Dynamics of a generic quantum system under a periodic perturbation

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We characterize the evolution operator of a quantum system with generic spectral properties (in agreement with random-matrix theory), with a time-dependent Hamiltonian. The results are applied to the independent-electron model for the dissipation of energy: an example of a physical application would be the absorption of low-frequency electromagnetic radiation by small metallic particles. We discuss the statistical properties of the various regimes of the model, which depend on whether the perturbation is large enough to mix levels, and on whether the frequency is low enough for the quantum adiabatic theorem to apply. We also show that the eigenstates of the evolution operator are Anderson localized in the adiabatic basis, and present results concerning the localization length. This localization causes a saturation of the energy absorption if the Hamiltonian is periodic in time. The addition of a small amount of noise to the Hamiltonian destroys this nonclassical saturation effect.

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I. INTRODUCTION

In this paper we analyze the response of a small quantum-mechanical system to a time-dependent perturbation and relate the results to a noninteracting-electron model for dissipation. Our results assume that the spectrum of the single-particle states has "generic" statistical properties, which can be described by a suitable random-matrix model. A typical physical application would be to the response of electrons in a small conducting particle to low-frequency electromagnetic radiation. Problems of this type are usually analyzed in terms of a linear-response theory (Kubo-Greenwood formula [1]), which assumes that the perturbation is not large enough to mix the single-particle levels. This condition is often not satisfied in situations of experimental interest, and we feel that this problem is of sufficiently general importance to warrant a thorough investigation, including the response to large perturbations. The results presented here expand upon those given in an earlier Letter [2].

We concentrate on the low-frequency response of the system, which is determined by properties of the spectrum and matrix elements involving small differences in energy. The statistical properties of the spectrum of complex quantum systems are universal at small energy scales and can be modeled by random-matrix ensembles. In Sec. II we discuss the use of random-matrix models and introduce the parametrized Gaussian orthogonal ensemble (GOE), which is a natural random-matrix ensemble for describing time-dependent Hamiltonians.

In Sec. III we characterize the evolution operator for a generic system subjected to a slowly varying perturbation. We show that there are several qualitatively different regimes, depending on whether the perturbation is large enough to mix levels and whether it is slow enough to satisfy the quantum adiabatic theorem. The most significant new results of this section concern the form of the evolution operator when the perturbation is *large* enough to mix states. In this case we model the statistical properties of the evolution operator using a stochastic differential equation (described in detail in Appendix A). The results are consistent with numerical computations and with semiclassical arguments (discussed in Appendix B).

In Sec. IV we apply our results on the evolution operator to the independent-electron model for dissipation. We show that the results are equivalent to the Kubo formula in two of the regimes of the model, including one in which the perturbation is large enough to mix levels.

We also characterize a situation in which the predictions of the Kubo formula fail: this happens when the perturbation is strictly periodic in time, so that the evolution operator for long times can be computed by successive multiplication by the Floquet operator (the evolution operator for a single cycle). The eigenstates of the Floquet operator exhibit Anderson localization, and this implies that the amount of energy the system can absorb exhibits a nonclassical saturation. A similar Anderson-localization effect in the eigenstates of an evolution operator has been discovered in the kicked quantum rotor [3] and in a model for the response of a microscopic loop to a time-dependent flux [4]. In Sec. V we use our results concerning the evolution operator to characterize this effect in generic systems.

We also show in Sec. VI that this saturation effect is very sensitive to deviations from exact periodicity and present estimates concerning the amount of noise required to recover the rate of dissipation predicted by linear-response theory.

II. RANDOM-MATRIX MODELS

To characterize the response of a system with Hamiltonian $\hat{H}(X)$ to a perturbation parameter X, it suffices to know the energy levels $E_n(X)$ and the matrix elements $\langle n|\partial\hat{H}/\partial X|m\rangle$. For a system with a large number of states contributing to the dynamics and in which there are no symmetries or constants of motion, the statistical

properties of the spectrum and matrix elements can be modeled by an appropriate random-matrix model.

In a generic quantum system, the statistical properties of the spectrum belong to one of three universality classes, which are discussed in detail in a reprint volume edited by Porter [5]. These universality classes are exemplified by three random-matrix models: the Gaussian orthogonal, symplectic, and unitary ensembles (GOE, GSE, and GUE). In the case of systems with time-reversal symmetry, the GOE describes bosons and fermions in which spin-orbit coupling is negligible. Fermions with a spin-dependent Hamiltonian are described by the GSE, and in the case of systems without time-reversal symmetry, the GUE applies. These models have been applied successfully to systems with a chaotic classical limit [6] as well as to nuclear spectra [5]. The application of these models to the response of small metallic particles to low-frequency radiation has been discussed by Gorkov and Eliashberg [7] (see also Ref. [8], which corrects a significant error in their work). The random-matrix models mentioned above apply to this system provided the electron states are not Anderson-localized on a length scale of the particle size or less and provided there are no symmetries or classical constants of motion.

In this paper we discuss only the GOE, which is the simplest model, but most of the results generalize in a straightforward manner. A realization of the Gaussian orthogonal ensemble is a real symmetric $\mathcal{N}\times\mathcal{N}$ matrix with independently Gaussian distributed elements with variance $(1+\delta_{ij})$. The density of states is $\rho=\sqrt{\mathcal{N}}/\pi$ at the center of the spectrum, $E=0$. The spectrum of almost any complex spin-independent system with time-reversal symmetry resembles that of a large-$\mathcal{N}$ GOE matrix, apart from an energy-dependent scaling of the smoothed density of states. One of the important properties of GOE matrices is that the probability density in the space of matrices is invariant under orthogonal transformations, so that changing the basis of a GOE matrix yields another typical GOE matrix [5].

Having described a random-matrix model for the spectrum, we must now consider a random-matrix model for the matrix elements of the perturbation, $\langle n|\partial\hat{H}/\partial X|m\rangle$, where $|n(X)\rangle$, $|m(X)\rangle$ are eigenstates of the Hamiltonian $\hat{H}(X)$. In a generic system these matrix elements can be modeled by independent, Gaussian distributed random variables with a local variance $\sigma^2(E,\Delta E)$:

$$\sigma^2(E,\Delta E)=\frac{1}{\rho^2}\sum_n\sum_m|\langle n|\frac{\partial\hat{H}}{\partial X}|m\rangle|^2 \times\delta_\epsilon(E-\tfrac{1}{2}(E_n+E_m)) \times\delta_\epsilon(\Delta E-(E_n-E_m)) , \quad (2.1)$$

where ρ is the smoothed density of states and $\delta_\epsilon(x)$ is a pseudo-δ-function with width ϵ large compared to the typical level spacing [9]. This function can be estimated using semiclassical arguments [10] and is nonsingular at $\Delta E=0$ if the classical correlation function decays faster than $1/t$. The diagonal elements do not have mean value zero, but their variance is closely related to that of the nearby off-diagonal matrix elements:

$$\left\langle\left|\left[\frac{\partial\hat{H}}{\partial X}\right]_{nn}-\left\langle\left[\frac{\partial\hat{H}}{\partial X}\right]_{nn}\right\rangle\right|^2\right\rangle\Bigg|_{E_n\simeq E_F} =\beta\left\langle\left|\left[\frac{\partial\hat{H}}{\partial X}\right]_{nm}\right|^2\right\rangle\Bigg|_{E_n\simeq E_m\simeq E_F} , \quad (2.2)$$

where $\beta=2$ for GOE systems and $\beta=1$ for the GUE. Arguments supporting this result are given in Ref. [9], Appendix B.

