ON THE 2-PARALLEL CHROMATIC INDEX OF STEINER TRIPLE SYSTEMS

P. Danziger
Department of Mathematics, Physics and Computer Science
Ryerson Polytechnical University
Toronto, Ontario M5B 2K3
CANADA

M.J. Grannell, T.S. Griggs
Department of Mathematics and Statistics
University of Central Lancashire
Preston PR1 2HE
UNITED KINGDOM

A. Rosa
Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario L8S 4K1
CANADA

0. Abstract

The 2-parallel chromatic index $\chi''(S)$ is the minimum number of colours required to colour the blocks of a Steiner triple system S so that any two parallel blocks receive different colours. The value of $\underline{\chi}''(v) = \min \{\chi''(S) : S \text{ is an } STS(v)\}$ is determined for all admissible v. It is further shown how the 2-parallel chromatic index is related to the independence number and a complete analysis for all $STS(v)$, $v \leq 15$ is given.

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1. Introduction

A Steiner triple system of order $v, STS(v)$, consists of a pair $(V, \mathcal{B})$ where V is a v-element set of points and $\mathcal{B}$ is a set of 3-element subsets of V (called blocks or triples) which collectively have the property that every 2-element subset of V occurs in precisely one block. Two distinct blocks are said to intersect if they have a point in common; they are said to be parallel if they are disjoint. The chromatic index of a Steiner triple system S, denoted by $\chi^{'}(S)$ is the minimum number of colours required to colour all the blocks of S, each with a single colour, in such a way that no two intersecting blocks receive the same colour.

A reasonable generalisation of this concept is obtained by considering any configuration T, say, consisting of n triples and seeking to colour the blocks of a Steiner triple system S avoiding monochromatic copies of T. The minimum number of colours required may be denoted by $\chi(T, S)$. When T consists of two intersecting triples $\chi(T, S)$ is just $\chi^{'}(S)$. For two triples, only one other configuration is possible in a Steiner triple system, namely the case when the two triples are parallel. In this case we shall denote $\chi(T, S)$ by $\chi^{''}(S)$ and refer to this as the <u>2-parallel chromatic index of S</u>. In the subsequent sections we shall obtain bounds on $\chi^{''}(S)$ in terms of v, the order of S. We hope to deal with certain other configurations T in a similar way in a further paper.

We make the following definitions:

(a) $\underline{\chi}^{''}(v) = \min\,\{\chi^{''}(S) : S \text{ is an } STS(v)\}$

(b) $\overline{\chi}^{''}(v) = \max\,\{\chi^{''}(S) : S \text{ is an } STS(v)\}$

Clearly $\underline{\chi}^{''}(v)$ and $\overline{\chi}^{''}(v)$ are only defined for those values of v for which an $STS(v)$ exists, namely $v \equiv 1$ or $3 \pmod 6$. Such values of v are called admissible. We shall establish the value of $\underline{\chi}^{''}(v)$ for all admissible v and an upper bound for $\overline{\chi}^{''}(v)$. For corresponding known results concerning $\chi^{'}$ we refer the reader to the survey [9]. Using analogous definitions to (a) and (b) above, these can be expressed as follows:

Theorem 1.1 [8] For $v \equiv 3 \pmod 6$, $\underline{\chi}'(v) = (v-1)/2$.

Theorem 1.2 [12] For $v \equiv 1 \pmod 6$ and $v \geq 19$, $\underline{\chi}'(v) = (v+1)/2$.

Theorem 1.3 [2] $\overline{\chi}'(v) \leq 3v/2$.

2. Bounds on $\chi''(S)$

Theorem 2.1

$$\underline{\chi}''(v) \leq \begin{cases} (v+1)/2 \text{ if } v \equiv 1 \text{ or } 9 (\text{mod } 12), \\ (v-1)/2 \text{ if } v \equiv 3 \text{ or } 7 (\text{mod } 12). \end{cases}$$

Proof
A subset $I \subseteq V$ in an $STS(v), S = (V, \mathcal{B})$, is said to be an independent set if no block of $\mathcal{B}$ is contained in I. It is well known (c.f., e.g., [10]) that for each admissible v there exists an $STS(v)$ with an independent set I of cardinality

$$|I| = \begin{cases} (v-1)/2 \text{ if } v \equiv 1 \text{ or } 9 (\text{mod } 12), \\ (v+1)/2 \text{ if } v \equiv 3 \text{ or } 7 (\text{mod } 12). \end{cases}$$

(Also, it is not possible for $|I|$ to exceed these values.)

