Y, Z_1, Z_1', t)$ be the PDF for $Y$ to increment by $\Delta Y$ and for $Z_1$ to reach $Z_1'$ after time $t$. The joint PDF of $Y$ and $Z_1$ evolves according to

$$\rho_{Y,Z_1}(Y,Z_1,t) = \int_{-\infty}^{\infty} \mathrm{d}\Delta Y \int_{-\infty}^{\infty} \mathrm{d}Z_1' \, K(\Delta Y, Z_1', Z_1, \Delta t)\, \rho_{Y,Z_1}(Y-\Delta Y, Z_1', t-\Delta t). \tag{21}$$

The steady-state probability density is expected to be a product,

$$\rho_{Y,Z_1}(Y,Z_1) = \rho_{Z_1}(Z_1)\exp(\alpha Y), \tag{22}$$

where the distribution $\rho_{Z_1}(Z_1)$ will be discussed shortly. Because Equation (21) is derived by linearisation of the equations of motion, Equation (22) is valid in the limit as $Y \to -\infty$. In order for the distribution to be normalisable, we require that $\rho_{Y,Z_1}$ approaches zero sufficiently rapidly as $Y \to -\infty$, implying that $\alpha > 0$.

Inserting (22) into (21), the steady-state distribution $\rho_{Z_1}(Z_1)$ and the exponent $\alpha$ must satisfy an integral equation

$$\rho_{Z_1}(Z_1) = \int_{-\infty}^{\infty} \mathrm{d}\Delta Y \int_{-\infty}^{\infty} \mathrm{d}Z_1' \, K(\Delta Y, Z_1', Z_1, \Delta t)\, \rho_{Z_1}(Z_1')\exp(-\alpha \Delta Y) \tag{23}$$

which is valid for all $\Delta t$.

Consider the distribution $\rho_{Z_1}(Z_1)$ in (22). It might be expected that this is the same as the distribution of $Z_1(t)$ obtained from Equation (18) or its multi-dimensional generalisation. We term this distribution $\rho_0(Z_1)$. However, the distribution $\rho_{Z_1}(Z_1)$ differs from $\rho_0(Z_1)$ because it is conditioned upon being at a particular value of $Y$.[20] If $\alpha \neq 0$, particles reaching a negative value of $Z_1$ have recently arrived from a larger value of $Y$, where the probability density is larger. This implies that the distributions are different, and moreover that the distribution $\rho_{Z_1}(Z_1)$ has a smaller mean value than $\rho_0(Z_1)$.

Now consider the application of this equation in two limiting cases.

### A. Short propagation time

Consider the limit $\Delta t \to 0$ in (23). In this limit the structure of the propagator can be simplified, because $\Delta Y = Z\Delta t + O(\Delta t^2)$. This implies that one of the integrals can be eliminated from (23), and we may write

$$\rho_{Z_1}(Z_1) = \int_{-\infty}^{\infty} \mathrm{d}Z_1' \, \mathcal{U}(Z_1', Z_1, \Delta t)\, \rho_{Z_1}(Z_1')\exp(-\alpha Z_1' \Delta t), \tag{24}$$

where $\mathcal{U}(Z_1', Z_1, \Delta t)$ is the propagator for the random process $Z_1(t)$ with equation of motion (18) (or its higher-dimensional generalisation) to reach $Z_1$ from $Z_1'$ in time $\Delta t$.

If a Markovian approximation is valid in the limit $\Delta t \to 0$, we have continuous-time Markov process for (23), and the probability density $\rho_{Z_1}(Z_1,t)$ obeys a Fokker-Planck equation,[29] where the evolution kernel $\mathcal{U}(Z_1', Z_1, t)$ is generated by a Fokker-Planck operator $\hat{\mathcal{F}}$,

$$\frac{\partial \rho_{Z_1}}{\partial t} = \hat{\mathcal{F}} \rho_{Z_1}. \tag{25}$$

We can represent functions as vectors using Dirac notation, so that (25) is notated as follows:

$$\partial_t |\rho_{Z_1}) = \hat{\mathcal{F}}\, |\rho_{Z_1}). \tag{26}$$

For small values of $\Delta t$, the action of the propagator kernel can then be approximated by $\hat{\mathcal{U}}(\Delta t) = \hat{\mathcal{I}} + \hat{\mathcal{F}}\Delta t + O(\Delta t^2)$, where $\hat{\mathcal{I}}$ is an identity operator, that is, for a function $f(Z_1)$ represented by a vector $|f)$, we have

$$\hat{\mathcal{U}}(\Delta t)|f) \equiv \int_{-\infty}^{\infty} \mathrm{d}Z_1' \, \mathcal{U}(Z_1', Z_1, \Delta t) f(Z_1') = |f) + \hat{\mathcal{F}}|f)\Delta t + O(\Delta t^2). \tag{27}$$

For small values of $\Delta t$ Equation (24) then reduces to

$$\rho_{Z_1}(Z_1) = \exp(-\alpha Z_1 \Delta t)\rho_{Z_1}(Z_1) + \Delta t \hat{\mathcal{F}} \rho_{Z_1}(Z_1) + O(\Delta t^2). \tag{28}$$

In the limit as $\Delta t \to 0$, this relation implies the condition

$$[\hat{\mathcal{F}} - \alpha Z_1]\, \rho_{Z_1}(Z_1) = 0 \tag{29}$$

which is a partial differential equation for $\rho_{Z_1}(Z_1)$ and $\alpha$.

