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Triangular constellations in fractal measures

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Abstract – The local structure of a fractal set is described by its dimension D, which is the exponent of a power-law relating the mass $\mathcal{N}$ in a ball to its radius ε: $\mathcal{N} \sim \varepsilon^D$. It is desirable to characterise the *shapes* of constellations of points sampling a fractal measure, as well as their masses. The simplest example is the distribution of shapes of triangles formed by triplets of points, which we investigate for fractals generated by chaotic dynamical systems. The most significant parameter describing the triangle shape is the ratio z of its area to the radius of gyration squared. We show that the probability density of z has a phase transition: $P(z)$ is independent of ε and approximately uniform below a critical flow compressibility β_c, which we estimate. For $\beta > \beta_c$ the distribution appears to be described by two power laws: $P(z) \sim z^{\alpha_1}$ when $1 \gg z \gg z_c(\varepsilon)$, and $P(z) \sim z^{\alpha_2}$ when $z \ll z_c(\varepsilon)$.

Introduction. – Fractal sets and measures play a pivotal role in many areas of physics [1,2]. Fractals are characterised by exploring their local structure. Consider, for example, a set of points obtained by sampling a fractal measure (these could be points representing trajectories in a phase space with a chaotic attractor). One commonly used approach is to pick one of the points at random, and then investigate the number of other points inside a sphere of radius ε centred on that test point. If the expectation value of the number of points $\mathcal{N}$ is a power law in ε,

$$\langle \mathcal{N}(\varepsilon) \rangle \sim \varepsilon^{D_2}, \tag{1}$$

then D_2 is the correlation dimension of the set [3] (throughout this paper $\langle X \rangle$ denotes the ensemble average of X). It is a characteristic feature of fractals that their local structure is characterised by power laws such as (1). More general definitions of dimension, involving different moments of the mass, are discussed in [4].

Fractals which have nearly identical values of the dimension can have a very different appearance. It is desirable to develop means to characterise the shape of the internal structure of fractal distributions, because differences in the local structure of fractal sets may have important implications for properties such as light scattering [5] or network connectivity. Light scattering, for example, may be strongly enhanced in some directions by specular effects if scatterers tend to align on planes or

lines. Recently, a "spectal dimension" was defined which characterises anisotropy in the local structure of the fractal measures [6], but it is desirable to find simpler descriptions of the local shapes of fractal sets.

Here we address the simplest question about the internal shape structure of a fractal set. Consider two randomly chosen particles in a ball of radius ε surrounding a reference point. Together with the test point, these define a triangle. The local structure can be described in greater detail by specifying the statistics of the shapes of these triangles. The shape of a triangle is described by a point in a two-parameter space (we could choose two of the angles, but a better choice is described later).

Many point-set fractals arise as a result of dynamical processes: examples are strange attractors [7], distributions of particles in turbulent flow [8], and possibly also the distributions of matter resulting from gravitational collapse [9]. In this paper we analyze the distribution of triangle shapes for a generic model of chaotic dynamics. We show that the distribution of triangle shapes is also associated with power laws. It might be expected that the stretching action of the dynamics will exaggerate the prevalence of thin, acute-angled triangles. This expectation is only partially correct. We consider a one-parameter family of point fractals on the plane, and show that as the dimension is reduced below two, the prevalence of acute triangles remains constant until a critical

dimension is reached. Below this critical dimension, the distribution of triangle shapes has a strong dependence upon dimension, and acute triangles become predominant.

Characterising shapes of triangles. – The statistics of the shapes of triangles drawn from a random scatter of points was addressed by Kendall [10]. He showed that there is a natural parametrisation of the shape of a triangle in terms of a point on the surface of a sphere [11], with coordinates θ (the polar angle) and ϕ (the azimuthal angle). In his coordinates, equilateral triangles of opposite orientation lie at the poles ($\theta = 0$ and $\theta = \pi$), and degenerate triangles consisting of co-linear points lie on the equator, $\theta = \pi/2$. Kendall observed that the image of a Brownian motion of the three corners is a Brownian motion on the surface of the sphere [10]. His paper does not give an explicit demonstration of this result. A derivation using computer algebra is discussed in [12], and a direct demonstration is given in [13].

