Enhanced transport when Anderson localization is destroyed

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We investigate the anomalous transport in optically-induced potentials that are random in both space and time. We find that the time variation destroys Anderson localization, replacing it by transport that is faster than diffusion, which in some cases can be even faster than ballistic. We relate this phenomenon to Chirikov's theory of overlapping resonances, and find radical differences between the anomalously-enhanced transport in one-dimensional and two-dimensional systems.

It is well known that a wavepacket (or a particle) moving in a spatially-disordered time-independent potential can exhibit Anderson localization, and that this is the generic situation in one or two dimensions [1, 2]. At the same time, it is also known that if the disordered potential is also fluctuating in time, localization is lost and transport is restored. That is, for localization to occur the disordered potential must be 'frozen' in time [1]. Consequently, it might naturally be expected that when Anderson localization is destroyed, transport will become diffusive. Over the years, several different mechanisms were proposed, for the breakdown of Anderson localization due to temporal fluctuations of the potential. Mott [3] considered the effect of phonons at low temperatures, and argued that this gives rise to a diffusive motion termed 'variable-range hopping' conductivity. Mott [4] also considered the effects of a weak AC field, and suggested that a resonant interaction dominates the low-frequency response. Alternatively, it has been argued that another limiting procedure leads to a distinct type of diffusive response to a low-frequency electric field, termed adiabatic transport [5]. On the other hand, there are reasons to expect that the response to time-dependent potentials may be different from diffusion. Specifically, the diffusion constants predicted by such mechanisms sensitively depend upon energy. Also, if the potential is time-dependent, the energies of the particles will not be constant. Hence, if the diffusion coefficient is a rapidly increasing function of energy, the response to a time-dependent perturbation can be a super-diffusive. The existence of such 'anomalous diffusion' has been demonstrated in classical ('effective particle') models with spatially and temporally fluctuating potentials [6–10]. However, experimental evidence for anomalous diffusion is still missing: electron-electron interactions render such phenomena, as well as Anderson localization effects, extremely difficult to observe via charge transport experiments in solids. In fact, Anderson localization is most easily demonstrated in optical and in matter-waves systems, where the potential is induced by light waves.

In this Letter, we investigate potentials that are induced by light waves, for example in cold atoms [11] and in optically-induced photonic potentials [12, 13]. We find that transport in these temporally-fluctuating and spatially-random potentials is faster than diffusion, and in some cases can be even faster than ballistic. When the potential is made up of a discrete set of momentum components, we establish connections with the Chirikov theory of overlapping resonances, and show how this approach maps into a diffusive approximation in the continuum limit. Finally, we find radical differences between the anomalously-enhanced transport in one-dimensional and two-dimensional systems.

The random potentials induced by optical waves in photonic lattices [13] and in atom optics experiments [11, 14], rely on transforming an intensity pattern into an effective potential for the light (the former) or the cold atoms (the latter). Such potentials are naturally described in terms of the Fourier spectrum of the waves inducing them, and their spectral coefficients are assumed to be independent random variables. Physically, the spectral content of optically-induced potentials is represented by a finite number, N, of Fourier coefficients, each representing a plane-wave component at some angle with respect to the optical axis. In typical experiments, this number may be large, hence we shall consider potentials with very dense Fourier transform, approximating the limit as $N \to \infty$. The problem is modeled by the Schrödinger equation with such type of potentials, where the initial condition is a localized wave-packet and the question is the rate of spreading of the wave-packet, when Anderson localization is destroyed. In the regime where the wave-packet has already acquired large values of momentum, one expects classical mechanics to provide a reasonable approximation. We will, therefore, explore here the classical dynamics for potentials, which are defined by their spectral content [15]. In particular, we consider the classical dynamics of a particle in potentials that are both random in space and in time, emphasizing the spreading of the momentum acquired by the particle as time evolves. When the potential is made up of a finite number of Fourier components, the theory can be formulated in terms of Chirikov's resonance overlap criterion [16–18]. In the limit as $N \to \infty$, we show how the Chirikov resonances are related to an expression for a diffusion coefficient D characterizing random changes in the momentum, p. For the potentials which arise in optical experiments, in one dimension this describes sub-diffusive behavior in momentum space for large values of momentum, different from the 'universal' behavior predicted by [6–9]. In two dimensions, we find an anomalous diffusion in momentum, in accord with earlier investigations [6–9]. The Chirikov resonances provide an intuitive understanding for the fundamental difference between one and two dimensions.