At this stage, it is useful to consider a concrete example. The one-dimensional model equation of motion for $Z_1$, Eq. (18), can be regarded as a stochastic differential equation in which the velocity gradient $E(t)$ is a random element. If the correlation time of $E(t)$ is sufficiently small, a Markovian approximation is justified, and $E(t)$ can be replaced by a multiple of a white noise signal, $\eta(t)$, which has the following statistical properties:

$$\langle \eta(t) \rangle = 0\,, \quad \langle \eta(t)\eta(t') \rangle = \delta(t - t'). \tag{30}$$

The equation of motion for $Z_1$ is replaced by

$$\dot{Z}_1 = -\gamma Z_1 - Z_1^2 + \sqrt{2\mathcal{D}}\eta(t), \tag{31}$$

where the diffusion coefficient is

$$\mathcal{D} = \frac{\gamma^2}{2} \int_{-\infty}^{\infty} \mathrm{d}t \, \langle E(t)E(0) \rangle. \tag{32}$$

The Fokker-Planck operator corresponding to the Langevin equation (31) is[29]

$$\hat{\mathcal{F}} = (\gamma Z_1 + Z_1^2)\frac{\partial}{\partial Z_1} + \mathcal{D}\frac{\partial^2}{\partial Z_1^2}, \tag{33}$$

so that for the one-dimensional model Equation (29) reduces to an ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}Z_1}\left[(\gamma Z_1 + Z_1^2)\rho_{Z_1}(Z_1) + \frac{\mathrm{d}\rho_{Z_1}}{\mathrm{d}Z_1}(Z_1)\right] - \alpha\, Z_1 \rho_{Z_1}(Z_1) = 0. \tag{34}$$

We require normalisable solutions $\rho_{Z_1}(Z_1)$, which only exist for particular values of $\alpha$. (Later, we give a prescription leading to a unique series solution of this equation.) Upon integrating over space, and using the fact that $\hat{\mathcal{F}}$ is a divergence, we have

$$\int_{-\infty}^{\infty} \mathrm{d}Z_1 \, Z_1 \, \rho_{Z_1}(Z_1) = \langle Z_1 \rangle = 0. \tag{35}$$

The Equations (29) and (35) together constitute a new and exact method for determining $D_2 = \alpha$, in two steps. First Equation (29) is solved to determine a one-parameter family of solutions. Second, the correct value of $D_2$ is determined by finding the value of $\alpha$ for which the mean value of $Z_1$ is zero.[20]

This approach has the attractive feature that it involves the analysis of differential equations, which are susceptible to many types of mathematical techniques.

### B. Long-time propagation

In the long-time limit, we expect that a Markovian approximation is always valid. In this limit, the propagator is expected to "forget" the initial distribution, so that

$$K(\Delta Y, Z_1', Z_1, \Delta t) = \rho_{Z_1}(Z_1')\, \rho_{\Delta Y}(\Delta Y, \Delta t) \tag{36}$$

independent of $Z_1$, where $\rho_{\Delta Y}(\Delta Y, \Delta t)$ is the probability of a displacement $\Delta Y$ in time $\Delta t$.

We now apply the large-deviation principle[30,31] to the statistics of $\Delta Y$. This principle concerns the statistics of time averages such as the finite-time Lyapunov exponent $\sigma(t) = \Delta Y/t$, Equation (14). It is expected that the tails of the distribution $\rho_{\Delta Y}(\Delta Y)$ satisfy

$$\rho_{\Delta Y}(\Delta Y, t) \sim \exp[-tI(\Delta Y/t)] \tag{37}$$

for some function $I(\sigma)$, which is termed the "entropy function" in the literature on large-deviation theory.

The displacement is $\Delta Y = \sigma t$. Changing the variable of integration in (23) from $\Delta Y$ to $\sigma$, we obtain

$$\rho_{Z_1}(Z_1) = t\int_{-\infty}^{\infty} \mathrm{d}\sigma \int_{-\infty}^{\infty} \mathrm{d}Z_1'\, \rho_{Z_1}(Z_1')\rho_{Z_1}(Z_1)\exp[-t(I(\sigma)+\alpha\sigma)]. \tag{38}$$

Assuming that $\rho_{Z_1}(Z_1)$ is a normalised distribution, this gives

$$1 = t\int_{-\infty}^{\infty} \mathrm{d}\sigma\ \exp[-t(I(\sigma)+\alpha\sigma)]. \tag{39}$$

This integral is an implicit relation between $\alpha$ and the large deviation function $I(\sigma)$. The integral is estimated using the Laplace principle: in the limit as $t \to \infty$, the integral is estimated by determining the value of the integrand at its maximum. The maximum is at position $\sigma^*$ determined by the condition

$$I'(\sigma^*) = -\alpha \tag{40}$$

and the integral is estimated as $t\exp[-t(I(\sigma^*)+\alpha\sigma^*)] \sim 1$, so that

$$I(\sigma^*) + \alpha\sigma^* = 0. \tag{41}$$

These Equations (40) and (41) can, in principle, be solved to determine $\alpha = D_2$. Similar approaches are discussed in Refs. 15, 16, and 3. The difficulty lies in determining the entropy function, $I(\sigma)$.

### C. An approximate expression for $D_2$

Before exploring the applications of Equation (29) in greater depth, we describe an approximate expression for $D_2$, previously discussed in Ref. 32, which is asymptotically correct in the limit as $D_2 \to 0$. Because of its simplicity, it is a natural benchmark against which other approaches can be compared.

We observe that the variable $Y$ has diffusive fluctuations,

$$\langle(\Delta Y(t) - \lambda t)^2\rangle = 2\mathcal{D}_Y \Delta t \tag{42}$$

with diffusion coefficient

$$\mathcal{D}_Y = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\,\left[\langle Z_1(t)Z_1(0)\rangle - \langle Z_1\rangle^2\right]. \tag{43}$$

On time scales which are large compared to the correlation time of $Z_1(t)$, we expect that the probability density $P(Y,t)$ satisfies a Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial Y}(vP) + \frac{\partial^2}{\partial Y^2}(\mathcal{D}_Y P). \tag{44}$$

The drift velocity $v$ and diffusion coefficient $\mathcal{D}_Y$ are defined by the relations

$$v = \frac{\langle \delta Y\rangle}{\delta t}, \quad \mathcal{D}_Y = \frac{\langle \delta Y^2\rangle}{2\delta t}. \tag{45}$$

When $v$ and $\mathcal{D}_Y$ are constant, this equation has an exponential solution,

$$P = A\exp\left(\frac{v}{\mathcal{D}_Y}Y\right). \tag{46}$$

Noting that the drift velocity $v$ is equal to the Lyapunov exponent, $\langle Z_1\rangle = \lambda$, comparison with (10) and (12) implies that

$$D_2 = \alpha = \frac{\lambda}{\mathcal{D}_Y}. \tag{47}$$

This approximation is only valid when $\lambda > 0$, because no normalisable solution can be constructed if $P(Y)$ is diverging as $Y \to -\infty$.