Our discussion of the triangle shapes will emphasise the coordinate $z = \cos\theta$, where θ is the polar angle on Kendall's sphere. This is related to the signed area $\mathcal{A}$ defined by writing $\delta\boldsymbol{r}_1 \wedge \delta\boldsymbol{r}_2 \equiv \mathcal{A}\mathbf{k}$ (where $\mathbf{k}$ is a unit normal to the plane), and to the radius of gyration $\mathcal{R}$:

$$z = \frac{2\mathcal{A}}{\sqrt{3}\mathcal{R}^2}, \quad \mathcal{R}^2 = \frac{1}{3}\left[(\delta\boldsymbol{r}_1)^2 + (\delta\boldsymbol{r}_2)^2 + (\delta\boldsymbol{r}_1 - \delta\boldsymbol{r}_2)^2\right],$$
(2)

where the $\delta\boldsymbol{r}_i$ are displacements of two points relative to the third, reference, point. Note that z may be negative, because $\mathcal{A}$ is defined via a vector product. The equilibrium distribution of diffusion on a spherical surface is a uniform probability density, corresponding to a uniform probability density for z. For a random scatter of points, Kendall's result [10] implies that z has a uniform distribution on $[-1, 1]$:

$$P(z) = \frac{1}{2}.$$
(3)

Note that thin, acute triangles correspond to small values of z. We concentrate upon the distribution $P(z)$ in the limit as $z \to 0$.

One of the motivations for Kendall's work was to test claims that sites of geographical and archeological interest are aligned on "ley lines". Kendall argued [10] that these apparent alignments were no more prevalent than those observed in random scatters of points. One possible criticism of Kendall's reasoning is that settlements and geographical features may not be randomly scattered, and that fractal measures might be a better model. We return to this point in our conclusion.

There is also earlier work on the shape distribution of triangles and tetrads in the fluid dynamics literature (examples are [14–18]). These papers are concerned with the shapes of large triangles or tetrads formed by sets of particles as they are advected away from each other by the action of a turbulent flow. This present work considers the opposite limit, namely the shapes of triangles formed by particles which are selected to be very close to each other.

Numerical studies. – As a concrete example of a dynamical process which generates a fractal measure we consider particles advected in a random flow in two dimensions [8]. This is the simplest case, but the techniques can be generalised to higher dimensions and more complex dynamical equations. The equation of motion is

$$\dot{\boldsymbol{r}} = \boldsymbol{u}(\boldsymbol{r}, t),$$
(4)

where $\boldsymbol{u}(\boldsymbol{r}, t)$ is a random velocity field. We consider only particles which are sufficiently close so that their separation $\delta\boldsymbol{r}$ has a linear equation of motion defined by a 2×2 matrix $\mathbf{A}$ with elements A_{ij}:

$$\delta\dot{\boldsymbol{r}} = \mathbf{A}(t)\delta\boldsymbol{r}, \qquad A_{ij}(t) = \frac{\partial u_i}{\partial r_j}(\boldsymbol{r}(t), t).$$
(5)

In our numerical investigations we have used the map

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \boldsymbol{u}_n(\boldsymbol{x}_n)\sqrt{\delta t}$$
(6)

rather than a continuous flow. The velocity field $\boldsymbol{u}_n$ is chosen independently at each timestep, labelled by an integer n. It is constructed from two scalar fields, namely a stream function $\psi_n(x, y)$ and a scalar potential $\chi_n(x, y)$:

$$\boldsymbol{u}_n = \left(\frac{\partial\psi_n}{\partial y} + \beta\frac{\partial\chi_n}{\partial x}, -\frac{\partial\psi_n}{\partial x} + \beta\frac{\partial\chi_n}{\partial y}\right).$$
(7)

The timestep δt is assumed to be sufficiently small to allow the use of diffusive approximations. This is equivalent to using a velocity field which is delta-correlated in time [19]. Note that the flow is incompressible ($\mathrm{div}(\boldsymbol{u}_n) = 0$) when $\beta = 0$. For this reason β is termed the compressibility parameter of the flow. The random fields ψ and χ have the same translationally invariant and isotropic statistics, and they are independent of each other as well as being chosen independently at each timestep. In our simulations, these fields had a Gaussian correlation function: $\langle\psi(\boldsymbol{r})\psi(\boldsymbol{0})\rangle \propto \exp[-\boldsymbol{r}^2/2\xi^2]$ and similarly for χ, where $\xi = 0.25$ is the correlation length of the flow. They were normalised so that $\langle(\partial_{xy}\psi)^2\rangle = 1$, and the coordinate space was a unit square with its edges joined to make a torus.