Because we are interested in the low-frequency limit, it is convenient to consider the Schrödinger equation expressed in terms of an adiabatic basis [i.e., the basis formed by the eigenstates $|n(t)\rangle$ of the instantaneous Hamiltonian $\hat{H}(X(t))$]. The conventions for the phases of the states $|n\rangle$ and expansion coefficients c_n of the states $|\psi(t)\rangle$ are defined by

$$|\psi(t)\rangle=\sum_n c_n(t)\exp[-i\varphi_n(t)]|n(X)\rangle ,$$
$$\varphi_n(t)=\frac{1}{\hbar}\int^t dt' E_n[X(t')] , \quad \left\langle n\left|\frac{\partial n}{\partial X}\right\rangle\right.=0 . \quad (2.3)$$

Inserting these definitions into the Schrödinger equation, the equation of motion for the expansion coefficients c_n is then

$$\dot{c}_n=\dot{X}\sum_{m(\neq n)}\frac{\left[\frac{\partial\hat{H}}{\partial X}\right]_{nm}}{E_n-E_m}\exp[i(\varphi_n-\varphi_m)]c_m . \quad (2.4)$$

The matrix elements $U_{nm}(t)$ of the evolution operator are given by solutions of (2.4) with initial condition $c_n(0)=\delta_{nm}$. We will be primarily interested in the solution of (2.4) when $X(t)$ is sinusoidal, $X(t)=X_0\cos(\omega t)$.

If we consider low-frequency perturbations, only the matrix elements $\langle n|\partial\hat{H}/\partial X|m\rangle$ with $E_n\simeq E_m$ are significant because of the cancellations introduced by the complex exponential in (2.4): this observation is the basis for the quantum adiabatic theorem. The matrix elements of $\partial\hat{H}/\partial X$ for large $|E_n-E_m|$ are therefore irrelevant and we can, in accordance with (2.2), take *both* the Hamiltonian and the matrix elements $\langle n|\partial\hat{H}/\partial X|m\rangle$ to be GOE matrices.

It follows that we can use the following model system:

$$\hat{H}=\cos[X(t)]\hat{H}_1+\sin[X(t)]\hat{H}_2, \quad X(t)=X_0\cos(\omega t) , \quad (2.5)$$

where $\hat{H}_1$ and $\hat{H}_2$ are two fixed GOE matrices. This is the natural model to choose for two reasons. Firstly, both $\hat{H}$ and $\partial\hat{H}/\partial X$ have the same statistical properties for all X. Secondly, because $\partial\hat{H}/\partial X$ is a GOE matrix, the property of invariance under orthogonal transformations implies that the variance σ^2 of the off-diagonal matrix elements in the basis formed by the eigenstates of $\hat{H}$ is the same as in the original basis: for choices other than the GOE, it would be difficult to compute these matrix elements. We will call this model the "parametrized GOE": it is the natural model to use for any system which has GOE spectral statistics and in which the Hamiltonian

varies slowly as a function of time. Specifically, we propose that statistical properties of the evolution operator of such a system can be modeled by this Hamiltonian, with suitable scaling of the density of states ρ and the typical size of the matrix elements, σ.

III. CHARACTERIZATION OF THE EVOLUTION OPERATOR

For the class of systems we consider, the matrix elements $\langle n|\partial\hat{H}/\partial X|m\rangle$ in the adiabatic Schrödinger equation (2.4) are Gaussian random variables, with mean zero (for $m\neq n$) and a variance $\sigma^2(E,\Delta E)$, which is a smooth function of $\Delta E=E_n-E_m$: the response to slow perturbations depends only on the variance at $\Delta E=0$, which will be denoted by σ^2. The other relevant parameters are the smoothed density of states ρ, $\hbar$, the size of the perturbation parameter, X_0, and its frequency ω. We can form two independent dimensionless groups from these parameters, e.g.,

$$\chi=\rho\sigma X_0, \quad \nu=\rho\hbar\omega \; ; \tag{3.1}$$

χ is a measure of the strength of the perturbation (energy levels are mixed if $\chi\gg1$) and ν is the ratio of the energy scale associated with the frequency of the perturbation, $\Delta E=\hbar\omega$, to the typical spacing between levels, $1/\rho$. Another important dimensionless parameter is given by

$$\kappa=\rho^2\sigma\hbar X_0\omega \; ; \tag{3.2}$$

the quantum adiabatic theorem [11] applies if $\kappa\ll1$. Note that κ is related to the other dimensionless parameters, $\kappa=\chi\nu$. The model has several qualitatively different regimes, depending on the values of χ and ν.

The evolution operator will be characterized by a function $P(n)$, which represents the probability of making a transition from the ith state to the $(i+n)$th state:

$$P(n)=\langle|U_{i,i+n}|^2\rangle_E \; , \tag{3.3}$$

where $\langle\ \rangle_E$ denotes an average over states i with energies E_i close to E. Now we discuss the various regimes of the model, depending on the values of χ and ν.

Case (i): Large ν, small χ

When χ is small, the energy levels and matrix elements in (2.4) can be regarded as independent of X. When $\nu\gg1$, the perturbation is able to induce transitions between states separated in energy by $\Delta E=\hbar\omega$, and we anticipate that the transition probability will be of the form

$$P(n)=(1-2p)\delta_\epsilon(n)+p\delta_\epsilon(n-\rho\hbar\omega)+p\delta_\epsilon(n+\rho\hbar\omega) \; . \tag{3.4}$$

Here $\delta_\epsilon(x)$ is a pseudo-δ-function, with a width ϵ which scales as $\epsilon=O(1/N)$, where N is the number of cycles over which the perturbation acts. Simple perturbation theory (a version of the Fermi golden rule) can be used to derive this result: we find that after N cycles, the transition probability is

$$p=\pi^2N\chi^2/\nu \; . \tag{3.5}$$

This result is valid when $N\gg1$ (so that the driving frequency is well defined). It ceases to be valid for very large N if p ceases to be small, and under other conditions discussed in Sec. V.

Case (ii): Large χ, large κ

When χ is large, the energy levels and matrix elements in (2.4) must be regarded as functions of the parameter X. The matrix elements can be modeled as independent Gaussian random functions of X with a correlation function $C(\Delta X)$, which decays over a range $\Delta X\simeq(\rho\sigma)^{-1}$. Equation (2.4) must then be treated as a stochastic differential equation: this is described in Appendix A. It is shown that the occupation probability P_n of the nth state satisfies

$$\frac{dP_n}{dt}=\sum_{m(\neq n)}R_{nm}(P_m-P_n) \; , \tag{3.6}$$

where the ensemble average of the rate constants R_{nm} can be expressed in terms of a joint correlation function $C^*(\Delta X)$ of the matrix elements and expansion coefficients c_n. If P_n is slowly varying, (3.6) can be approximated by a diffusion equation,

$$\frac{\partial P}{\partial t}=D\frac{\partial^2P}{\partial n^2} \; . \tag{3.7}$$

The diffusion constant is given by

$$D=\pi\hbar\rho^3\sigma^2\dot{X}^2 \tag{3.8}$$

so that the probability distribution after one cycle of the perturbation is a Gaussian, with variance

$$\Delta=2\pi^2\chi^2\nu \; . \tag{3.9}$$

The value of $P(n)$ at $n=0$ is double the value for other small n if the system has time-reversal symmetry. This effect can be understood in terms of constructive interference between time-reversed pairs of paths leading to scattering back into the original state: essentially the same effect is well known in the context of "weak localization" and is discussed in Ref. [12].