The use of the Fokker-Planck equation is only justified when the gradient of $P(Y,t)$ is sufficiently small. The condition is that $\partial P/\partial Y$ should be small compared to $1/\delta Y_0$, where $\delta Y_0$ is the scale over which $Y$ varies during its correlation time. The condition for the validity of (47) is therefore $\langle Z_1\rangle/\langle|Z_1|\rangle \ll 1$, which is equivalent to $D_2 \ll 1$.

Consider how Equation (47) relates to the long-time limit of the propagator. The statistics of the displacement $Y(t)$ are directly related to the finite-time Lyapunov exponent: $\Delta Y(t) = \sigma(t)$. The variance of $\sigma(t)$ is $2\mathcal{D}_Y t$. In the case where $I(\sigma)$ can be adequately approximated by a quadratic function, we see that $I(\sigma)$ may be approximated by

$$I(\sigma) = \frac{(\sigma-\lambda)^2}{4\mathcal{D}_Y}. \tag{48}$$

Using this approximation in (40) and (41) we recover Eq. (47).

## IV. THREE-DIMENSIONAL MODEL

In this section, we consider how to compute the correlation dimension for inertial particles suspended in a three-dimensional flow. In order to make it possible to perform the analysis, we consider particles in a random velocity field with known statistical properties. This approach has been successful in modelling the Lyapunov exponents of particles in turbulent flows: the leading Lyapunov exponent was obtained in Ref. 33, and all three Lyapunov exponents for the spatial separation of particles in Ref. 25, showing excellent agreement with the numerical simulations of particles in turbulent flows described by Bec.[7] Here, we build upon the results of these earlier calculations by analysing the correlation dimension for the same random-flow model.

The flow underlying turbulent aerosols is usually incompressible, $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. But in order to analyse the properties of the perturbation theory employed in this paper, it is of interest to also consider partially compressible flows. We use the following decomposition of the flow velocity into solenoidal and potential contributions:

$$\boldsymbol{u} = C_3\,(\nabla \wedge \boldsymbol{A} + \beta\nabla\psi), \tag{49}$$

where $C_3$ is a constant. This model was used in Ref. 33 to compute the maximal Lyapunov exponent of inertial particles in random, partially compressible flows. The parameter $\beta$ determines the relative magnitude of the potential and solenoidal contributions. A convenient measure of the relative importance of these two contributions is

$$\Gamma = \frac{4+\beta^2}{2+3\beta^2}. \tag{50}$$

Since the parameter $\beta$ assumes values between zero and infinity, we have that $\frac{1}{3} \le \Gamma \le 2$. The case $\Gamma = 2$ corresponds to solenoidal flow ($\beta = 0$). For $\Gamma = \frac{1}{3}$, by contrast, the flow is purely potential ($\beta \to \infty$). A special case of interest discussed below corresponds to $\Gamma = 1$, where the solenoidal and potential contributions are of equal strengths.

We take the components of $\boldsymbol{A}$ and $\psi = A_0$ to be Gaussian homogeneous isotropic random functions with zero mean values and correlation functions,

$$\langle A_i(\boldsymbol{r},t)A_j(\boldsymbol{r}',t')\rangle = \delta_{ij}\,C(|\boldsymbol{r}-\boldsymbol{r}'|,|t-t'|). \tag{51}$$

The correlation function $C$ is assumed to decay to zero for spatial separations much larger than the correlation length $\eta$ of the flow, and for time differences much larger than the correlation time $\tau$. The typical fluctuation size of the flow is denoted by $\langle \boldsymbol{u}^2\rangle = u_0^2$. This implies that the normalisation constant in (49) must be chosen as

$$C_3^2 = u_0^2\big[3(2+\beta^2)\,|C''(0,0)|\,\big]^{-1}. \tag{52}$$

Following the approach in Refs. 33 and 25, we analyse this model in the "white noise" limit $\tau \to 0$, which justifies the use of the Markovian approximation considered in Section III A. The fluctuations of the velocity gradients $\mathbb{E}(t)$ are characterised by specifying a set of diffusion coefficients, analogous to Equation (32). The diffusion coefficients are expressed in terms of the correlation functions of the elements of $\mathbb{E}$,

$$\mathcal{D}_{ii} = \frac{\gamma^2}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, \langle E_{i1}(t)E_{i1}(0)\rangle \tag{53}$$

(the factor of $\gamma^2$ is a consequence of the fact that $E(t)$ is multiplied by $\gamma$ in (18)). There are some technical complications involved in calculating the Fokker-Planck operator appearing in (29), which were discussed in detail in Ref. 33. We can read off the Fokker-Planck operator from the results in that paper. The version of (29) which is applicable to our model is

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \frac{\partial}{\partial Z_1}[(\gamma Z_1 + Z_1^2 - Z_2^2 - Z_3^2)\rho] + \mathcal{D}_{11}\frac{\partial^2 \rho}{\partial Z_1^2}\\ &+ \frac{\partial}{\partial Z_2}[(\gamma Z_2 + 2Z_1 Z_2)\rho] + \mathcal{D}_{22}\frac{\partial^2 \rho}{\partial Z_2^2}\\ &+ \frac{\partial}{\partial Z_3}[(\gamma Z_3 + 2Z_1 Z_3)\rho] + \mathcal{D}_{33}\frac{\partial^2 \rho}{\partial Z_3^2} - \alpha Z_1 \rho.\end{aligned} \tag{54}$$

The steady-state form of this equation is analogous to the one-dimensional Equation (34). In three dimensions, condition (35) takes the form

$$\int_{-\infty}^{\infty}\mathrm{d}Z_1\int_{-\infty}^{\infty}\mathrm{d}Z_2\int_{-\infty}^{\infty}\mathrm{d}Z_3\, Z_1\,\rho(Z_1,Z_2,Z_3) \equiv \langle Z_1\rangle = 0. \tag{55}$$

The correlation dimension (equal to $\alpha$) is obtained by finding a value of $\alpha$ for which a normalisable solution of (54) can be obtained for which the mean value of $Z_1$ is zero. The Equations (54) and (55) together constitute an exact method for determining the correlation dimension in the white-noise limit.