Figure 1 shows the numerically determined distribution of z for small triangular constellations formed by triplets of randomly chosen points inside a disc of radius $\varepsilon \ll \xi$. The plots show the probability distribution $P(z)$ for particles advected in six different random flows, with increasing values of the compressibility parameter β. In each case the probability distributions $P(z)$ are shown on double-logarithmic scales, for eight different values of ε. For small compressibility β, the distribution is approximately independent of the value of ε and uniform (apart from a cusp at $z = 1$ which arises because our sampling criterion is different from Kendall's, in that we require that the three points lie inside a disc of radius ε). When β exceeds a critical value β_c, $P(z)$ becomes dependent upon ε. It appears to be asymptotic to two power laws in the limit as $\varepsilon \to 0$: $P(z) \sim z^{\alpha_1}$ when z is small, but exceeds a
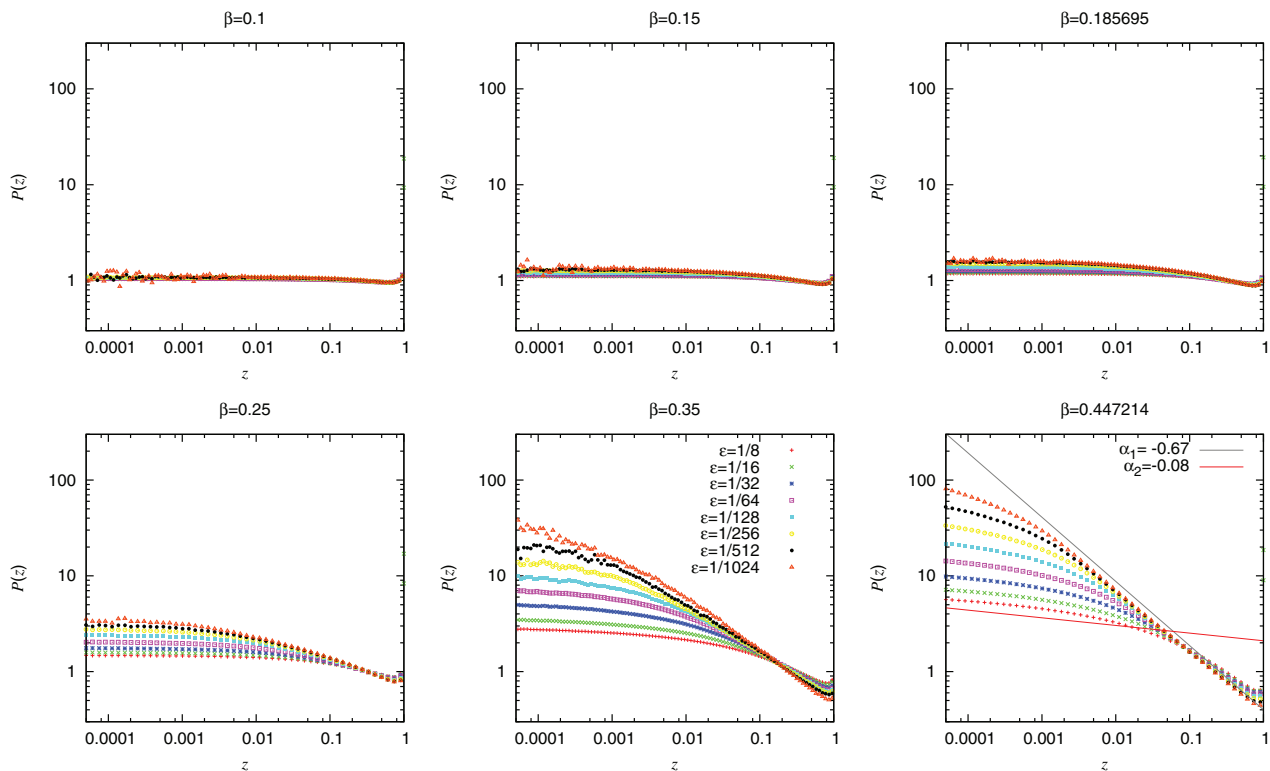
Fig. 1: (Colour on-line) Probability density $P(z)$ for various compressibilities β including two special values: $\beta_c = 1/\sqrt{29} = 0.185\ldots$ is our estimate of the critical compressibility, and $\beta_1 = 1/\sqrt{5} = 0.447\ldots$ is the compressibility at which the correlation dimension satisfies $D_2 = 1$. Straight lines indicate estimates for α_1 and α_2 when $\beta = 1/\sqrt{5}$. Note that $P(z)$ is normalisable ($\alpha_1 > -1$) even at $\beta_1 = 1/\sqrt{5}$, where $D_2 = 1$.