Before further specification of the model, consider the model defined by the Hamiltonian,

$$H = \frac{p^2}{2} + V(x,t), \tag{1}$$

where $V(x,t)$ is a one-dimensional quasi-periodic potential of the form,

$$V(x,t) = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} A_m \exp\left[\mathrm{i}\left(k_m x - \omega_m t\right)\right] + \text{c.c.}. \tag{2}$$

Here A_m are independent, identically distributed complex random variables. The expectation values of these variables satisfy $\langle A_m \rangle = \langle A_m A_n \rangle = 0$, $\langle A_m A_n^* \rangle = A^2 \delta_{mn}$. For finite N, the potential is a quasi-periodic function of x and t, and in the limit of $N \to \infty$, for fixed x and t, it is a Gaussian random variable with the variance $2A^2$. The random variables k_m and ω_m are distributed with the probability density, or spectral content, $P(k,\omega)$, which includes the cases where $P(k,\omega)$ is unbounded (discrete spectrum). The motion of a particle in a potential given by (2), for sufficiently small $|A_m|$, was analyzed by Chirikov [17]. It was predicted that the phase-space is built up of chains of non-overlapping resonances, which are given by the condition $\frac{\mathrm{d}}{\mathrm{d}t}\left(k_m x - \omega_m t\right) = k_m p - \omega_m = 0$,which is just the stationary phase requirement. This reduces to the condition,

$$p_m^{\mathrm{res}} = \frac{\omega_m}{k_m}. \tag{3}$$

Assuming that the resonances are isolated, as is the case for sufficiently small A, starting a particle with an initial momentum near a resonance will produce a bounded pendulum-like motion near that resonance. This can be seen by neglecting all non-resonant terms in the potential, and making a Galilean transformation to the frame of reference of the specific resonance. The half-width of the resonances, Δ_m, can be assessed from energy conservation, $\Delta_m = \sqrt{8\left|A_m\right|/\sqrt{N}}$. We now order the resonances such that $p_1^{\mathrm{res}} \le p_2^{\mathrm{res}} \le \cdots \le p_N^{\mathrm{res}}$, and define the distance between adjacent resonances by $\delta_m = p_m^{\mathrm{res}} - p_{m-1}^{\mathrm{res}}$. Therefore, the precise condition on the amplitudes of the potential, for isolated resonances becomes $(\Delta_m + \Delta_{m-1}) \le \delta_m$ for all m; namely, that there are no overlapping resonance chains.

The dynamics for weakly overlapping resonances for the model (11) is presented in Fig. 1. When the amplitudes of the potential, $|A_m|$, are not small, namely, $2\Delta_m \sim \delta_m$, some of the resonance chains overlap. It is established [16, 17] that in places where resonances overlap, stochastic regions will form, which results in a random walk between resonances and therefore diffusion in the momentum. In order to observe such diffusion, the number of resonances has to be large, since for the diffusion approximation to be valid a large number of jumps between the resonances has to occur. We will now obtain the diffusion coefficient, adapting a technique developed in [6–9] to the case where the potential is described by the statistics of its Fourier components. In the limit $N \to \infty$ the Chirikov resonances become dense in momentum, which appears at first sight to complicate the problem. It is instructive to study the force-force correlation function