In Appendix B we show that if the Hamiltonian we are modeling has a classical limit, the case we are currently considering corresponds to this limit, because both χ and κ increase as $\hbar\to0$ with all classical quantities held fixed. The Gaussian distribution can therefore also be derived from a purely classical argument in this case: this is discussed in Appendix B.

Case (iii): Large χ, small κ

In this case the adiabatic theorem is applicable because κ is small, but the perturbation causes energy levels to approach each other closely at events called avoided crossings. The adiabatic theorem breaks down when the energy levels become nearly degenerate at these avoided crossings, and electrons can undergo Landau-Zener transitions [13] between neighboring levels. The occupation probability spreads diffusively if the perturbation is large enough that many avoided crossings are encountered:

this has been discussed in detail in Refs. [9] and [14]. The distribution $P(n)$ is therefore Gaussian:

$$P(n)=(2\pi\langle\Delta n^2\rangle)^{-1/2}\exp(-n^2/2\langle\Delta n^2\rangle)\ ,$$
$$\langle\Delta n^2(t)\rangle=2\int_0^t d\tau D(\tau)\ . \tag{3.10}$$

The diffusion constant D can be computed analytically: it is different for the different universality classes, and for the GOE it takes the value

$$D=2^{5/4}\pi\Gamma(\tfrac{3}{4})\frac{1}{\rho\hbar}(\rho^2\sigma^2\hbar\dot{X})^{3/2}\ . \tag{3.11}$$

This result is derived in Ref. [14].

Case (iv): Small ν, small χ

When ν is small, the frequency of the perturbation is small compared to the typical beat frequency between quantum states. In this case the quantum adiabatic theorem applies, and there are no transitions between states unless energy levels are nearly degenerate. When $\chi\ll 1$, the perturbation is too small to cause near degeneracies between levels, and in this case the matrix representing the evolution operator is nearly diagonal.

In addition to the four regimes described above, there are two intermediate regimes.

Case (v): Intermediate between cases (i) and (ii)

In case (i) the distribution $P(n)$ has the form given by (3.4) corresponding to resonant absorption, whereas in case (ii) it is essentially Gaussian. There cannot be a sudden transition between these forms: as χ increases with $\kappa\gg 1$ held fixed, the peaks in (3.4) start to broaden at $\chi\simeq 1$ until they merge into a single Gaussian peak. The three peaks merge when the width of the Gaussian distribution predicted by (3.9), $\Delta n\simeq\chi\sqrt{\nu}$, is equal to the spacing of the three peaks, $\Delta n=\nu$. The three peaks are therefore only fully merged when

$$\chi^3/\kappa\gg 1\ . \tag{3.12}$$

Case (vi): Intermediate between cases (iii) and (iv)

In case (iii) the Gaussian form for $P(n)$ will only be observed if the width Δn of the distribution satisfies $\Delta n\gg 1$. Estimating Δn using the diffusion constant (3.11) gives $\Delta n\simeq(D/\omega)^{1/2}\simeq\chi^2\kappa$. The condition for the Gaussian solution to be valid is therefore

$$\chi^2\kappa\gg 1\ . \tag{3.13}$$

If this condition is not satisfied, but $\chi\gg 1$, there will be some Landau-Zener transitions between isolated states, but not enough to give a significant diffusion of the occupation probability.

The forms of the distribution $P(n)$ in the various cases are illustrated schematically in Fig. 1, and their asymptotic regions of validity are summarized in Fig. 2. The boundaries between these regimes are not, of course, rigid: there are smooth transitions between the different cases.

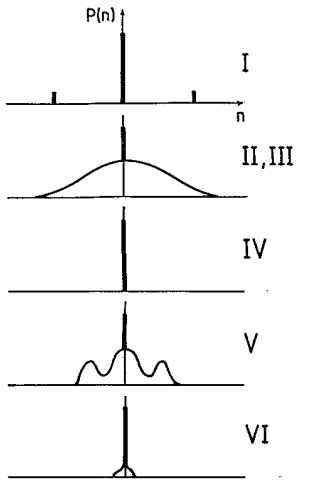

FIG. 1. Schematic illustration of the form of the transition probability for the various regimes discussed in Sec. III: (i) Resonant form, corresponding to small perturbations. (ii), (iii) Gaussian form found when perturbation is large enough to mix states. (iv) Diagonal form, when adiabatic theorem applies. (v) Intermediate between cases (i) and (ii). (vi) Intermediate between cases (iii) and (iv).

We have performed some numerical experiments to verify these predictions, using the parametrized GOE model, Eq. (2.5). We computed the evolution operator for this system using the fourth-order Runge-Kutta method, for matrices of dimension $\mathcal{N}=100$, with $\hbar=1$. We chose the energy E at which $P(n)$ was evaluated to be at the center of the band, where the density of states is $\rho=\sqrt{\mathcal{N}}/\pi$. We computed the evolution operator numerically for various values of X_0 and ω, with several realizations of the random matrices. Figure 3 shows the distribution $P(n)$ computed for two different evolution operators, with different values of χ and ν, chosen to typify the cases (i) and (ii) above. Figure 3(a) shows $P(n)$ with X_0, ω chosen so that $\chi=0.1$, $\nu=20$, after $N=20$ cycles of the perturbation: the position and area of the resonant peak are in good agreement with (3.4) and (3.5). Figure 3(b) shows $P(n)$ for $\chi=2$, $\nu=2$, after $N=1$ cycles: the results have been averaged over five realizations of the GOE.

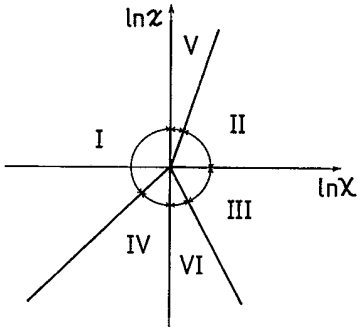

FIG. 2. A plot illustrating the asymptotic regions of validity of the various regimes of the model discussed in Sec. III. The horizontal axis represents $\ln(\chi)$, where χ is the dimensionless strength of the perturbation, and the vertical axis represents $\ln(\kappa)$, where $1/\kappa$ is a dimensionless measure of the degree of adiabaticity.