## V. PERTURBATION THEORY

Here, we derive a perturbation expansion for the correlation dimension. It is convenient to introduce dimensionless variables,

$$x_i = \sqrt{\gamma/\mathcal{D}_{ii}}\, Z_i. \tag{56}$$

The expansion parameter of the perturbation expansion is given by $\epsilon$, where

$$\epsilon^2 = \frac{\mathcal{D}_{11}}{\gamma^3}. \tag{57}$$

Because $\epsilon \to 0$ in the over-damped limit, this perturbation parameter plays a role which is analogous to the Stokes number. The connection between $\epsilon$ and St is discussed in detail in Ref. 25. We denote the joint probability density of $x_1,\ldots,x_3$ in the steady state by $P(x_1,x_2,x_3)$. It follows from Eq. (54) that $P$ satisfies the equation,

$$\begin{aligned}0 = \hat{\mathcal{F}}\,P &\equiv \frac{\partial}{\partial x_1}[(x_1 + \epsilon(x_1^2 - \Gamma(x_2^2 + x_3^2)))P]\\ &+ \frac{\partial}{\partial x_2}[x_2(1+2\epsilon x_1)P] + \frac{\partial}{\partial x_3}[x_3(1+2\epsilon x_1)P]\\ &+ \frac{\partial^2 P}{\partial x_1^2} + \frac{\partial^2 P}{\partial x_2^2} + \frac{\partial^2 P}{\partial x_3^2} - \epsilon\alpha x_1 P.\end{aligned} \tag{58}$$

This equation defines the Fokker-Planck operator $\hat{\mathcal{F}}(\epsilon,\alpha,\Gamma)$. Following,[20] we now develop its solution as a series expansion in $\epsilon$, using a system of annihilation and creation operators which are analogous to those used in quantum mechanics. We employ a notation similar to the Dirac notation: a function $f(x_1,x_2,x_3)$ is denoted by a vector $|f)$. The scalar product between two states $|f)$ and $|g)$

is given by

$$(f|g) = \int_{-\infty}^{\infty}\mathrm{d}x_1 \int_{-\infty}^{\infty}\mathrm{d}x_2 \int_{-\infty}^{\infty}\mathrm{d}x_3\, f(x_1,x_2,x_3)\, g(x_1,x_2,x_3). \tag{59}$$

We expand both the solution $|P)$ of (58) and the value of $\alpha$ for which the solution of this equation exists and satisfies $\langle x_1\rangle = 0$ as power series in $\epsilon$,

$$|P) = \sum_{k=0}^{\infty} \epsilon^k\, |P_k)\,, \quad \alpha = \sum_{k=0}^{\infty} \epsilon^k\, \alpha_k. \tag{60}$$

The Fokker-Planck operator in Eq. (58) is written as

$$\hat{\mathcal{F}} = \hat{\mathcal{F}}_0 + \epsilon(\hat{\mathcal{G}} - \alpha\hat{x}_1) \tag{61}$$

which defines the operators $\hat{\mathcal{F}}_0$ and $\hat{\mathcal{G}}$. The unperturbed steady-state $|P_0)$ satisfies

$$\hat{\mathcal{F}}_0|P_0) = 0. \tag{62}$$

It is given by

$$P_0(x_1,x_2,x_3) = \frac{\exp[-(x_1^2+x_2^2+x_3^2)/2]}{(2\pi)^{3/2}}. \tag{63}$$

Other eigenfunctions of $\hat{\mathcal{F}}_0$ are generated by creation operators $\hat{a}_i$ and annihilation operators $\hat{b}_i$,

$$\begin{aligned}\hat{a}_i &= -\partial_{x_i}\\ \hat{b}_i &= \partial_{x_i} + x_i.\end{aligned} \tag{64}$$

These operators generate eigenfunctions satisfying

$$\hat{\mathcal{F}}_0|\phi_{pnm}) = -(n+m+p)|\phi_{pnm}) \tag{65}$$

according to the rules

$$\begin{aligned}\hat{a}_1|\phi_{p,n,m}) &= |\phi_{p+1,n,m}),\\ \hat{b}_1|\phi_{p,n,m}) &= p|\phi_{p-1,n,m}),\\ \hat{a}_2|\phi_{p,n,m}) &= |\phi_{p,n+1,m}),\\ \hat{b}_1|\phi_{p,n,m}) &= n|\phi_{p,n-1,m}),\\ \hat{a}_3|\phi_{p,n,m}) &= |\phi_{p,n,m+1}),\\ \hat{b}_1|\phi_{p,n,m}) &= m|\phi_{p,n,m-1}),\end{aligned} \tag{66}$$

with $|\phi_{000}) = |P_0)$, normalised as a probability density. The states $|P_k)$ in (60) are expressed as linear combinations of the eigenfunctions $|\phi_{pnm})$,

$$|P_k) = \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{pnm}^{(k)}\, |\phi_{pnm}). \tag{67}$$

The eigenfunctions generated by repeated applications of $\hat{a}$ are neither normalised nor do they form an orthogonal set. This is different from earlier perturbation theories for the Lyapunov exponent of inertial particles.[34,25]

We first consider how the condition $\langle x_1\rangle = 0$ constrains the coefficients $p_{pnm}^{(k)}$ in (67). Using (64), we find

$$\langle x_1\rangle = (x_1|P) = \sum_k \epsilon^k \sum_{p,n,m} p_{pnm}^{(k)}(x_1|\phi_{pnm}) = \sum_k p_{100}^{(k)}\epsilon^k, \tag{68}$$

so that the condition $\langle x_1\rangle = 0$ is satisfied by requiring that

$$p_{100}^{(k)} = 0 \tag{69}$$

for all values of $k$. Substituting (60) into (61) leads to a recursion for $|P_k)$. The term of order $\epsilon^k$ is given by

$$0 = \hat{\mathcal{F}}_0|P_k) + \hat{\mathcal{G}}|P_{k-1}) - \sum_{l=0}^{k-1} \alpha_l (\hat{a}_1 + \hat{b}_1)|P_{k-l-1}), \tag{70}$$

with

$$\hat{\mathcal{F}}_0 = -\hat{a}_1\hat{b}_1 - \hat{a}_2\hat{b}_2 - \hat{a}_3\hat{b}_3 \tag{71}$$

and

$$\begin{aligned}\hat{\mathcal{G}} = &-\hat{a}_1[((\hat{a}_1 + \hat{b}_1)^2 - \Gamma((\hat{a}_2 + \hat{b}_2)^2 + (\hat{a}_3 + \hat{b}_3)^2))] \\ &- 2(\hat{a}_1 + \hat{b}_1)[\hat{a}_2(\hat{a}_2 + \hat{b}_2) + \hat{a}_3(\hat{a}_3 + \hat{b}_3)].\end{aligned} \tag{72}$$