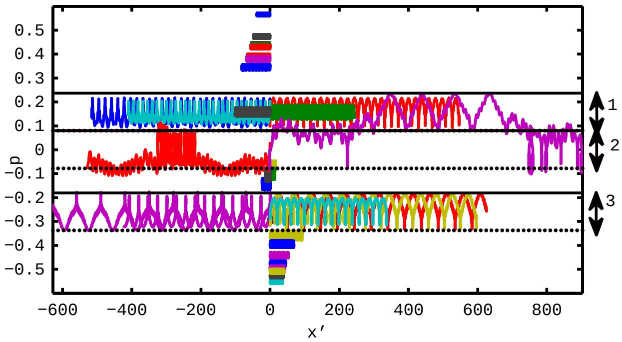

Figure 1: (color online) Phase-space diagram, $(x' = x - p^{\mathrm{res}}t, p)$, for two overlapping resonances. The double two-sided arrows indicate the region of the resonances. Resonances (1) and (2) overlap, but none overlaps with resonance (3). Different colors (shades) describe different trajectories. The horizontal black solid and dashed lines designate the edges of the resonance chains. The initial conditions were uniformly distributed on the p axis $(x\,(t=0)=0)$. The system was integrated up-to $t = 10^4$ (dimensionless variables).

$$C(x_1,t_1;\, x_2,t_2) = \left\langle F(x_1,t_1) F(x_2,t_2) \right\rangle, \tag{4}$$

where $x_1 = x_1(t_1)$ and $x_2 = x_2(t)$ are points on the trajectory of the particle, which depend on the realization of the potential. It is assumed that the correlations decay sufficiently fast such that C vanishes in a small time interval $|t_1 - t_2|$, while the momentum in this interval is practically fixed [9]. This is, essentially, the Markovian approximation leading to the Fokker-Planck equation, that turns into a solvable problem. The resulting diffusion coefficient in momentum is:

$$D(p) = \frac{1}{2}\int_{-\infty}^{\infty} C(p\tau,\tau)\,\mathrm{d}\tau \qquad (5)$$
$$= 4\pi A^2 \int \mathrm{d}k \int \mathrm{d}\omega\, k^2\, P(k,\omega)\delta(\omega - kp).$$

Notice, that this expression is closely connected to the resonance probability density, which can be defined as

$$P(p^{\mathrm{res}}) \;=\; \int \mathrm{d}k \int \mathrm{d}\omega\; |k|\; P(k,\omega)\,\delta(\omega - kp^{\mathrm{res}}). \quad (6)$$

These two expressions quantify the connection between Chirikov resonances and phase space diffusion. We can obtain the asymptotic behavior of (5) for large p following a similar procedure done in [9], by rescaling the variables, $k' = kp$,

$$D(p) = \frac{4\pi A^2}{p^3}\int_{-\infty}^{\infty} \mathrm{d}k'\, k'^2 P(\frac{k'}{p}, k'). \qquad (7)$$

Therefore if $P(0,k') \neq 0$, which is the condition for the universal behavior, in the limit of large p we have,

$$D(p) \sim \frac{D_0}{p^3}, \qquad (8)$$

where, $D_0 = \int_{-\infty}^{\infty} \mathrm{d}k'\, k'^2\, P(0,k')$. The spreading in momentum is controlled by the Fokker-Planck equation,

$$\frac{\partial\rho}{\partial t} = \left(\frac{\partial}{\partial p} D(p) \frac{\partial}{\partial p}\right)\rho, \qquad (9)$$

for the momentum density ρ. The scaling (8) of the diffusion coefficient predicts anomalous diffusion in momentum, in line with [6, 7, 9, 10], with $\langle p^2\rangle \sim t^{2/5}$, as can be easily seen by rescaling the Fokker-Planck equation (9). The resulting spread in the position is $\langle x^2\rangle \sim t^{12/5}$. This result implies that the width of the region of high probability of finding the particle, $\langle x^2\rangle^{1/2}$, is increasing with $t^{6/5}$, which is clearly faster than ballistic transport ($\sim t$). This result applies for one dimensional systems where the spectrum of the disordered potential is unbounded. Physically, however, in all cases where the potential is induced by light waves, the range of frequencies in the potential is limited, namely, there is some $p_{\max}$, such that, $P(k,\omega) = 0 \quad : \quad \omega \geq p_{\max} k$. Namely, the function $P(k,\omega)$ is restricted to some region. It then follows from (5) that

$$D(p) = \begin{cases} 4\pi A^2 \int \mathrm{d}k\, k^2\, P(k,pk) & |p| \leq p_{\max} \\ 0 & |p| > p_{\max}. \end{cases} \qquad (10)$$