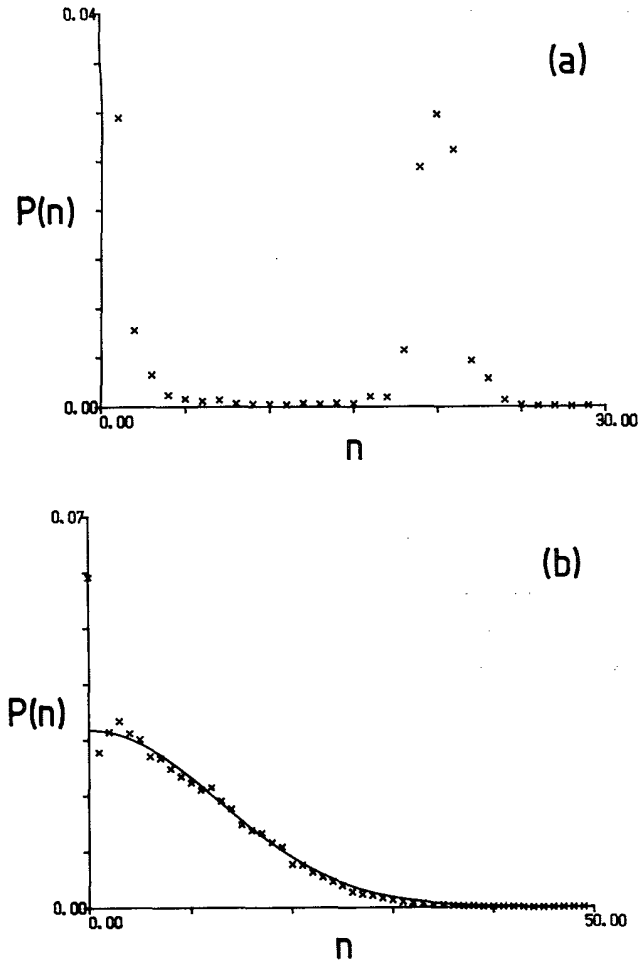

FIG. 3. Numerical results on the transition probability $P(n)$ for the parametrized GOE model. (a) $\chi=0.1$, $\nu=20$, corresponding to the resonant absorption case discussed in Sec. III [case (i)] after $N=20$ cycles. (b) $\chi=2$, $\nu=2$, corresponding to the Gaussian behavior discussed in Sec. III [case (ii)], after one cycle of the perturbation.

The solid line is a Gaussian, with variance given by (3.9). The small deviation from the Gaussian curve near $n=2$ appears to be a relic of the resonant absorption illustrated in Fig. 3(a).

IV. INDEPENDENT-ELECTRON MODEL FOR DISSIPATION

It is often justifiable to model a many-electron system by a system of independent quasiparticles, each of which is described by a one-electron Schrödinger equation [15]. The energy of the system is the sum of the single-particle energies. In this section we apply our results on the evolution operator to the independent-electron model: the perturbation $X(t)$ causes an irreversible increase in the energy of the system. An example of a physical problem where these results are applicable is the absorption of low-frequency electromagnetic radiation by small conducting particles. The usual approach to this problem is via a linear-response formalism ("Kubo formula"). Below we derive a formula (4.8) which relates the change in the total energy of the system to the transition probability $P(n)$ defined by (3.3). Using this result, we show that the Kubo formula applies in the nonperturbative regime discussed in Sec. III [case (ii)], as well as the perturbative case (Sec. III [case (i)]).

Assume that the system starts in its ground state, with all of the energy levels up to $\mathcal{N}_F$ filled. After a time t, the energy of the electron initially in the ith state is

$$\langle i|\hat{U}^+(t)\hat{H}\hat{U}(t)|i\rangle=\sum_j |U_{ji}(t)|^2E_j(t)\ , \tag{4.1}$$

where the $E_j(t)$ are the energies of the single-electron Hamiltonian $\hat{H}(X(t))$ and the $U_{ij}(t)$ are matrix elements of the evolution operator: $U_{ij}(t)$ is the amplitude to reach the ith state starting from the jth state. The total energy of the system of electrons is therefore

$$E_T(t)=\sum_{i=1}^{\mathcal{N}_F}\sum_j |U_{ji}(t)|^2E_j(t)\ . \tag{4.2}$$

The energy absorbed by the system is related to the distribution $P(n)$ characterizing the probability of making a transition through n states, (3.3). To show this, we define the filling probability f_i of the ith state:

$$f_i(t)=\sum_{j=1}^{\mathcal{N}_F}|U_{ji}(t)|^2\ . \tag{4.3}$$

The total energy is then

$$E_T=\sum_i f_iE_i=\int_0^\infty dn\, f(n)E(n)\ , \tag{4.4}$$

where in the second equality we approximate the sum by an integral. Integrating by parts gives

$$E_T=-\int_0^\infty dn\frac{\partial f}{\partial n}E_T^0(n)\ ,\quad E_T^0(n)=\int_0^n dn'E(n')\ , \tag{4.5}$$

where $E_T^0(n)$ is the ground-state energy of a system of n electrons. The total energy is most conveniently calculated in terms of (4.5): the first term in the integrand gives

$$\frac{\partial f}{\partial n}=\frac{\partial}{\partial n}\int_0^{\mathcal{N}_F}dn'P(n-n')=-P(n-\mathcal{N}_F) \tag{4.6}$$

and the ground-state energy can be expanded about $\mathcal{N}_F$:

$$E_T^0(n)=E_T^0(\mathcal{N}_F)+E_F(n-\mathcal{N}_F)+\tfrac{1}{2}\rho_F(n-\mathcal{N}_F)^2+\cdots\ . \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5) gives

$$\Delta E_T=\Delta_F/2\rho_F,\quad \Delta_F=\int_{-\infty}^\infty dn\, n^2P(n)\ , \tag{4.8}$$

where $\Delta E_T=E_T-E_T^0(\mathcal{N}_F)$ is the energy absorbed and Δ_F is the second moment of the distribution $P(n)$. Equation (4.8) is the principal result of this section.

The value of Δ_F can be obtained from the results of Sec. III: in particular, in both cases (i) and (ii) we find that $\Delta_F=2\pi^2\rho^3\sigma^2\hbar X_0^2\omega$, so that the energy dissipated per cycle is

$$\Delta E_T=\pi^2\rho^2\sigma^2\hbar X_0^2\omega\ . \tag{4.9}$$

We can compare this with the conventional way of calculating the energy dissipated using the Kubo-Greenwood formula. In this approach the expectation value of the force operator is computed, which has a term proportional to the velocity $\dot{X}$:

$$\langle F\rangle=\mathrm{tr}\left[\hat{\rho}\frac{\partial\hat{H}}{\partial X}\right]=F(X)+\mu\dot{X}\ , \tag{4.10}$$

where $\hat{\rho}$ is the single-particle density matrix. The viscosity coefficient μ is computed to be [1]

$$\mu=\pi\hbar\sum_n\sum_m\left|\left[\frac{\partial\hat{H}}{\partial X}\right]_{nm}\right|^2\delta(E_n-E_F)\delta(E_m-E_F)$$
$$=\pi\hbar\rho_F^2\sigma_F^2\ . \tag{4.11}$$

The change in energy ΔE_T over a cycle of the periodic perturbation can be computed from the force

$$\Delta E_T=\int_0^{2\pi/\omega}dt\,\langle F\rangle\dot{X}=\pi^2\hbar\rho_F^2\sigma_F^2X_0^2\omega=\pi^2\kappa\chi\ , \tag{4.12}$$

which agrees with (4.9). This shows that the Kubo formula correctly predicts the rate of dissipation in the nonperturbative case discussed in Sec. III [case (ii)], as well as in the perturbative case (Sec. III [case (i)]).

V. LOCALIZATION OF EIGENVECTORS

Several studies on periodically driven quantum systems have shown that the eigenstates of the Floquet operator can be localized, leading to a suppression of energy transfer to the system: this has been demonstrated for the quantized standard map [3], the response of a microscopic ring to a time-dependent flux [4], and other systems. In this section we demonstrate that a similar localization effect occurs in our very general model.