Equation (70) is a recursion for the state $|P_k)$, and the coefficient $\alpha_{k-1}$ in terms of the states $|P_j)$ and coefficients $\alpha_j$ determined in previous iterations. By considering the coefficient of $|\phi_{pnm})$, we obtain $p^{(k)}_{pnm}$ in terms of the coefficients $p^{(k-1)}_{p',n',m'}$ and $\alpha_l$, with $l = 0, \ldots, k-1$. In order to determine the coefficients $\alpha_l$, consider the case $p = n = m = 0$, where the coefficient of the state $|\phi_{000})$ in Eq. (70) reduces to the condition

$$\sum_{l=0}^{k-1} \alpha_l \, p^{(k-l-1)}_{100} = 0. \tag{73}$$

This condition can be fulfilled in at least two ways. One solution is obtained by setting all $\alpha_k$ equal to zero. This case corresponds to calculating the Lyapunov exponent. Using $\lambda = \langle Z_1 \rangle = \sqrt{\mathcal{D}_{11}/\gamma}\,\langle x_1 \rangle$, we find Eqs. (67) and (68) in Ref. 33. A second possibility is to require $\langle x_1 \rangle = 0$ corresponding to $p^{(k)}_{100} = 0$ for all values of $k$, as explained above. This is the case relevant for calculating the correlation dimension. Using the initial condition

$$p^{(0)}_{pnm} = \delta_{p0}\delta_{n0}\delta_{m0}, \tag{74}$$

we can iterate Eq. (70) to determine the coefficients $p^{(k)}_{pnm}$ in terms of the $\alpha_l$. The first few non-vanishing coefficients obtained by recursion of (70) are listed in Table I. Note that Eq. (70) only relates coefficients $p^{(k)}_{pnm}$ with indices $n$ and $m$ to coefficients with indices $n'$ and $m'$ provided that $n' - n$ and $m' - m$ are even integers. This implies that only coefficients with even values of $n$ and $m$ are non-zero (see Table I). Note also that if all odd-order coefficients $\alpha_{2n+1}$ vanish, then Eq. (70) does not mix the parity of $p + k$ in $p^{(k)}_{pnm}$. We find that in this case, Eq. (70) provides two independent recursions: one for $p^{(k)}_{pnm}$ with even $p + k$ and the initial condition $p^{(0)}_{2p,n,m} = \delta_{p0}\delta_{n0}\delta_{m0}$ when $k = 0$ and one for $p^{(k)}_{pnm}$ with odd $p + k$ and the initial condition $p^{(0)}_{2p+1,n,m} = 0$ when $k = 0$. Using the boundary condition $p^{(k)}_{100} = 0$ (Eq. (75)), we find that indeed all odd order $\alpha_{2n+1}$ vanish, and that all coefficients with odd values of $p + k$ vanish. This is illustrated to the lowest order in

TABLE I. Non-zero expansion coefficients $p^{(k)}_{pnm}$ in Eq. (67) for $k = 0, \ldots, 3$. Obtained by recursive solution of Eq. (70).

$p^{(0)}_{000} = 1$

$p^{(1)}_{100} = 2\Gamma - 1 - \alpha_0$, $p^{(1)}_{102} = p^{(1)}_{120} = \frac{\Gamma-2}{3}$, $p^{(1)}_{300} = -\frac{1}{3}$

$p^{(2)}_{002} = p^{(2)}_{020} = \frac{8\Gamma-7}{3} + \frac{\Gamma-8}{6}\alpha_0$, $p^{(2)}_{022} = \frac{\Gamma-2}{3}$, $p^{(2)}_{004} = p^{(2)}_{040} = \frac{\Gamma-2}{6}$, $p^{(2)}_{200} = -\frac{(4\Gamma-5)(4\Gamma-3)}{6} + \frac{4\Gamma-5}{2}\alpha_0 - \alpha_0^2/2$,
$p^{(2)}_{202} = p^{(2)}_{220} = \frac{(-13+17\Gamma-6\Gamma^2)}{6} + \frac{\Gamma-2}{3}\alpha_0$, $p^{(2)}_{204} = p^{(2)}_{240} = -\frac{(\Gamma-2)^2}{18}$, $p^{(2)}_{222} = -\frac{(\Gamma-2)^2}{9}$, $p^{(2)}_{400} = \frac{4\Gamma-5}{6} - \frac{\alpha_0}{3}$, $p^{(2)}_{402} = p^{(2)}_{420} = \frac{\Gamma-2}{9}$,

$p^{(3)}_{100} = -5 + 20\Gamma - 16\Gamma^2 + 2(-5 + 10\Gamma - 3\Gamma^2)\alpha_0 + 2(2\Gamma - 3)\alpha_0^2 - \alpha_0^3 - \alpha_2$,
$p^{(3)}_{102} = p^{(3)}_{120} = \frac{-130+247\Gamma-118\Gamma^2}{9} + \frac{-20+19\Gamma-2\Gamma^2}{2}\alpha_0 + \frac{5\Gamma-28}{18}\alpha_0^2$, $p^{(3)}_{104} = p^{(3)}_{140} = \frac{-122+167\Gamma-56\Gamma^2}{30} + \frac{-22+13\Gamma-\Gamma^2}{18}\alpha_0$,
$p^{(3)}_{122} = \frac{-122+167\Gamma-56\Gamma^2}{15} + \frac{-22+13\Gamma-\Gamma^2}{9}\alpha_0$, $p^{(3)}_{124} = p^{(3)}_{142} = -\frac{(\Gamma-2)^2}{6}$,
$p^{(3)}_{300} = \frac{-135+290\Gamma-212\Gamma^2+56\Gamma^3}{18} + \frac{-67+77\Gamma-24\Gamma^2}{9}\alpha_0 + \frac{9\Gamma-19}{9}\alpha_0^2 - \frac{\alpha_0^3}{6}$,
$p^{(3)}_{302} = p^{(3)}_{320} = \frac{-715+1194\Gamma-724\Gamma^2+172\Gamma^3}{90} + \frac{-65+61\Gamma-18\Gamma^2}{18}\alpha_0 + \frac{\Gamma-2}{6}\alpha_0^2$, $p^{(3)}_{304} = p^{(3)}_{340} = \frac{2}{9}(-6 + 9\Gamma - 5\Gamma^2 + \Gamma^3) - \frac{(\Gamma-2)^2}{18}\alpha_0$,
$p^{(3)}_{322} = \frac{4}{9}(-6 + 9\Gamma - 5\Gamma^2 + \Gamma^3) - 1/9(-2 + \Gamma)^2\alpha_0$, $p^{(3)}_{324} = p^{(3)}_{342} = \frac{(\Gamma-2)^3}{54}$,