value $z_c(\varepsilon)$ which decreases as $\varepsilon \to 0$, and $P(z) \sim z^{\alpha_2}$ for $z \ll z_c$. In the remainder of this paper we explain why $P(z)$ has power-law behaviour, why there is a critical compressibility β_c, and why the distribution $P(z)$ may have two exponents for $\beta > \beta_c$.

Relation to an advection-diffusion equation. – Now it will be shown that the triangle-shape statistics can be related to an advection-diffusion process. Although the calculation is illustrated for the simplest case (advective motion in two dimensions), the results are readily generalised to three-dimensional systems, and to more complex equations of motion, such as those describing inertial particles. The size and shape of a triplet of points $(\boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{r}_2)$ may be described by three parameters: R_1, R_2, and $\delta\varphi$, which are defined by parametrising the separations $\delta\boldsymbol{r}_i = \boldsymbol{r}_i - \boldsymbol{r}_0$ as follows:

$$\delta\boldsymbol{r}_1 = R_1\mathbf{n}_1, \qquad \delta\boldsymbol{r}_2 = R_2(\mathbf{n}_1 + \delta\varphi\mathbf{n}_2), \qquad (8)$$

where $\mathbf{n}_1$, $\mathbf{n}_2$ are two orthogonal, time-dependent unit vectors and the particles are labelled such that $R_1 \geq R_2$. The equations of motion for R_1, R_2 and $\delta\varphi$ are obtained by substituting (8) into (5) and projecting the equations of motion for $\delta\boldsymbol{r}_i$ onto the $\mathbf{n}_j$. Using the notation

$$F_{ij} = \mathbf{n}_i \cdot \mathbf{A}\,\mathbf{n}_j \qquad (9)$$

we obtain

$$\frac{\dot{R}_1}{R_1} = F_{11}(t), \qquad \frac{\delta\dot{\varphi}}{\delta\varphi} = F_{22}(t) - F_{11}(t) \qquad (10)$$

and a similar equation for $\dot{R}_2/R_2$, as well as an equation describing the rotation of the pair of orthogonal unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$:

$$\dot{\mathbf{n}}_1 \cdot \mathbf{n}_2 = F_{21}(t) = -\mathbf{n}_1 \cdot \dot{\mathbf{n}}_2. \qquad (11)$$

The important point about (10) is that the logarithmic derivative is expressed in terms of the randomly fluctuating quantities $F_{ij}(t)$, so that it is advantageous to transform to logarithmic variables. The variables $(R_1, R_2, \delta\varphi)$ can be replaced by

$$X_1 = -\ln\frac{R_1}{\xi}, \quad X_2 = -\ln\delta\varphi, \quad X_3 = \ln\left(\frac{R_1}{R_2}\right), \quad (12)$$

where ξ is the correlation length of the velocity field. When X_1 and X_2 satisfy $X_1 \gg 0$ and $X_2 \gg 0$, (that is we are dealing with small, acute-angled triangles), the dynamics is trivial: X_3 is frozen and X_1, X_2 obey stochastic equations of motion $\dot{X}_i = \eta_i(t)$, where $\eta_i(t)$ are random functions of time, with statistics independent of position in (X_1, X_2, X_3) space. Because X_3 is frozen, we have

$\eta_3(t) = 0$. The probability density $P(X_1, X_2, X_3)$ obeys a steady-state advection-diffusion equation,

$$-v_i \partial_i P + \mathcal{D}_{ij} \partial_i \partial_j P = 0 \tag{13}$$

(with $\partial_i = \partial/\partial X_i$, and summation over repeated indices). The drift velocities, correlation functions and diffusion coefficients are

$$v_i = \langle \dot{X}_i(t) \rangle,$$
$$C_{ij}(t) = \langle [\dot{X}_i(t) - v_i][\dot{X}_j(0) - v_j] \rangle,$$
$$\mathcal{D}_{ij} = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t \ C_{ij}(t). \tag{14}$$