Let S denote such a system of order v and I an independent set with this cardinality. Consider the points in $V \backslash I$. Assign a distinct colour to each one of these points. Colour each block incident with just one of these points with the colour allocated to that point. For blocks incident with two (or three) of these points, arbitrarily select one of the two (or three) colours and colour the block with that colour. The number of colours used is at most

$$|V \setminus I| = \begin{cases} (v+1)/2 \text{ if } v \equiv 1 \text{ or } 9 (\text{mod } 12), \\ (v-1)/2 \text{ if } v \equiv 3 \text{ or } 7 (\text{mod } 12). \end{cases}$$

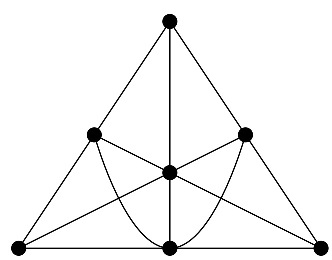
Because I is an independent set, all blocks contain a point of $V \setminus I$ and are therefore coloured. If two blocks have the same colour then they must intersect. Hence

$$\chi''(S) \leq \begin{cases} (v+1)/2 \text{ if } v \equiv 1 \text{ or } 9 (\text{mod } 12), \\ (v-1)/2 \text{ if } v \equiv 3 \text{ or } 7 (\text{mod } 12). \end{cases}$$

and the result follows.

Note

A configuration of blocks intersecting a single common point is called a star; if there are n blocks it will be called an n-star. The common point is called the star-centre. A colouring of the form described in the above proof will be called a star-colouring; the points of $V \setminus I$ form the star-centres. Since the cardinality of I cannot exceed the values stated, no star-colouring of S can lead to a lower estimate for $\chi''(S)$. However, stars are not the only configurations which avoid parallel blocks. In particular, an $STS(7)$ (Fano plane) has the form shown below and this avoids parallel blocks.



Configurations obtained from this by deleting lines (Fano subconfigurations) will also have the same property. All of these are potential colour classes for a colouring of an $STS(v)$. Any configuration which avoids parallel blocks must be either a Fano subconfiguration or a star. Hence the only configurations with more than 7 blocks which avoid parallel blocks are stars.

If it were possible to colour an $STS(v)$ in less colours than the estimate given in Theorem 2.1 then the average number of blocks per colour class would be at least $v/3$. This suggests that, for large v, the star-colouring estimate of Theorem 2.1 may be best possible. Theorem 2.2 establishes this conjecture.

Theorem 2.2

If S is any $STS(v)$ with $v \geq 39$ then

$$\chi''(S) \geq \begin{cases} (v+1)/2 \text{ if } v \equiv 1 \text{ or } 9 (\text{mod } 12), \\ (v-1)/2 \text{ if } v \equiv 3 \text{ or } 7 (\text{mod } 12). \end{cases}$$

Proof

Suppose that S is an $STS(v)$ with $\chi''(S) = k$ where

$$k < \begin{cases} (v+1)/2 \text{ if } v \equiv 1 \text{ or } 9 (\text{mod } 12), \\ (v-1)/2 \text{ if } v \equiv 3 \text{ or } 7 (\text{mod } 12). \end{cases}$$

We shall prove that for $v \geq 39$ there is a k-colouring in which all the colour classes are necessarily stars and that this leads to a contradiction.

Let s denote the number of colour classes which are stars. We may assume that none of the blocks in the $(k-s)$ non-star colour classes contains one of the star centres. (If it did then it could be transferred to the appropriate star colour class. This might create a new star colour class and the process might have to be repeated; however, it will eventually terminate.)