This indicates there would be a saturation in the diffusive growth of the kinetic energy, which is determined by the region where Chirikov resonances are found. This results in a deviation from the universal scaling for such potentials. Therefore, a particle started inside the part of space with non–vanishing resonance density will first display enhanced diffusion, but it will not diffuse to regions of zero resonance density, and eventually the asymptotic spread in position space will be ballistic $\langle x^2\rangle \sim t^2$. We conclude that the spreading in phase-space results from a resonance between the motion of the particle and the time-dependence of the driving force, and takes place where the density of resonances (6) does not vanish. For strong enough driving (temporal fluctuations) spreading out of this region can take place, but it is eventually suppressed by an effective adiabaticity of the driving potential. This is demonstrated in Fig. 2 for the potential described by (11).

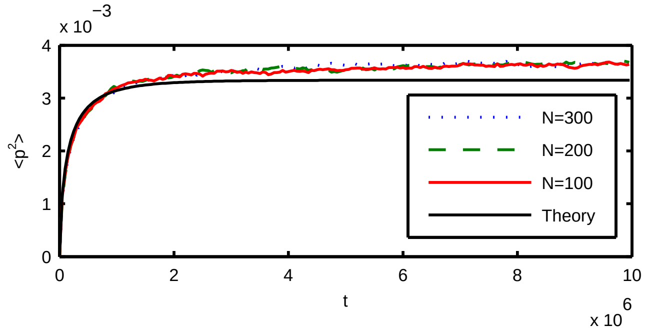

Figure 2: (color online) Mean kinetic energy $\langle p^2\rangle$ as a function of time, t. The dashed gray line is the numerical solution of Eq. (1) for 1000 initial conditions taken from a Gaussian distribution and averaged over 10 realizations of the potential V. Different shades and styles of lines designate different N (see legend). The solid thick line is the numerical solution of the Fokker-Planck equation (Eq. (9)), with a diffusion constant given by Eq. (14). The parameters are $k_R = 0.1$ and $A = 10^{-4}$.

We now turn specifically to the optical and atom optics settings [11], where the potential is proportional to the intensity of the optical field $|E_0|^2$. In atom optics, this potential results from the interaction of the induced dipole of the atoms with the electric field [11], while, in the domain of optics, the effective potential is the variation in the refractive index, which is proportional of to the intensity (the interference pattern) inducing it [12, 19]. For the sake of clarity, we confine ourselves to the optical system used to study localization [13], where the potential was induced by the interference pattern of several paraxial waves [13, 20]. In these systems, transport is manifested in the propagation dynamics of an optical probe beam (not to be confused with the waves inducing the potential) through a disordered dielectric medium. The role of time is played by the propagation axis of the beam, whereby ballistic transport corresponds to the regime where the width of the diffracting beams is proportional to the propagation distance, and localization occurs when the expansion in real space comes to a halt. The potential resulting from the interference pattern is,

$$V(\mathbf{x},t) = \frac{1}{N}\sum_{n,m=1}^{N} A_m A_n^* \exp\left[\mathrm{i}\left((\mathbf{k}_m - \mathbf{k}_n)\cdot\mathbf{x} - (\omega_m - \omega_n)t\right)\right] \tag{11}$$