$k$ in Table I: all displayed coefficients which are of different parities in $p$ and $k$ are multiplied by $\alpha_1 = -p^{(2)}_{100} = 0$.

These considerations do not determine the normalisation of the distribution $|P)$. Expanding the normalisation condition in terms of Eqs. (60) and (67) yields

$$p^{(k)}_{000} = \delta_{k0}. \tag{75}$$

Once the coefficients $p^{(k)}_{mnp}$ have been determined to each order $k$, we use Eq. (69) to compute $\alpha_{k-1}$. From Table I, we find the first three coefficients in expansion (60) of $\alpha$ in powers of $\epsilon$,

$$\alpha_0 = 2\Gamma - 1\,, \quad \alpha_1 = 0\,, \quad \alpha_2 = -2(\Gamma-1)\Gamma(2\Gamma+1). \tag{76}$$

This gives the correlation dimension for the three-dimensional random-flow model in the white-noise limit

$$D_2 = 2\Gamma - 1 - 2\Gamma(\Gamma-1)(2\Gamma+1)\epsilon^2 + \cdots \tag{77}$$

to second order in $\epsilon$. As mentioned in the Introduction, the two leading non-zero coefficients of this expansion for incompressible flows ($\Gamma = 2$) were computed in Ref. 19. The coefficient of $\epsilon^2$ in that work differs from our result.

## VI. RESULTS AND DISCUSSION

We use an algebraic manipulation program to obtain the series expansion of $D_2(\epsilon)$ in powers of $\epsilon$ from Eq. (70) to higher orders in $k$. To order $\epsilon^{10}$, the result is

$$\begin{aligned} D_2 = {} & 2\Gamma - 1 + 2\Gamma(\Gamma-1)(2\Gamma+1)[-\epsilon^2 + (-11 - 2\Gamma + 6\Gamma^2)\epsilon^4 \\ & + \tfrac{1}{3}(-588 + 5\Gamma + 391\Gamma^2 - 16\Gamma^3 - 144\Gamma^4)\epsilon^6 \\ & + \tfrac{1}{9}(-42\,579 + 18\,573\Gamma + 22\,727\Gamma^2 - 19\,284\Gamma^3 - 12\,648\Gamma^4 + 3464\Gamma^5 + 3960\Gamma^6)\epsilon^8 \\ & + \tfrac{1}{27}(-3\,863\,052 + 3\,918\,303\Gamma + 288\,351\Gamma^2 - 4\,153\,120\Gamma^3 + 47\,186\Gamma^4 \\ & \qquad + 1\,409\,736\Gamma^5 + 277\,928\Gamma^6 - 216\,448\Gamma^7 - 117\,936\Gamma^8)\epsilon^{10}] \\ & + O(\epsilon^{12}). \end{aligned} \tag{78}$$

In special cases, we have obtained expansions to higher orders. For incompressible flows ($\Gamma = 2$), for example, we find to order $\epsilon^{28}$,

$$\begin{aligned} D_2 = {} & 3 - 20\epsilon^2 + 180\epsilon^4 - 9640\epsilon^6 + 206\,940\epsilon^8 \\ & - 16\,548\,920\epsilon^{10} + 477\,315\,000\epsilon^{12} - 50\,149\,424\,368\epsilon^{14} \\ & + 1\,692\,947\,357\,004\epsilon^{16} - 5\,614\,110\,582\,647\,928/25\epsilon^{18} \\ & + 209\,543\,657\,412\,608\,424/25\epsilon^{20} \\ & - 860\,424\,252\,594\,210\,743\,568/625\epsilon^{22} \\ & + 241\,528\,608\,428\,504\,721\,258\,888/4375\epsilon^{24} \\ & - 8\,471\,768\,050\,800\,513\,607\,578\,954\,992/765\,625\epsilon^{26} \\ & + 2\,514\,450\,499\,347\,358\,305\,045\,823\,304\,592/5\,359\,375\epsilon^{28} \\ & + \cdots. \end{aligned} \tag{79}$$

For incompressible flows, we have obtained the first 33 non-vanishing coefficients as fractions of integers and the following 17 coefficients to ten significant digits.

The corresponding series in two spatial dimensions was derived by[20]

$$D_2 = \Gamma - 1 - \Gamma(\Gamma^2-1)\epsilon^2 + \Gamma(\Gamma^2-1)(3\Gamma^2 + 2\Gamma - 11)\epsilon^4 + O(\epsilon^6). \tag{80}$$

Iterating the recursions derived by Ref. 20, we have obtained the first non-vanishing 110 coefficients to ten significant digits in the incompressible case ($\Gamma = 3$).