We shall suppose $s < k$ and obtain a contradiction. Firstly we count the blocks in the star colour classes. Since no block lying in a non-star colour class contains a star-centre, each star-centre is incident with $(v-1)/2$ blocks lying in the star colour classes. Counting these by star-centres gives a total of $s(v-1)/2$ blocks; however some blocks will contain two or three star-centres. If a blocks contain exactly two star-centres and b blocks contain exactly three star-centres then counting pairs gives

$$a + 3b = \frac{s(s-1)}{2}$$

Eliminating the multiple counting of such blocks gives the number of blocks in the star colour classes as

$$\frac{s(v-1)}{2} - a - 2b = \frac{s(v-1)}{2} - \frac{s(s-1)}{3} - \frac{a}{3}.$$

The number of blocks, say l, in the non-star colour classes is therefore

$$l = \frac{v(v-1)}{6} - \frac{s(v-1)}{2} + \frac{s(s-1)}{3} + \frac{a}{3}$$

$$= \frac{(v-s)(v-1-2s)}{6} + \frac{a}{3}$$
$$\geq \frac{(v-s)(v-1-2s)}{6}$$

On the other hand $l \leq 7(k-s)$.

Thus $\dfrac{(v-s)(v-1-2s)}{6} \leq 7(k-s)$.

We will make extensive use of this inequality in section 3.

Put $f(x) = (v-x)(v-1-2x)/6 - 7(k-x)$.Then $f'(x) = (4x-3v+43)/6$.

Case (i) $v \equiv 1$ or 9 (mod 12).
We have $k < (v+1)/2$ and so $k \leq (v-1)/2$. Also $s < k$, so $s \leq (v-3)/2$.
For $0 < x < (v-3)/2$, $f'(x) < (37-v)/6 \leq 0$ if $v \geq 37$.
Thus f is strictly decreasing on $[0,\ (v-3)/2]$ if $v \geq 37$. Hence

$$f(s) \geq f((v-3)/2) = \frac{(11v-30)}{3} - 7k$$
$$\geq \frac{11v-30}{3} - \frac{7(v-1)}{2} = \frac{v-39}{6}$$
$$> 0, \ \text{if } v > 39.$$

Thus, if $v > 39$, the average number of blocks in the $(k-s)$ non-star colour classes exceeds 7, which is impossible. Consequently $s = k$.

Case (ii) $v \equiv 3$ or 7 (mod 12).
We have $k < (v-1)/2$ and so $k \leq (v-3)/2$. Also $s < k$, so $s \leq (v-5)/2$.
For $0 < x < (v-5)/2$, $f'(x) < (33-v)/6 \leq 0$ if $v \geq 33$.
Thus f is strictly decreasing on $[0, (v-5)/2]$ if $v \geq 33$. Hence

$$f(s) \geq f((v-5)/2) = \frac{23v-95}{6} - 7k$$
$$\geq \frac{23v-95}{6} - \frac{7(v-3)}{2} = \frac{v-16}{3}$$
$$> 0, \ \text{if } v \geq 33.$$

Thus if $v \geq 33$ the average number of blocks in the $(k-s)$ non-star colour classes exceeds 7, which is a contradiction. Consequently $s = k$.

Combining cases (i) and (ii) we find that $s = k$ for all admissible $v \geq 39$. The complement of the star-centres therefore forms an independent set I of cardinality

$$v - k > \begin{cases} (v-1)/2 \text{ if } v \equiv 1 \text{ or } 9 (\text{mod } 12), \\ (v+1)/2 \text{ if } v \equiv 3 \text{ or } 7 (\text{mod } 12). \end{cases}$$

This is a contradiction and, consequently, for $v \geq 39$

$$\chi^{''}(S) \geq \begin{cases} (v+1)/2 \text{ if } v \equiv 1 \text{ or } 9 (\text{mod } 12), \\ (v-1)/2 \text{ if } v \equiv 3 \text{ or } 7 (\text{mod } 12). \end{cases}$$

Corollary
For $v \geq 39$,

$$\underline{\chi}^{''}(v) = \begin{cases} (v+1)/2 \text{ if } v \equiv 1 \text{ or } 9 (\text{mod } 12), \\ (v-1)/2 \text{ if } v \equiv 3 \text{ or } 7 (\text{mod } 12). \end{cases}$$