It is well known that the eigenvectors of a banded Hermitian matrix with random elements are localized [16,17]. The matrix need not be strictly banded; the matrix elements only need to decay sufficiently quickly as we move away from the diagonal [18]. Many of the arguments supporting these results are also valid for banded, random, unitary operators, and we will give theoretical estimates of the localization length and maximum energy absorption, supported by numerical results.

The solution of the localization problem depends on the form of the distribution $P(n)$. We consider the two most important cases, when $P(n)$ is Gaussian and when $P(n)$ is of the form (3.4), corresponding to resonant absorption.

A. Gaussian case

We consider first the cases in which the distribution $P(n)$ obeys a diffusion equation and is a Gaussian with variance Δ after one cycle of the perturbation. The variance Δ_N after N cycles is clearly

$$\Delta_N=N\Delta \tag{5.1}$$

in the diffusive approximation. If the eigenvectors of the Floquet operator are localized, however, (5.1) must break down for large N. Consider the Nth power of the Floquet operator. If the eigenvectors of $\hat{U}$ are $|u_n\rangle$, with eigenvalues $\exp(i\varphi_n)$, matrix elements of $\hat{U}^N$ can be written

$$\langle i|\hat{U}^N|j\rangle=\sum_n\langle i|u_n\rangle\exp(iN\varphi_n)\langle u_n|j\rangle\ . \tag{5.2}$$

If L is the localization length, then all the terms in (5.2) will be small if $|i-j|\gg L$. The second moment Δ_N therefore saturates at $\Delta_N\simeq L^2$ after initially increasing linearly. For small N, we expect that the correlations between the matrix elements will not exert much influence, and (5.1) remains valid.

Now consider the implications of this argument for the absorption of energy by a system of noninteracting electrons. Recall that the energy transferred to the system is proportional to Δ_N [Eq. (4.8)], so that (5.1) predicts that the energy absorbed is proportional to the number of cycles of the perturbation. This is in accordance with our expectation that the responses to each cycle are independent (the Markovian approximation). Localization of the eigenvectors of $\hat{U}$ implies that after some number of cycles N^*, the total energy transferred saturates at a value

$$\Delta E_{\max}\simeq\mathrm{const}\times L^2/\rho\ . \tag{5.3}$$

The localization length can be estimated using an adaptation of an argument first given by Chirikov, Shepelyansky, and Izraelev [19] in an analysis of localization of the eigenstates of the Floquet operator for the kicked quantum rotor. If an electron is initially in an eigenstate of the Hamiltonian, this state $|\psi_0\rangle$ can be written as a superposition of $\simeq L$ eigenstates of the evolution operator. The state $|\psi_N\rangle=\hat{U}^N|\psi_0\rangle$ is a quasiperiodic function of N, but its quasiperiodic nature only becomes apparent after $N^*\simeq L$ cycles. From (5.1), we argue that after N^* cycles,

$$\Delta_N\simeq N^*\Delta\simeq L^2\ , \tag{5.4}$$

hence we estimate

$$L\simeq\mathrm{const}\times\Delta\ , \tag{5.5}$$

since $N^*\simeq L$.

We tested this localization hypothesis numerically and found that the eigenvectors of the evolution operator are localized and that the energy absorption saturates in agreement with the results above. The saturation of the energy absorption is illustrated in Fig. 4, curves (a) and (b). Here we computed the Floquet operator of the parametrized GOE model, (2.5), with dimension $\mathcal{N}=100$ and $\hbar=1$ with the Fermi level at the center of the band ($\mathcal{N}_F=50$), and with X_0 and ω chosen so that $\chi=1$, $\kappa=2$. The energy absorption saturates after initially increasing linearly at a rate predicted by the Kubo formula (the straight line in Fig. 4). Curve (a) shows the results of one realization, curve (b) the average over ten realizations.

It is noteworthy that the law for calculating the variance of $P(n)$ after N cycles, Eq. (5.1), is the same as would be obtained for a product of N independent Gaussian-banded matrices with independent matrix elements. This suggests that the evolution operators could be modeled by randomly generated, Gaussian-banded unitary matrices: a method for generating these matrices is

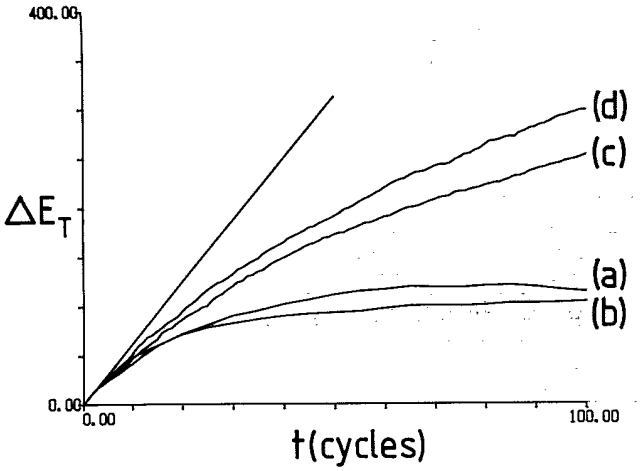

FIG. 4. Plots showing saturation of the energy absorbed as a function of time for the parametrized GOE model. (a) One realization of the GOE evolution operator. (b) Average of ten realizations. (c) and (d): effects of adding a small amount of noise to the GOE Hamiltonian (discussed in Sec. VI). The straight line is the prediction of the Kubo formula.

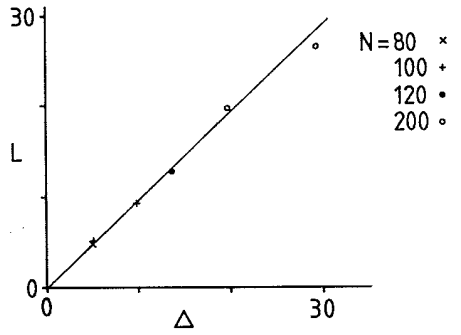

FIG. 5. Localization length as function of second moment Δ of $P(n)$ for random Gaussian-banded unitary matrices: the results verify Eq. (5.5). Different styles of point indicate different values of the dimension $\mathcal{N}$ of the matrices.

described in Appendix C. We find that the mean localization length for these matrices is comparable to that of the evolution operator, but for all the cases we examined, it is significantly higher. As an example, the mean value of the maximum energy transferred is approximately 50% higher for the random unitary operators than for the evolution operators, implying [from (5.3)] that the localization length is approximately 25% larger. We believe that the discrepancy is related to the finite sizes of the matrices involved, but we have not been able to test this conclusively because this would require evolution operators of very large dimension $\mathcal{N}$.

It is difficult to test the relationship (5.5) for the localization length of the eigenstates of the evolution operator directly because the results need to be averaged over several realizations, and each realization of the evolution operator requires the integration of the Schrödinger equation for $\mathcal{N}$ different initial states (where $\mathcal{N}$ is the dimension of the matrix). For this reason we tested (5.5) using the randomly generated Gaussian-banded unitary matrices produced using the method described in Appendix C. The results for these simulated evolution operators (each averaged over 25 realizations) are shown in Fig. 5. The localization length (defined as the square root of the second moment of the probability density of the eigenfunction) was shown to be proportional to Δ: with this definition the constant in (5.5) is approximately unity.