The series quoted above are asymptotically divergent: they diverge but every partial sum of the series approaches $D_2$ as $\epsilon \to 0$. Evaluating the coefficients $\alpha_k$ for a given value of $\Gamma$ shows that they grow factorially as a function of $k$,

$$\alpha_k \sim a_\Gamma S_\Gamma^{-k}(k-1)!(1 - b_\Gamma/k + \cdots). \tag{81}$$

This is a typical asymptotic behaviour of the coefficients $\alpha_k$ for large values of $k$.[35] Here, the "action" $S_\Gamma$ and the constant $b_\Gamma$ are obtained by fitting of the ansatz (81) to the coefficients. For the fit, we use a non-linear least-squares method, assuming that the relative magnitude of subsequent coefficients is on the form

$$\frac{\alpha_{k+1}}{\alpha_k} \sim S_\Gamma^{-1} \frac{k^2}{k+1} \frac{k+1-b_\Gamma}{k-b_\Gamma}. \tag{82}$$

Fig. 1(a) illustrates the asymptotic behaviour of the coefficients, using the case $\Gamma = 3/2$ in three spatial dimensions as an example. The action $S_\Gamma$ extracted from fits such as the one in Fig. 1(a) is shown in Fig. 1(b), in both two and three spatial dimensions. The resulting action is found to depend upon $\Gamma$ as follows:

$$S_\Gamma = \min[1/6, 1/(6|\Gamma - 1|)], \tag{83}$$

in both two and three spatial dimensions. We note that the coefficients of the perturbation series for the maximal Lyapunov exponent in two spatial dimensions[34] give rise to the action $1/(6|\Gamma - 1|)$ for all values of $\Gamma$.

We also note that the two- and three-dimensional cases shown in Fig. 1(b) differ from each other. In three dimensions, the action is always given by $1/6$ in the allowed range of $\Gamma$ (this is not the case in two spatial dimensions). As opposed to the perturbation expansions for the Lyapunov exponent and the two-dimensional correlation dimension, the three-dimensional perturbation expansion for the correlation dimension is determined by one action only, $S = 1/6$.

We have resummed perturbation series (78) and (80) using Padé-Borel resummation: to sum the series

$$D_2(\epsilon^2) \sim \sum_{l=0}^{\infty} \alpha_{2l}\epsilon^{2l}, \tag{84}$$

consider the modified series, the so-called "Borel sum" (assumed to have a finite radius of convergence due to the extra factor of $1/l!$),

$$B(\epsilon^2) = \sum_{l=0}^{\infty} \frac{\alpha_{2l}}{l!}\epsilon^{2l}. \tag{85}$$

Then, the sum is estimated by

$$D_2(\epsilon^2) = \mathrm{Re} \int_C \mathrm{d}t\, \mathrm{e}^{-t} B(\epsilon^2 t). \tag{86}$$

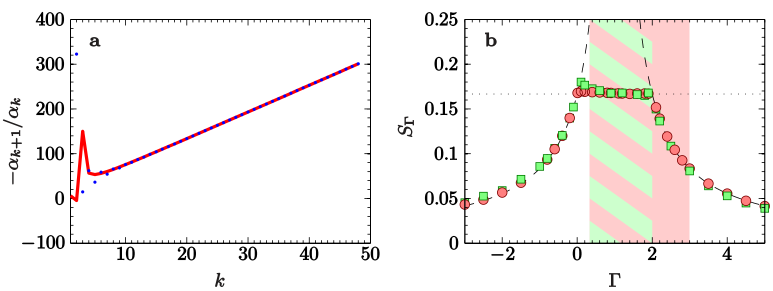


FIG. 1. (a) Fit to asymptotic form (82) (red solid line) for $\Gamma = 3/2$ and $d = 3$ in the range $25 < k \le 50$. The quotients $-\alpha_{k+1}/\alpha_k$ are shown as blue symbols. The resulting values of the fitted parameters are $S_{\Gamma=3/2} = 0.167\, b_{\Gamma=3/2} = 4.8$. (b) Shows actions obtained from the perturbation coefficients as a function of $\Gamma$, in two and in three spatial dimensions. The first 110 non-vanishing coefficients in two dimensions and the first 50 non-vanishing coefficients in three spatial dimensions are fitted to (82) using a non-linear least-squares method. The fitted actions $S_\Gamma$ are shown as symbols: red filled circle in two dimensions and green filled square in three dimensions. Also shown are the curves $S = 1/6$ (dotted black) and $S = 1/(6|\Gamma - 1|)$ (dashed black). The coloured regions indicate the allowed ranges of the parameter $\Gamma$: $1/3 \le \Gamma \le 3$ in two spatial dimensions (red) and $1/3 \le \Gamma \le 2$ in three spatial dimensions (hashed green).

The integration path $C$ is taken to be a ray in the upper right quadrant of the complex plane. In order to perform the integral, an approximation of the Borel sum outside its radius of convergence is required. One possibility is to approximate $B$ by "Padé approximants"[36] of order $[n,n]$ (or $[n,n+1]$). For $\Gamma = 2$, the Padé approximations of order $[2,2]$ and $[3,3]$ are (with $x = \epsilon^2$)

$$B_{[2,2]}(x) = 3 + \frac{\frac{48\,180}{721}\,x^2 - 20\,x}{1 + \frac{1671}{1442}\,x - \frac{649\,927}{8652}\,x^2}, \tag{87}$$

$$B_{[3,3]}(x) = 3 + \frac{-\frac{728\,234\,642\,879}{562\,607\,103}\,x^3 + \frac{91\,933\,567\,500}{187\,535\,701}\,x^2 - 20\,x}{1 - \frac{7\,505\,535\,441}{375\,071\,402}\,x - \frac{99\,077\,893\,373}{937\,678\,505}\,x^2 + \frac{11\,726\,142\,610\,857}{7\,501\,428\,040}\,x^3}.$$

Higher orders are too lengthy to write down here. Fig. 2 shows the results we obtained for $D_2$ for $\Gamma = 2$ in three spatial dimensions by integrating $B_{[4,4]}$, $B_{[8,8]}$, and $B_{[16,16]}$ according to Eq. (86). The corresponding contour $C$ in the complex $t$-plane was chosen along a ray from the origin at angle $\pi/4$. For small values of $\epsilon$, the results depend only negligibly on the precise choice of the contour. Also shown are results for $D_2$ obtained by direct numerical simulations of equations of motion (9). We observe that the Padé-Borel resummations converge quickly for not too large values of $\epsilon$, and we find excellent agreement with results of direct numerical simulations of the random-flow model.