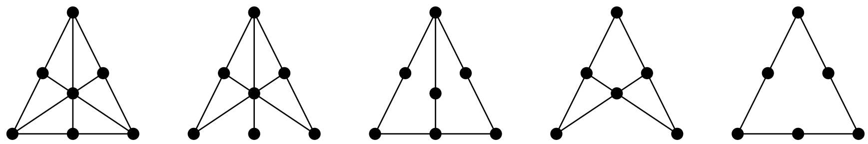
In section 3 we determine $\underline{\chi}^{''}(v)$ for all the admissible values of $v < 39$. To conclude section 2 we look at the upper bound for $\chi^{''}(S)$.

Theorem 2.3
$\overline{\chi}^{''}(v) \leq v - c\sqrt{v \text{ log } v}$ for some absolute constant c.

Proof
Phelps and Rodl [7] established that every $STS(v)$ has an independent set of cardinality $n \geq c\sqrt{v \text{ log } v}$ for some absolute constant c. A star-colouring of an $STS(v)$ based on the complement of its maximal independent set will therefore have at most $v - c\sqrt{v \text{ log } v}$ colours and avoid monochromatic parallel blocks.

3. Admissible values of v less than 39

A colour class which avoids parallel blocks and is not a star must be an $STS(7)$ or one of the following five Fano subconfigurations



Semihead Mia Sail Pasch Triangle

The names of these configurations have, with the exception of the sail, become traditional.

3.1 <u>$v = 3{,}7$ or 9</u>

For v=3 or 7 the system is unique (up to isomorphism) and contains no parallel blocks. Hence $\chi''(S) = 1$ in each case and consequently, for $v = 3$ or 7, $\underline{\chi}''(v) = \overline{\chi}''(v) = 1$. For $v = 9$ the system is unique and may be resolved into parallel classes each containing three blocks. Thus $\chi''(S) \geq 3$. Because $STS(9)$ is anti-Pasch, no colour class can contain more than 4 blocks. Hence if $\chi''(S) = 3$, each colour class must contain precisely 4 blocks. This can be achieved only by (a) three sails or (b) a 4-star and two sails. An example of each type of colouring is given below.

(a)

Sail	Sail	Sail
$\{2,1,3\}$	$\{6,9,3\}$	$\{1,4,7\}$
$\{2,5,8\}$	$\{6,4,5\}$	$\{1,6,8\}$
$\{2,9,4\}$	$\{6,2,7\}$	$\{1,5,9\}$
$\{3,8,4\}$	$\{3,5,7\}$	$\{7,8,9\}$

(b)

4-Star	Sail	Sail
$\{1,2,3\}$	$\{5,7,3\}$	$\{9,3,6\}$
$\{1,4,7\}$	$\{5,2,8\}$	$\{9,4,2\}$
$\{1,5,9\}$	$\{5,6,4\}$	$\{9,8,7\}$
$\{1,6,8\}$	$\{3,8,4\}$	$\{6,2,7\}$

Hence $\chi''(S) = 3$ and therefore $\underline{\chi}''(9) = \overline{\chi}''(9) = 3$.

3.2 <u>$v = 13$</u>

For $v = 13$ there are precisely two non-isomorphic systems. One of these systems is cyclic and may be generated by the starters $\{0,1,4\}$, $\{0,2,7\}$ under $i \mapsto i + 1 \pmod{13}$. A 6-colouring is provided by taking three star-centres at the points 0, 2 and 7, a Pasch configuration, a triangle and a star-centre at the point 3.