B. Resonant absorption

Now we consider the saturation of energy absorption when $P(n)$ is given by (3.4). This form corresponds to the resonant absorption of energy in quanta of $\Delta E=\hbar\omega$. If the frequency of the perturbation were precisely defined, only states separated by precisely $\hbar\omega$ would experience transitions. The frequency has an uncertainty $\Delta\omega$, however, because the perturbation is only applied for a finite time: $\Delta\omega/\omega\simeq 1/N$, where N is the number of cycles over which the perturbation has been applied. Deviations from the predictions of the Kubo formula are therefore expected when $N\geq N^*\simeq\nu=\rho\hbar\omega$, after which time the frequency is sufficiently well defined that the quasicontinuum approximation breaks down.

When $N\gg\nu$, the response of the system has to be considered in terms of Rabi oscillations of electrons between pairs of states separated in energy by $\hbar\omega$. Each of these pairs is characterized by a detuning parameter, $\delta E=(\Delta E-\hbar\omega)$. The amplitude to reach the upper state of the two-state system, starting from the lower state, is given by

$$|a(t)|=\frac{VX_0}{(\delta E^2+V^2X_0^2)^{1/2}}\sin\left[\frac{1}{2\hbar}(\delta E^2+V^2X_0^2)^{1/2}t\right] \tag{5.6}$$

in the rotating-wave approximation [20], where V is the matrix element of $\partial\hat{H}/\partial X$ between the nearly resonant states.

To estimate the energy absorbed, we note that there are approximately $\nu=\rho\hbar\omega$ pairs of states for which the lower level is filled, and the upper level separated by $\Delta E=\hbar\omega$ is empty. Only these pairs of states can contribute to the absorption of energy. Each pair of states has a different value of the detuning parameter δE, which can be regarded as a random variable with uniform probability distribution. Integrating over this distribution, the dominant contribution to the energy absorption comes from those pairs of states for which δE is sufficiently small that the amplitude (5.6) is of order unity. These are the pairs of states for which the detuning δE satisfies

$$\delta E\leq VX_0\ . \tag{5.7}$$

The proportion of states satisfying this condition is $P\simeq\rho\sigma X_0=\chi$, where we have used the fact that σ is the typical size of the matrix element V. The number of contributing pairs of states is therefore

$$N\simeq P\nu\simeq\chi\nu \tag{5.8}$$

and each of these pairs accounts for an absorption of energy of order $\hbar\omega$. The maximum energy transfer is there-

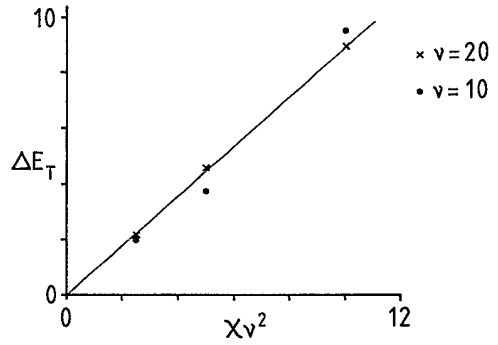

FIG. 6. Maximum energy absorption $\Delta E_{\max}$ in the resonant absorption regime as a function of the dimensionless parameter $\chi\nu^2$: the results verify Eq. (5.9).

fore expected to be

$$\Delta E_{\max}=\text{const}\times\chi\nu\hbar\omega=\text{const}\times\chi\nu^2/\rho \ . \tag{5.9}$$

Some numerical results verifying (5.9) for the parametrized GOE model are shown in Fig. 6, which shows the maximum energy absorbed for a variety of values of $\chi\ll 1$ and $\nu\gg 1$. All of the data in this plot were for matrices of dimension $\mathcal{N}=100$, with $\hbar=1$ and Fermi level $\mathcal{N}_F=50$, and the values of $\Delta E_{\max}$ were averaged over 15 or more realizations.

It is apparent from these results that the saturation of energy absorption is most readily observable (i.e., the number of cycles required before saturation becomes apparent, N^*, is small) in systems where neither χ nor ν is large.

VI. EFFECTS OF NOISE

The saturation of the energy absorption considered in Sec. V depends on the Hamiltonian being a periodic function of time, and we argue that the effect is extremely sensitive to deviations of the Hamiltonian from exact periodicity. In the case of the application to the absorption of electromagnetic radiation by small metallic particles, the deviations from periodicity can be caused by the random perturbation of the small particles by phonons in the supporting material [2].

If $|u_i^N\rangle$ is the ith eigenvector of the evolution operator at the Nth cycle of the motion, the evolution operator for the Nth cycle can be written

$$\hat{U}_N=\sum_i |u_i^N\rangle\exp(i\phi_i^N)\langle u_i^N| \ . \tag{6.1}$$

The evolution operator for N cycles can therefore be written

$$\hat{U}(N)=\sum_k\sum_{k'} |u_k^N\rangle M_{kk'}\langle u_{k'}^1| \ , \tag{6.2}$$

where $M_{kk'}$ is the matrix formed from the product of all the N overlap matrices with elements

$$m_{kk'}^M=\langle u_k^M|u_{k'}^{M+1}\rangle\exp(i\phi_k^M) \ . \tag{6.3}$$

If the suppression of energy absorption is to be maintained, the off-diagonal matrix elements $M_{kk'}$ in (6.2) should be small for pairs of states $|u_k^N\rangle$ and $|u_{k'}^1\rangle$, which are localized in different regions of the spectrum: this will certainly be satisfied if all the off-diagonal elements of the matrices $m_{kk'}^M$ are small. Assuming the matrix elements in the product of the $m_{kk'}^M$ are of the form

$$m_{kk'}^M=\delta_{kk'}\exp(i\phi_k^M)+\epsilon r_{kk'}^M \ , \tag{6.4}$$

where ϵ is small and the $r_{kk'}^M$ are independent random variables, with values of order unity, after N cycles the matrix elements $M_{kk'}$ are of the form

$$|M_{kk'}|=\delta_{kk'}+O(\epsilon\sqrt{N}) \ . \tag{6.5}$$

If the number of cycles required for the saturation to become apparent in the absence of noise is N^*, the saturation effect is still observable as a reduction in the rate of absorption if

$$\epsilon\sqrt{N^*}\ll 1 \ . \tag{6.6}$$

If $\epsilon\sqrt{N^*}\gg 1$, the quantum coherence required for Anderson localization is completely destroyed and (provided $\epsilon\ll 1$) the energy absorption continues to increase at the rate predicted by the Kubo formula after N^* cycles. These arguments are related to those used by Ott *et al.* [21] in a discussion of the effects of noise on the kicked quantum rotor.

Now consider the problem of estimating ϵ, the typical size of the off-diagonal matrix elements of (6.3). It does not appear to be possible to give a general result here: we consider the case where the correlation time of the noise is comparable to the period of the periodic perturbation. If the noise contribution to the Hamiltonian , $\hat{H}_n$, were time independent within each cycle of the perturbation, the eigenstates of the Floquet operator $\hat{U}_M$ would be the same as those of $\hat{H}$ and the typical size of the off-diagonal matrix elements in (6.4) would be

$$\epsilon_1\simeq\rho_F\delta E \ , \tag{6.7}$$

where δE is the typical shift in the eigenvalues due to the perturbation $\hat{H}_n$. If, in addition, H_n varies as a function of time within each cycle, there are also contributions to the matrix elements of (6.4) that arise because phase relationships are not preserved under the perturbation: these are of order

$$\epsilon_2\simeq\frac{1}{\hbar}\int_0^T dt\ \delta E(t) \ , \tag{6.8}$$

where $T=2\pi/\omega$ is the period. The larger of the two contributions ϵ_1 and ϵ_2 should be used in (6.6). Small amounts of noise are sufficient to make either (6.7) or (6.8) large enough that the inequality (6.6) does not hold. The energy-localization effect described in Sec. V is therefore very easily destroyed by noise.

