Wilson-Schreiber Colourings of Cubic Graphs

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\begin{abstract}
An $S$-colouring of a cubic graph $G$ is an edge-colouring of $G$ by points of a Steiner triple system $S$ such that the colours of any three edges meeting at a vertex form a block of $S$. In this note we present an infinite family of point-intransitive Steiner triple systems $S$ such that (1) every simple cubic graph is $S$-colourable and (2) no proper subsystem of $S$ has the same property. Only one point-intransitive system satisfying (1) and (2) was previously known.

\textit{Keywords:} Cubic graph, edge-colouring, Steiner triple system.
\end{abstract}

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1 Introduction

A Steiner triple system $\mathcal{S} = (V, B)$ of order $v$ consists of a set $V$ of $v$ elements, called points, and a collection $B$ of 3-element subsets of $V$, called blocks, such that every 2-element subset of $V$ is contained in exactly one block. It is well known that a Steiner triple system of order $v$ exists if and only if $v \equiv 1$ or 3 (mod 6) (see Kirkman [5]).

Given a Steiner triple system $\mathcal{S}$, an $\mathcal{S}$-colouring of a cubic graph $G$ is an edge-colouring of $G$ by points of $\mathcal{S}$ such that the colours of any three edges meeting at a vertex form a block of $\mathcal{S}$. This kind of colouring was introduced by Archdeacon [1] in 1986 and later studied by a number of authors (for example, see [2,4,6,8]). Interesting connections between Steiner colourings and several difficult conjectures, such as the cycle double cover conjecture and the Fulkerson conjecture, are discussed in [1,7,8].

One of the questions that naturally arise in this area is whether a given Steiner triple system $\mathcal{S}$ is universal, that is, whether every simple cubic graph admits an $\mathcal{S}$-colouring. Somewhat surprisingly, the best known geometric examples of Steiner triple systems, the projective systems $PG(n, 2)$ and the affine systems $AG(n, 3)$, include no universal member [4]. The first universal Steiner triple system (of order 381) was found by Grannell et al. [2]. Pál and Škoviera [9] improved this result by identifying a subsystem of the previous system of order 21 that is also universal. Further significant progress was made by Kráľ’ et al. [6] who proved that every non-trivial point-transitive Steiner triple system that is neither projective nor affine is universal. In particular, the smallest order of a universal system is 13. In contrast, very little is known about colourings by point-intransitive Steiner triple systems. In fact, only one universal point-intransitive system is currently known [6].

In this note we describe an infinite family of point-intransitive universal Steiner triple systems based on the Wilson-Schreiber construction [10,11]. The smallest member of the family has order 15. Infinitely many of these systems are minimally universal, that is, they do not contain a proper universal subsystem. A detailed discussion and proofs will appear in a further paper [3].

2 Wilson-Schreiber Systems and Colourings

Let $A$ be an Abelian group of order $n$, written additively. We construct a Steiner triple system $\mathcal{S}$ of order $v = n + 2$ whose points are the elements of $A$ and two additional points $\alpha$ and $\beta$. The construction applies only when, for every prime divisor $p$ of $n$, the order of $-2 \pmod{p}$ is even; we call such
a group admissible. Since \( v \) is the order of a Steiner triple system, we have \( v \equiv 1 \) or \( 3 \) (mod 6), so \( n \equiv 1 \) or \( 5 \) (mod 6), and therefore neither 2 nor 3 divides \( n \).

Let us list all unordered triples \( \langle a, b, c \rangle \) of elements of \( A \) with \( a + b + c = 0 \) and repetitions allowed. For each triple \( \langle a, b, c \rangle \) with pairwise distinct entries we include the set \( \{a, b, c\} \) as a block of \( S \). The triples of the form \( \langle a, a, -2a \rangle \) where \( a \in A - 0 \) can be partitioned into orbits under the action of the mapping \( z \mapsto -2z, z \in A \). Since \( A \) is admissible, the number of triples in each orbit is even. Pick one of the orbits and replace the repeated element in each triple by \( \alpha \) and \( \beta \) alternately along the orbit. Process each orbit similarly, and include all sets \( \{\alpha, a, -2a\} \) and \( \{\beta, b, -2b\} \) obtained in this way as blocks of \( S \). Finally, replace the triple \( \langle 0, 0, 0 \rangle \) with \( \{0, \alpha, \beta\} \) and include it in \( S \).