with ω_n and k related by a dispersion relation, which is approximated by $\omega_n = k_n^2/(2k)$ and $k = 2\pi n_0/\lambda$ [15], n_0 is the bulk refractive index and λ is the wavelength of light in vacuum (in what follows we will use the units where, $k = 1$). First, the analysis is performed in one (transverse) dimension, replacing $\mathbf{k}$ by $k_x = |\mathbf{k}|$. The resulting Chirikov resonances are,

$$p_{nm}^{\mathrm{res}} = \frac{\omega_m - \omega_n}{k_m - k_n} = \frac{k_m + k_n}{2}, \tag{12}$$

and p_{nm}^{res} is bounded. The number of resonances is $N(N-1)/2$, and their half-width is, $\Delta_{mn} = \sqrt{8|A_m A_n|/N}$. The overlap condition of the resonances is acquired in a similar way as above, by comparing the width of the resonances with the distance between adjacent resonances. Assuming, $\langle A_m A_n^* A_i^* A_j\rangle = A^4\left(\delta_{nm}\delta_{ij} + \delta_{mi}\delta_{nj}\right)$, one finds,

$$D(p) = 4\pi A^4 \int \mathrm{d}k_2\, P(2p - k_2)\, P(k_2)\, |k_2 - p|\,, \tag{13}$$

where $P(k)$ is the distribution of resonances in Fourier space, determined by the experimental set-up. In particular, for $P(k) = \theta(k_R - |k|)/(2k_R)$, that is the 1D counterpart of the 2D distribution used in experiments, where $\theta(x)$ is a step function, using (13) we compute the diffusion coefficient

$$D(p) = \begin{cases} \frac{\pi A^4}{k_R^2}(k_R - |p|)^2 & |p| \le k_R \\ 0 & |p| > k_R. \end{cases} \tag{14}$$

The resulting growth of momentum is presented in Fig. 2. It is clear that the universal behavior predicted by (8) is not satisfied here.

In dimensions higher than one, the Chirikov resonance condition (3) is

$$(\mathbf{k}_m - \mathbf{k}_n)\cdot\mathbf{p}_{mn}^{\mathrm{res}} = \omega_m - \omega_n. \tag{15}$$

Because two wavevectors can be selected to give a resonance at any given value of the momentum, this condition allows resonance overlap more easily than in one dimension. In the limit $N \to \infty$, the resonances become dense and their effect is described by diffusion of momentum, with an anisotropic diffusion tensor. We analyze the component of the diffusion tensor in the direction of the momentum p, namely $D_{\parallel}(p)$, and find that, in contrast to the one-dimensional case, the diffusion in momentum is not bounded, even if k is restricted so that $k < k_R$. A particularly simple result is found for a Gaussian distribution in $d-$dimensions, $P(k) = \left(2\pi k_R^2\right)^{-d/2}\exp - \left[\left(k_1^2 + k_2^2 + \cdots + k_d^2\right)/\left(2k_R^2\right)\right]$, leading to

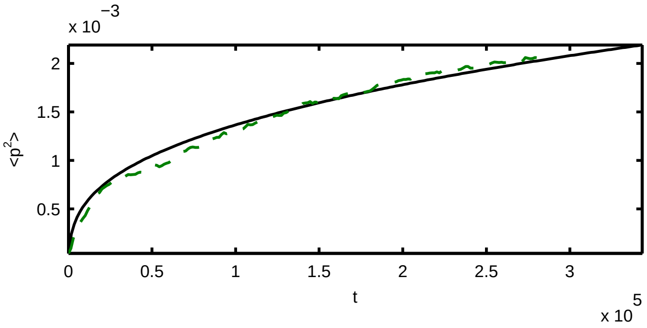

Figure 3: (color online) Same as Fig. 2, but for a two-dimensional system, for $N = 800$ and parameters commonly used in the experiment [22], $k_R = 0.0294$ and $A = 2.06\times10^{-4}$.