It is clear, on the other hand, that the resummation fails for larger values of $\epsilon$. We suspect that a non-analytical contribution of the form $A\exp[-1/(6\epsilon^2)]$ is not captured and must be added to the perturbation series. In Ref. 34, it is shown that a corresponding term must to be added to the perturbation result for the maximal Lyapunov exponent. The situation here is similar. This is most easily seen by considering the case $\Gamma = 1$. Eq. (78) shows that the first twenty perturbation coefficients vanish for $\Gamma = 1$. We hypothesise that all coefficients vanish at $\Gamma = 1$ and that the correlation dimension exhibits a non-analytic dependence on $\epsilon$, of the form

$$D_2 \sim A_1 \exp\Big(-\frac{1}{6\epsilon^2}\Big). \tag{88}$$

This is shown in two and three spatial dimensions in Fig. 3. These results complement earlier studies of the information dimension $D_1$, discussed in detail in Ref. 25: the Borel summation technique was more successful in that case, but $D_1$ has less direct physical significance.

We conclude this section with two further comments. First, for small values of $\epsilon$,we see that in incompressible flows $3 - D_2 \propto \epsilon^2 \propto \mathrm{St}$. This is a consequence of the fact that we considered the white-noise limit. In this limit, the fractal information dimension exhibits the same scaling.[25] In flows with finite correlation time, by contrast, the correlation dimension deficit behaves as $3 - D_2 \propto \mathrm{St}^2$ for small Stokes numbers.[37–39]

Second, we note that the correlation dimension exhibits a singularity in the advective limit ($\epsilon = 0$) as the compressibility parameter approaches $\Gamma = 1/2$, corresponding to a path-coalescence transition where the maximal Lyapunov exponent changes sign.[33] The perturbation theory gives correct results for $\Gamma \geq 1/2$, it fails for $\Gamma < 1/2$.

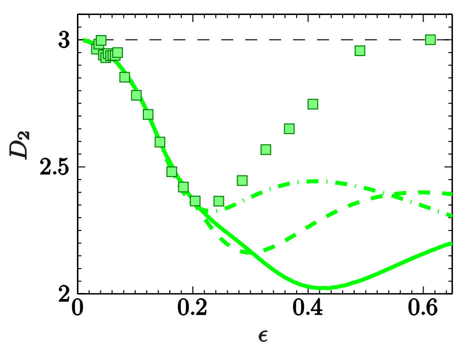


FIG. 2. Correlation dimension for the white-noise model in three spatial dimensions for $\Gamma = 2$ as a function of $\epsilon$. Shown are results of direct numerical simulations of equation of motion (16), symbols, and results of Padé-Borel resummations of the perturbations series for $D_2$, of order $[4,4]$ (dashed-dotted line) $[8,8]$ (dashed line), $[16,16]$ (solid line).

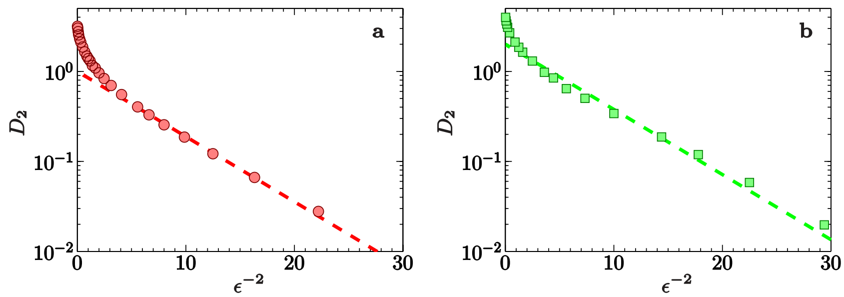


FIG. 3. Correlation dimension for $\Gamma = 1$ as a function of $\epsilon^{-2}$ in two spatial dimensions (a) and in three spatial dimensions (b). Also shown is non-analytical law (88) with prefactors $A_1 = 1$ in two dimensions and $A_1 = 2$ in three dimensions (dashed lines).

## VII. CONCLUSIONS

In this paper we have derived a general method for calculating the correlation dimension of random dynamical systems, which complements DNS (direct numerical simulation) studies of particles in turbulence[7,8] and numerical studies of stochastic models.[22–24] The method is formulated in terms of a propagator describing the time evolution of particle separations and particle-velocity gradients. In special cases, known methods for computing the correlation dimension are obtained.[15,16,3,32]

A short-time expansion of the propagator yields a solvability condition on a partial differential equation, leading to a perturbative expansion of the correlation dimension, for which the coefficients can be obtained exactly and to any order. We derived the exact first 33 coefficients in a series expansion of the correlation dimension for inertial particles in three-dimensional spatially smooth random flows that are white noise in time. Related series expansions have been presented for Lyapunov exponents of inertial particles in such flows in earlier works.[33,25,34]

We have obtained accurate results for the correlation dimension of inertial particles in three-dimensional white-noise flows by Padé-Borel resummation of the perturbation series for not too large values of $\epsilon$. However, for the correlation dimension $D_2$, the resummation method is not as successful as for the information dimension $D_1$, which was considered in Ref. 25. It would be desirable to develop a more direct analytical approach to extracting information about $D_2$ from Equation (54).

In a particular case, for $\Gamma = 1$, we find that the perturbation coefficients vanish and the correlation dimension exhibits a non-analytical dependence upon $\epsilon$. We conjecture that there is a corresponding non-analytical contribution also for $\Gamma > 1$.

Finally, we remark that it is possible to extend the method presented here to treat velocity fields with finite correlation time. This can be achieved by considering a temporally smooth velocity gradient obtained from a stochastic process which is driven by a white noise signal. Numerical studies of velocity gradient statistics in turbulence show that they have correlation functions which are well approximated by exponentials.[40] Velocity gradients of turbulent flows can, therefore, be modelled by an Ornstein-Uhlenbeck process,[41] as described in Ref. 40. The operator methods used in this present paper have been extended to temporally smooth velocity gradients,[42] but some care may be required in their application and interpretation.[43,44]

## ACKNOWLEDGMENTS

Michael A. Morgan (Seattle University) helped in the initial stages of exploring the series expansion of $D_2$ discussed in Section V. Support from Vetenskapsrådet and the Göran Gustafsson Foundation for Research in Natural Science and Medicine is gratefully acknowledged. K.G. acknowledges partial funding from the European Research Council under the European Community's Seventh Framework Programme, ERC Grant Agreement N. 339032.

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