It is easy to see that, with the above collection of blocks, \( S \) is a Steiner triple system. Since there exist infinitely many primes \( p \) such that \(-2\) has even order (mod \( p \)), there are infinitely many such Wilson-Schreiber systems. Furthermore, it can be shown that the systems constructed from the prime groups \( \mathbb{Z}_p \) do not contain any non-trivial proper subsystem [3].

Every Wilson-Schreiber system \( S \) constructed from an admissible group \( A \) of order greater than 9 is point-intransitive, that is, it contains two points that cannot be mapped onto each other by an automorphism of the system. This follows from the fact that the number of mitres having 0 as an apex differs from the number having \( x \in A - 0 \) as an apex, where a mitre is partial subsystem of \( S \) having the form \( \{\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, f\}, \{c, e, g\}\} \) and the apex of the mitre is the point \( a \).

Our main result is the following theorem.

**Theorem 2.1** Let \( S \) be a Wilson-Schreiber system obtained from an admissible Abelian group of order greater than 9. Then \( S \) is universal.

To show that \( S \) is universal we employ a sufficient condition based on the existence of certain substructures in \( S \). A rooted configuration is a configuration \( C \) of points and 3-element blocks with one distinguished point, the root. A rooted homomorphism of \( C \) into \( S \) is a homomorphism \( C \to S \) such that the root of \( C \) is mapped to a given point of \( S \).

The following result, based on ideas from [4] and [6], will be proved in [3].

**Theorem 2.2** Let \( P \) be a set of points of a Steiner triple system \( S \). Suppose that for every configuration \( C_i \in U = \{C_0, C_1, \ldots, C_7\} \) (see Fig. 1) and for every point \( y \in P \) there exists a rooted homomorphism \( C_i \to S \) taking the points of \( C_i \) to \( P \) and the root to \( y \). Then \( S \) is universal.
Sketch of proof of Theorem 2.1. First observe that every non-trivial subgroup of an admissible group is admissible, and that the Wilson-Schreiber system constructed from a subgroup is a subsystem of that constructed from the whole group. By the classification of finite Abelian groups, and since \( \mathbb{Z}_2, \mathbb{Z}_3, \text{ and } \mathbb{Z}_{11} \) are not admissible groups, it suffices to prove the result for all admissible cyclic groups of prime order \( p \geq 13 \) as well as for three individual groups \( \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{Z}_7, \text{ and } \mathbb{Z}_7 \times \mathbb{Z}_7 \).

Let \( p \geq 13 \) be a prime, and let \( S \) be a Wilson-Schreiber system based on \( \mathbb{Z}_p \). To apply Theorem 2.2, we take \( P = (\mathbb{Z}_p - 0) \cup \{\alpha, \beta\} \) and define \( D \) to be the partial subsystem of \( S \) induced by the points from \( P \). For each \( C_i \) we construct two particular rooted homomorphisms \( C_i \to D \), one taking the root to \( \alpha \) or \( \beta \) and the other taking the root to some element of \( \mathbb{Z}_p - 0 \). All other rooted homomorphisms \( C_i \to D \) required by Theorem 2.2 are then obtained from these two by applying automorphisms of \( S \). As an example, in Table 1 we display the two homomorphisms for the configuration \( C_3 \). The remaining configurations, as well as the three small groups will be dealt with in [3].

Fig. 1. Set of configurations \( U \) from Theorem 2.2
Table 1

<table>
<thead>
<tr>
<th>group $\mathbb{Z}_p, p \geq 13$</th>
<th>root</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
</tr>
</thead>
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<tr>
<td>$\beta$</td>
<td>1</td>
<td>2</td>
<td>$-3$</td>
<td>$-6$</td>
<td>5</td>
<td>4</td>
<td>$-2$</td>
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References