$$\begin{aligned} D_{\parallel}(p) &= \frac{A^4}{2}\int_{-\infty}^{\infty}\mathrm{d}\tau \frac{1}{\left(1 + k_R^4\tau^2\right)^{d/2}} \times \\ &\times \left(-\frac{1}{\tau^2}\frac{\partial^2}{\partial p^2}\right)\exp\left[-\frac{k_R^2\tau^2 p^2}{1 + k_R^4\tau^2}\right], \end{aligned} \tag{16}$$

where τ is the difference in time between the points of the force-force correlation function. The above expression is in good agreement with the numerical simulations, as is clear from Fig. 3. Asymptotically, $D_{\parallel}(p) \sim 1/p^3$, which yields an universal asymptotic expansion in momentum $\langle p^2\rangle \sim t^{2/5}$, and a ballistic expansion rate in position space $\langle x^2\rangle \sim t^2$. [21].

In this Letter, we analyzed the classical dynamics in optically-induced random potentials. Such potential, that are commonly used for optics and atom optics experiments, are naturally described in terms of their spectral content. We find that the diffusion coefficient in momentum space is related to the Chirikov resonances. In 1D systems, if the momentum is unbound the diffusion coefficient is proportional to the inverse of momentum cubed. On the other hand, if the spectral content is bounded in momentum (as is always the case for optically-induced potentials), the anomalous transport can go on for some finite time but eventually the diffusion in momentum is suppressed, and the system asymptotically converges to ballistic transport. For systems with dimension of two or higher, we find that diffusion in momentum is unbounded for any spectral content of the potential and universality always holds. This is clearly explained in terms of the Chirikov resonances. The anomalous diffusion in momentum, at least for some range of parameters, results in a super-ballistic spreading in the coordinate space at least for some time scales that may be very long. As a result of the spreading in momentum, both waves and particles will reach high momentum. This Letter studied the classical dynamics of particles, hence providing a qualitative description for the spreading of waves for these potentials (see [22] for relevant optics experiments and theory). The correspondence to the dynamics of waves in such potentials, especially in the asymptotic regime of very long evolution should be a subject of further studies. Likewise, it would be intriguing to explore the dynamics for small number of spectral components, N , which could yield mixed dynamics involving chaotic and regular regimes.

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[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[3] N. Mott and E. A. Davis, *Electronic Processes in Non-Crystalline Materials* (Oxford University Press, USA, 1979), 2nd ed.
[4] N. F. Mott, Phil. Mag. **19**, 835 (1969).
[5] M. Wilkinson, J. Phys. A: Math. Gen. **24**, 2615 (1991).
[6] L. Golubović, S. Feng, and F.-A. Zeng, Phys. Rev. Lett. **67**, 2115 (1991).
[7] M. N. Rosenbluth, Phys. Rev. Lett. **69**, 1831 (1992).
[8] E. Arvedson et al., Phys. Rev. Lett. **96**, 030601 (2006).
[9] V. Bezuglyy et al., J. Math. Phys. **47**, 073301 (2006).
[10] B. Aguer et al., J. Stat. Phys. **138**, 780 (2009).
[11] L. Sanchez-Palencia et al., Phys. Rev. Lett. **98**, 210401 (2007).
[12] N. Efremidis et al., Phys. Rev. E **66**, 046602 (2002).
[13] T. Schwartz et al., Nature **446**, 52 (2007).
[14] J. E. Lye et al., Phys. Rev. Lett. **95**, 070401 (2005).
[15] Y. Krivolapov et al. (2011), to be submitted to Phys. Rev. A.
[16] G. M. Zaslavsky and B. V. Chirikov, Sov. Phys. Usp. **14**, 549 (1972).
[17] B. V. Chirikov, Phys. Rep. **52**, 263 (1979).
[18] A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics* (Springer, 2010), 2nd ed.
[19] J. W. Fleischer et al., Phys. Rev. Lett. **90**, 023902 (2003).
[20] L. Levi et al., Science **332**, 1541 (2011).
[21] Y. Krivolapov et al. (2011), in preparation.
[22] L. Levi et al. (2011), to be submitted to Nature.