At this point we can already see why z may have a power-law distribution. Note that, because $z \sim \delta\varphi$, if $\delta\varphi$ has a power-law distribution, then z also has a power-law distribution with the same exponent. Because the equation of motion for (X_1, X_2, X_3) is translationally invariant in the sector $X_1, X_2 \gg 0$, any function invariant under translation (up to a change of normalisation) gives a steady-state solution for $P(X_1, X_2, X_3)$. Because the exponential function is translationally invariant, solutions exist in the form

$$P(X_1, X_2, X_3) = \exp(\gamma_1 X_1 + \gamma_2 X_2) P_3(X_3) \tag{15}$$

(where normalisation requires that the constants γ_i are negative). The corresponding distributions of R_1 and $\delta\varphi$ have probability densities proportional to $R_1^{\gamma_1 - 1}$ and $\delta\theta^{\gamma_2 - 1}$, respectively, so that $P(z) \sim z^\alpha$ with $\alpha = \gamma_2 - 1$. Also, we identify R_1 with ε and note that ε^{γ_1} is the probability density for both points being within a ball of radius ε surrounding the reference point, implying that $\gamma_1 = 2D_3$, where D_3 is the third Renyi dimension [4]. This argument about translational invariance indicates why $P(z)$ may have a power-law distribution, but it is necessary to consider the boundary conditions in order to identify the exponent. In fact it will be argued that the boundary conditions imply the existence of two different asymptotic power laws for $P(z)$, depending upon the ordering of the limits $z \to 0$ and $\varepsilon \to 0$.

In order to identify which power-law solutions should be obtained, we must consider the boundary conditions on the lines $X_1 = 0$ and $X_2 = 0$. The boundary $X_1 = 0$ is a distributed source, corresponding to random triplets of points with a separation approximately equal to the correlation length ξ. Some of these are "squeezed" by the linearised flow so that they enter the region $X_1 > 0$. Phase points $\boldsymbol{X}$ representing these triplets are created on the line $X_1 = 0$ at a rate $J(X_2)$ which corresponds to $\delta\theta$ having a uniform probability density in the limit as $\delta\theta \to 0$ (as required by Kendall's result [10]), implying that the source density on the boundary $X_1 = 0$ is

$$J(X_2) = \begin{cases} J_0 \exp(-X_2), & X_2 > 0, \\ 0, & X_2 \le 0, \end{cases} \tag{16}$$

(where J_0 is a constant which determines the normalisation of the joint probability density, $P(X_1, X_2, X_3)$). The boundary at $X_2 = 0$ is more complicated: it is non-absorbing, and the approximations that X_3 decouples and (X_1, X_2) obeys a simple advection-diffusion equation fail close to $X_2 = 0$. This makes the evaluation of γ_1 (or equivalently of D_3) using our approach difficult (however, formulae for D_3 which are applicable to our model are given in [20]).

Because the probability density is expected to vary over a wide range of values, the large deviation principle [21] indicates that it is conveniently expressed using an exponential form

$$P(\boldsymbol{X}) \sim \exp[-\Phi(\boldsymbol{X})]. \tag{17}$$

Expressed in terms of $\Phi(\boldsymbol{X})$, the exponents describing $P(z)$ are, respectively,

$$\alpha_1 = \frac{\partial \Phi}{\partial X_2}(X_1, 0) - 1,$$
$$\alpha_2 = \lim_{X_2 \to \infty} \frac{\partial \Phi}{\partial X_2}(X_1, X_2) - 1. \tag{18}$$

A model for the shape distribution. – It is not possible to determine the steady-state solution of the advection-diffusion equation exactly. In the following we use an approximate propagator to make a quantitative theory for the critical compressibility β_c, and give a qualitative explanation of the reason why there are two exponents, α_1 and α_2.