To investigate the effects of noise, we added a noise term to the parametrized GOE Hamiltonian (2.5):

$$\hat{H}=\cos[X(t)]\hat{H}_1+\sin[X(t)]\hat{H}_2+\alpha f(t)\hat{H}_n \ , \tag{6.9}$$

where α is a constant, $f(t)$ is a Gaussian random function with autocorrelation function $C(t)$, and $\hat{H}_n$ is a fixed GOE matrix. Because the numerical evaluation of the

evolution operators is very computationally intensive, we generated two evolution operators, each for one cycle of the perturbation, with the same realization of the matrices $\hat{H}_1$ and $\hat{H}_2$ but with different realizations of $\hat{H}_n$. We then simulated the evolution by taking a random product of these unitary matrices, each occurring with probability $\frac{1}{2}$.

Two examples are shown in Figs. 4(c) and 4(d), which show the suppression of the energy-localization effect by the addition of a small amount of noise. In these examples the parameters χ and ν were the same as for the other examples in Fig. 4, the correlation function $C(t)$ was a Gaussian with variance 0.125, normalized so that $C(0)=1$, and the amplitude of the noise term was $\alpha=10^{-4}$ in case (c) and $\alpha=10^{-3}$ in case (d). Note that a tenfold increase in the amplitude of the noise has little effect on the rate of transfer of energy, indicating that even this very small amount of noise is sufficient to destroy the energy-localization effect.

ACKNOWLEDGMENTS

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APPENDIX A

When $\chi \gg 1$, the matrix elements cannot be assumed constant, and we can model the X dependence of the matrix elements $\langle n|\partial\hat{H}/\partial X|m\rangle$ by assuming that they are Gaussian random functions with an autocorrelation function $C(\Delta X)$,

$$\left\langle \left[\frac{\partial\hat{H}}{\partial X}\right]_{nm}(X)\left[\frac{\partial\hat{H}}{\partial X}\right]_{n'm'}(X+\Delta X)\right\rangle=\delta_{nn'}\delta_{mm'}C(\Delta X)\ . \tag{A1}$$

The expansion coefficients $c_n(t)$ also change very rapidly. We therefore treat the equation of motion for the coefficients $c_n(t)$ as a stochastic differential equation. In the following calculation we compute the rate of increase of the occupation probability P_n of the nth state in terms of those of the surrounding states:

$$\left.\frac{dP_n}{dt}\right|_{in}=\sum_{n(\neq m)}R_{nm}P_m\ . \tag{A2}$$

Reciprocity implies that the averaged transition rate for transitions $n\to m$ is the same as for $m\to n$. The total rate of change of P_n is therefore

$$\frac{dP_n}{dt}=\sum_{n(\neq m)}R_{nm}(P_m-P_n)\ . \tag{A3}$$

Now we integrate (2.4), treating the energy differences E_n-E_m as constant: this approximation is valid for all terms except those where $|n-m|$ is small. The change in the expansion coefficient c_n over a time interval Δt can be written

$$\Delta c_n=\frac{\dot{X}}{E_n-E_m}\sum_{m(\neq n)}\int_0^{\Delta t}dt\,\exp[i(E_n-E_m)t/\hbar]\times\left[\frac{\partial\hat{H}}{\partial X}\right]_{nm}(t)c_m(t) \tag{A4}$$

and the change in the occupation probability after time Δt is

$$\Delta P_n=|c_n+\Delta c_n|^2-|c_n|^2=c_n^*\Delta c_n+c_n\Delta^* c_n+|\Delta c_n|^2\ . \tag{A5}$$

Only the last term in the right-hand side of this equation need be retained, because the other two terms are rapidly oscillating and average to zero. Hence,

$$\langle\Delta P_n\rangle=\langle|\Delta c_n|^2\rangle=\sum_{m(\neq n)}|A_{nm}|^2+\sum_m\sum_{m'}A_{nm}A_{nm'}\ , \tag{A6}$$

where

$$A_{nm}=\frac{\dot{X}}{E_n-E_m}\int_0^{\Delta t}dt\,\exp[i(E_n-E_m)t/\hbar]\times\left[\frac{\partial\hat{H}}{\partial X}\right]_{nm}(t)c_m(t)\ . \tag{A7}$$

The matrix elements $\langle n|\partial\hat{H}/\partial X|m\rangle$ are independent for different values of m, implying that the second term in (A6) is the sum of uncorrelated random variables and has mean value zero. If Δt is long compared to the correlation time of the matrix elements of $\partial\hat{H}/\partial X$, we have

$$\langle|A_{nm}|^2\rangle=\frac{\dot{X}^2}{(E_n-E_m)^2}\langle|c_m|^2\rangle\Delta t\times\int_{-\infty}^{\infty}d\tau\exp[i(E_n-E_m)\tau/\hbar]C^*(\tau)\ , \tag{A8}$$

where $C^*(\tau)$ is a joint correlation function of the matrix elements and the expansion coefficients defined as follows:

$$C^*(\tau)=\frac{\left\langle\left[\frac{\partial\hat{H}}{\partial X}\right]_{nm}(0)\left[\frac{\partial\hat{H}}{\partial X}\right]_{nm}(\tau)c_m(0)c_m(\tau)\right\rangle}{\langle|c_m(0)|^2\rangle}\ . \tag{A9}$$

Combining (A6) and (A8), we find that the rate of increase of P_n due to transitions from other states is given by (A2), with the rate constant given by

$$R_{nm}=\frac{\dot{X}^2}{(E_n-E_m)^2}\int_{-\infty}^{\infty}d\tau\exp[i(E_n-E_m)\tau/\hbar]C^*(\tau)\ . \tag{A10}$$

The correlation function $C^*(\tau)$ has a discontinuity in its first derivative at $\tau=0$, due to nearly degenerate pairs of states which vary very rapidly in $X(\tau)$. This implies that R_{nm} decays as ΔE^{-4} as $\Delta E \to \infty$.