Consider first the form of the exponent $\Phi(\boldsymbol{X})$ for the advection-diffusion equation with drift velocity $\boldsymbol{v}$ and diffusion tensor $\mathbf{D}$ when there is a point source at the origin. In this case the exponent in (17) will be denoted by $\Phi_0(\boldsymbol{X})$. If the source is localised at $\boldsymbol{X} = \boldsymbol{0}$ and at $t = 0$, the solution of the diffusion process in d dimensions is a Gaussian centred at $\boldsymbol{X} = \boldsymbol{v}t$. The steady-state probability density from a constant intensity source at $\boldsymbol{X} = \boldsymbol{0}$ can be obtained by integration of this Gaussian propagator over t:

$$P(\boldsymbol{X}) = A \int_0^\infty \mathrm{d}t \ [4\pi \det(\mathbf{D})t]^{-d/2} \exp[-S(\boldsymbol{X}, t)],$$
$$S(\boldsymbol{X}, t) = \frac{1}{4t}(\boldsymbol{X} - \boldsymbol{v}t) \cdot \mathbf{D}^{-1}(\boldsymbol{X} - \boldsymbol{v}t). \tag{19}$$

Here A is a normalisation constant. To estimate $\Phi_0(\boldsymbol{X})$, we determine the time t^* where the propagator is maximal, and set $\Phi_0(\boldsymbol{X}) = S(\boldsymbol{X}, t^*)$, where $\partial S/\partial t(\boldsymbol{X}, t^*) = 0$. The equation for t^* is $\boldsymbol{X} \cdot \mathbf{D}^{-1} \boldsymbol{X} - t^{*2} \boldsymbol{v} \cdot \mathbf{D}^{-1} \boldsymbol{v} = 0$. Hence

$$\Phi_0(\boldsymbol{X}) = \frac{1}{2}\left[\sqrt{\boldsymbol{X} \cdot \mathbf{D}^{-1} \boldsymbol{X}} \sqrt{\boldsymbol{v} \cdot \mathbf{D}^{-1} \boldsymbol{v}} - \boldsymbol{X} \cdot \mathbf{D}^{-1} \boldsymbol{v} \right]. \tag{20}$$

Note that $\Phi_0(X_1, X_2)$ is the height of a tilted conical surface, which touches the (X_1, X_2)-plane along the ray $\boldsymbol{X} = \lambda \boldsymbol{v}$ with parameter λ ("downwind" of the source), but increases linearly along any other ray starting from the source point. Correspondingly, there is an asymptotically exponential reduction of $P(X_1, X_2)$ along any ray from the source.

Next we estimate $P(\boldsymbol{X}) = \exp[-\Phi(\boldsymbol{X})]$ taking account of the condition that there is a distributed source on the line $X_2 > 0$, with intensity $J(X_2) = \exp(-X_2)$. The probability is obtained by integrating the point-source solution $\exp[-\Phi_0(\boldsymbol{X})]$ over the initial point $(X_1, X_2) = (0, X_0)$, with $X_0 > 0$:

$$P(X_1, X_2) = \int_0^\infty \mathrm{d}X_0 \ \exp(-X_0) \exp[-\Phi_0(X_1, X_2 - X_0)]. \tag{21}$$

We may assume that the integral is dominated by contributions from the vicinity of a stationary point X^*. The exponent in (17) is then determined by

$$\Phi(X_1, X_2) = \Phi_0(X_1, X_2 - X^*) + X^*,$$
$$0 = 1 - \frac{\partial \Phi_0}{\partial X_2}(X_1, X_2 - X^*). \tag{22}$$

Because of the conic structure of the function $\Phi_0(X_1, X_2)$, its gradient is constant along any ray from the source point $(X_1, X_2) = (0, X^*)$. For any given value of the compressibility parameter β the stationary point condition is satisfied on a ray which leaves the source point with a slope $s(\beta)$ (which is determined for our specific model below). The stationary point is therefore

$$X^* = X_2 - s(\beta)X_1. \tag{23}$$

Hence (22) gives

$$\Phi(X_1, X_2) = \Phi_0(X_1, s(\beta)X_1) + X_2 - s(\beta)X_1. \tag{24}$$

Note that for constant X_1, we have $\Phi(X_1, X_2) \sim X_2$, so that (18) gives the exponent $\alpha = 0$, consistent with the results shown in fig. 1 for $\beta < \beta_c$. This analysis is only correct if there is a valid stationary point, that is when (23) predicts $X^* > 0$. If $s(\beta) < 0$, there is always a positive solution to eq. (22), so that $P(z) \sim z^0$. If $s(\beta) > 0$, however, stationary points (23) do not exist when (X_1, X_2) lies below the line of slope $s(\beta)$. In this case the integral (21) is dominated by the contribution from $X_0 = 0$, and we have $\Phi(X_1, X_2) \sim \Phi_0(X_1, X_2)$. In the case in which $s(\beta) > 0$, there are expected to be two exponents (α_2 and α_1, respectively) which characterise $P(z)$, depending upon whether $(X_1, X_2) = (\ln(\xi/\varepsilon), -\ln(\delta\theta))$ lies above or below the line of slope $s(\beta)$.