Equation (A3) can be thought of as a discretized diffusion equation with a long-ranged interaction: if P_n is represented by a slowly varying function $P(n)$, this equation can be approximated by a diffusion equation,

$$\frac{\partial P(n)}{\partial t}=\int_{-\infty}^{\infty} dn' R(n')[P(n+n')-P(n)]$$
$$\simeq \tfrac{1}{2}\int_{-\infty}^{\infty} dn' R(n') n'^2 \frac{\partial^2 P(n)}{\partial n^2}=D\frac{\partial^2 P}{\partial n^2}\ , \qquad \text{(A11)}$$

where we have used

$$R(n')=R_{n,n+n'}=R(-n')\ .$$

The diffusion constant is

$$D=\tfrac{1}{2}\int_{-\infty}^{\infty} dn' n'^2 R(n')=\tfrac{1}{2}\int_{-\infty}^{\infty}\rho d\Delta E(\rho\Delta E)^2\frac{\dot X^2}{\Delta E^2}\int_{-\infty}^{\infty} d\tau \exp(i\Delta E\tau/\hbar)C^*(\tau)$$
$$=\tfrac{1}{2}\rho^3\dot X^2\int_{-\infty}^{\infty} d\tau\, C^*(\tau)\int_{-\infty}^{\infty} d\Delta E \exp(i\Delta E\tau/\hbar)$$
$$=\pi\hbar\rho^3\dot X^2\int_{-\infty}^{\infty} d\tau\, C^*(\tau)\delta(\tau)=\pi\hbar\rho^3\dot X^2 C^*(0)=\pi\hbar\rho^3\sigma^2\dot X^2\ , \qquad \text{(A12)}$$

in agreement with (3.8). The distribution $P(n)$ is therefore essentially a Gaussian, with variance given by (3.9). In the tails of the distribution, the approximation used in deriving (A11) from (A3) breaks down, and the rate of decay of $P(n)$ is controlled by the decay of the rate constants $R(n')$: the tails of $P(n)$ therefore decay as $|n|^{-4}$.

APPENDIX B

In this appendix we consider a classical system with a slowly varying Hamiltonian and relate the results to one of the cases discussed in Sec. III. The classical adiabatic theorem is different for systems with integrable and chaotic motion: we consider only the latter case, because we assumed that the spectrum of the quantum system is of the GOE type, which is found in chaotic systems (with time-reversal symmetry) but not in integrable ones [6].

For an ergodic classical system the adiabatic quasi-invariant is the phase-space volume $\mu(E)$ of the energy shell at energy E. Ott [22] considered the changes in the adiabatic invariant, and found that over time scales that are long compared to the correlation time of the classical motion, they increase diffusively, with variance

$$\langle \Delta\mu^2\rangle=\Omega\int_0^T dt\,\dot X^2\int_{-\infty}^{\infty} dt' C(E(t),t')\ , \qquad \text{(B1)}$$

where $E(t)$ is the energy at time t, computed using the relation $\mu(E)=$const and $C(E,t)$ is the classical correlation function of $\partial H/\partial X$:

$$C(E,t)=\int d\alpha\left[\frac{\partial H}{\partial X}(\alpha)-\left\langle\frac{\partial H}{\partial X}\right\rangle\right]$$
$$\times\left[\frac{\partial H}{\partial X}(\alpha_t)-\left\langle\frac{\partial H}{\partial X}\right\rangle\right]\delta(E-H(\alpha))\ . \qquad \text{(B2)}$$

Here Ω is the weight of the energy shell

$$\Omega(E)=\int d\alpha\,\delta(E-H(\alpha)) \qquad \text{(B3)}$$

and in the formulas above α represents a point in phase space and α_t is the point reached by evolving Hamilton's equations for time t, starting at α.

If the motion of the system is chaotic, the function $\partial H/\partial X(\alpha_t)$ can be modeled by a random function, so that in the limit $t\to\infty$ the changes in μ are Gaussian distributed. The distribution of the energies of trajectories which initially started on the same energy shell is therefore also a Gaussian, because $\Delta\mu$ and ΔE are related by

$$\Delta\mu=\Omega\Delta E\ . \qquad \text{(B4)}$$

We can identify this distribution of energies with the distribution $P(n)$ for making quantum transitions through N states. In the semiclassical limit we therefore expect that the distribution $P(n)$ is a Gaussian, with variance

$$\Delta=\rho^2\langle\Delta\mu^2\rangle/\Omega^2\ . \qquad \text{(B5)}$$

It can be shown that this result is consistent with the purely quantum-mechanical results (3.8) and (3.9). To make this connection, we need a relationship between the typical size of the matrix elements, σ, which appears in (3.8), and the classical correlation function appearing in (B2). It has been shown [10] that $\sigma^2(E,\Delta E)$, defined by (2.1), and the correlation function (B2) are related as follows:

$$\sigma^2(E,\Delta E)=\frac{1}{2\pi\hbar\rho\Omega}\int_{-\infty}^{\infty} dt\, C(E,t)\exp(i\Delta Et/\hbar)\ . \qquad \text{(B6)}$$

Substituting this result for $\sigma=\sigma(E,0)$ into (B1) and (B5) gives

$$\Delta=2\pi\hbar\rho^3\sigma^2\int_0^{2\pi/\omega} dt\,\dot X^2=2\pi^2\hbar\rho^3\sigma^2X_0^2\omega\ , \qquad \text{(B7)}$$

which agrees with the quantum-mechanical expression (3.9).

Now we consider the values of the dimensionless parameter χ and κ corresponding to the semiclassical limit and verify that these do correspond to the regime discussed in Sec. III [case (ii)] if we take the limit $\hbar\to 0$, with all classical quantities held fixed. The density of states, and the typical size of matrix elements of $\partial\hat H/\partial X$, scale in the following way for a system with d degrees of freedom:

$$\rho=O(\hbar^{-d}),\quad \sigma=O(\hbar^{(d-1)/2})\ . \qquad \text{(B8)}$$

The first of these results follows from the Weyl rule [6] for the semiclassical density of states, the second from (B6). Given these results, we find that

$$\chi=O(\hbar^{-(d+1)/2}),\quad \kappa=O(\hbar^{-(3d-1)/2})\ , \qquad \text{(B9)}$$

and $\ln(\chi)/\ln(\kappa) > 1/3$, so that the semiclassical limit does indeed correspond to the case in Sec. III [case (ii)].

APPENDIX C

The Gaussian-banded random unitary matrices used for the numerical simulations discussed in Sec. V A were generated by taking a product of M block-diagonal matrices of dimension $\mathcal{N}$,

$$\tilde{U} = \prod_{i=1}^{M} \tilde{u}_i \ , \tag{C1}$$

with each of the $\tilde{u}_i$ of the form

$$\tilde{u}_i = \begin{pmatrix} \tilde{I}_1 & \tilde{0} & \\ & z_1\cos\vartheta & -z_2\sin\vartheta \\ & z_1\sin\vartheta & z_2\cos\vartheta \\ \tilde{0} & & \tilde{I}_2 \end{pmatrix} . \tag{C2}$$

Here $\tilde{I}_1, \tilde{I}_2$ are identity matrices, with the dimension of the matrix $\tilde{I}_1$ chosen randomly with uniform probability density between 0 and $\mathcal{N}-2$, and z_1, z_2 are complex numbers of modulus unity with arguments uniformly distributed random numbers between 0 and 2π. The transition probability $P_t = \sin^2\vartheta$ was taken to be $P_t = \exp(-x^2)$, where x is a random variable uniformly distributed between 0 and $x_{\max} \gg 1$. When M is large, the product matrix $\tilde{U}$ is Gaussian-banded [i.e., $\langle |U_{ij}|^2 \rangle \propto \exp(-|i-j|^2/2\Delta)$], with variance Δ given by

$$\Delta = M\bar{P}_t/\mathcal{N} \ , \tag{C3}$$

where $\bar{P}_t$, the mean value of the transition probability P_t, is given by $\sqrt{\pi}/2x_{\max}$ for our choice of the probability distribution of P_t.

We remark that this method of constructing a random unitary matrix models the evolution operator obtained for a system which undergoes a sequence of Landau-Zener transitions between neighboring levels.

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