Estimate of critical compressibility. – Now we consider how to estimate the critical compressibility for the phase transition illustrated in fig. 1. This is determined by the condition $s(\beta_c) = 0$. Consider the source of small, non-acute triangles (represented by $X_1 \gg 1$, $X_2 \approx 0$). When $\beta < \beta_c$, there is a solution of (22) and these triangles are predominantly formed by squeezing of acute triangles from $(0, X^*)$ along their axis. When $\beta > \beta_c$, they are formed by approximately isotropic squeezing of non-acute triangles.

For the flow defined by (6) and (7), the elements of the diffusion tensor $\mathbf{D}$ are

$$\mathcal{D}_{11} = \frac{1}{2}(1 + 3\beta^2),$$
$$\mathcal{D}_{12} = \mathcal{D}_{21} = -(1 + \beta^2),$$
$$\mathcal{D}_{22} = 2(1 + \beta^2), \tag{25}$$

and the components of the drift velocity are

$$v_1 = -(1 - \beta^2),$$
$$v_2 = 2(1 + \beta^2). \tag{26}$$

The correlation dimension is

$$D_2 = -\frac{v_1}{\mathcal{D}_{11}} = \frac{2(1 - \beta^2)}{1 + 3\beta^2} \tag{27}$$

[20,22] (so that $D_2 = 1$ for $\beta = 1/\sqrt{5}$). Writing

$$K = \frac{1 + 3\beta^2}{2\beta\sqrt{2(1 + \beta^2)}}, \qquad \Lambda = \frac{4(1 + \beta^2)}{1 + 3\beta^2}, \tag{28}$$

the exponent (20) for the point-source solution is

$$\Phi_0(\boldsymbol{X}) = K\sqrt{X_2^2 + \Lambda(X_1^2 + X_1 X_2)} - X_1 - X_2. \tag{29}$$

Substituting this into (22), we find that the stationary point satisfies (23) with slope

$$s(\beta) = \frac{2(1 + \beta^2)}{1 + 3\beta^2} \left[\frac{8\beta^2}{\sqrt{1 - 26\beta^2 - 23\beta^4}} - 1 \right] \tag{30}$$

so that $s(\beta) = 0$ at $\beta = 1/\sqrt{29} = 0.185\ldots$, where $D_2 = 7/4$. This implies that $P(z) \sim z^0$, independently of β, for $\beta \leq \beta_c = 1/\sqrt{29}$. This prediction for the critical compressibility is in good agreement with our numerical results, illustrated in fig. 1. When $\beta > 1/\sqrt{29}$, there are two different exponents for $P(z)$ because the dominant contribution to the propagator depends upon the position in the (X_1, X_2)-plane. We remark that a quantitative treatment of the exponents α_1, α_2 when $\beta > \beta_c$ requires a more sophisticated model for the propagator, taking account of the different boundary condition at $X_2 = 0$.

Concluding remarks. – We investigated the distribution of shapes of triangular constellations in fractal sets arising from compressible chaotic flow. We find that the distribution of the acute angle is approximately independent of compressibility β up to a critical point β_c, which we were able to determine analytically for a simple model flow. For $\beta > \beta_c$ the exponent α in $P(z) \sim z^\alpha$ takes two different values, depending upon how small z is. While the distribution of $P(z)$ reported in fig. 1 is particular to our model, the techniques do extend to three-dimensional systems and to other equations of motion. In particular, the logarithms of the angles and lengths defining a small simplex in a random flow satisfy simple stochastic equations of motion, for which the corresponding probability

density satisfies an advection-diffusion equation with constant coefficients.

We conclude with a remark about Kendall's discussions of "ley lines". While Kendall's analysis of triangle shapes in a random scatter of points appears to provide a satisfactory refutation of ideas about ley lines, it should be noted that human settlements are not randomly scattered. Their distribution may well be described by a fractal measure, in which case it might be expected that the distribution of acute triangles described by $P(z)$ would be different from that of a random scatter. However, our results show that the distribution $P(z)$ can remain almost unchanged until the fractal dimension passes below a threshold value ($D_c = 1.75$ for our model). This is an unexpected vindication of Kendall's arguments.

$* * *$

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