6-Star	5-Star	5-Star	Pasch	Triangle	3-Star
$\{0,2,7\}$	$\{2,4,9\}$	$\{5,7,12\}$	$\{4,6,11\}$	$\{4,5,8\}$	$\{1,3,8\}$
$\{6,8,0\}$	$\{8,10,2\}$	$\{7,9,1\}$	$\{10,12,4\}$	$\{5,6,9\}$	$\{3,5,10\}$
$\{11,0,5\}$	$\{1,2,5\}$	$\{3,4,7\}$	$\{12,1,6\}$	$\{8,9,12\}$	$\{9,11,3\}$
$\{0,1,4\}$	$\{2,3,6\}$	$\{6,7,10\}$	$\{10,11,1\}$		
$\{9,10,0\}$	$\{11,12,2\}$	$\{7,8,11\}$			
$\{12,0,3\}$					

If we replace the Pasch configuration by the opposite Pasch configuration (i.e. by the blocks $\{1,10,12\}$, $\{1,6,11\}$, $\{4,10,11\}$, $\{4,6,12\}$) then we obtain the other $STS(13)$, again with a 6-colouring. It follows that $\chi''(S) \leq 6$ for each of the $STS(13)$s; this is better than the bound of 7 given by Theorem 2.1.

Next we show that neither $STS(13)$ can be coloured using just 5 colours. If it were possible to use just 5 colours then, necessarily, the sizes of the 5 colour classes would be 6,5,5,5,5. This is because there are 26 blocks to colour, the largest possible star is a 6-star, neither system contains an $STS(7)$ or a semihead, and the inclusion of a 6-star precludes the inclusion of any further 6-stars. The class containing 6 blocks must be a 6-star; removing this from the system leaves 12 points each lying in 5 blocks. Of the four remaining colour classes, at most two can be 5-stars.

Suppose that exactly two of these remaining classes were 5-stars. Removing them would leave 10 points each of valency 3 (i.e. lying in 3 blocks). The valencies of the 7 points of a mia are 3,3,2,2,2,2,1. It is clear that it is impossible for the remaining two classes to be mias. Hence, at most one of the remaining four classes is a 5-star.

Suppose that exactly one of the classes were a 5-star. Removing this would leave 1 point of valency 5 and 10 points each of valency 4. Again it is easy to see that this cannot be accomplished using three mias. Thus <u>none</u> of the four classes containing 5 blocks can be a 5-star, and so all must be mias.

There are just five non-isomorphic designs obtained from an $STS(13)$ by deleting a 6-star [6]. None of these designs contains four block-disjoint mias; in fact none even contains four block-disjoint Pasch configurations. We list below the five designs. Against each block which appears in a Pasch configuration we identify with letters $A, B, C, \ldots$ the Pasch configuration(s) of which it forms part. Thus the four blocks labelled A form one Pasch configuration, the four labelled B form a second, and so on. It is then easy to check from the tabulation that none of the five designs contains four block-disjoint Pasch configurations.

Design ♯1 (7 Pasch configurations)

1	2	3		2	4	6	ADE	3	5	12	G	5	8	11	CG
1	4	5	A	2	5	7	AF	3	6	10	BD	5	9	10	CF
1	6	7	AB	2	8	10	D	3	7	11	B	6	8	12	
1	8	9	C	2	9	12	EF	4	7	9		6	9	11	E
1	10	11	BC	3	4	8	DG	4	11	12	EG	7	10	12	F

Design ♯2 (4 Pasch configurations)

1	2	3	A	2	4	6	BC	3	5	12	D	5	6	10	
1	4	5	B	2	5	7	B	3	7	11		5	8	11	D
1	6	7	B	2	8	10	A	3	9	10	A	6	8	12	
1	8	9	A	2	9	12	C	4	7	9		6	9	11	C
1	10	11		3	4	8	D	4	11	12	CD	7	10	12	

Design ♯3 (4 Pasch configurations)

1	2	3	A	2	4	6	B	3	6	13	D	5	6	10	D
1	4	5	B	2	5	7	BC	3	7	11		5	8	11	C
1	6	7	B	2	8	10	A	3	9	10	AD	5	9	13	D
1	8	9	A	2	11	13	C	4	7	9		6	9	11	
1	10	11		3	4	8		4	10	13		7	8	13	C

Design ♯4 (5 Pasch configurations)

1	2	3		2	4	6	AC	3	5	12	E	5	8	11	DE
1	4	5	A	2	5	7	AD	3	6	13		5	9	13	
1	6	7	AB	2	9	12	C	3	7	11		6	8	12	B
1	8	9		2	11	13	D	4	7	9		6	9	11	C
1	12	13	B	3	4	8	E	4	11	12	CE	7	8	13	BD

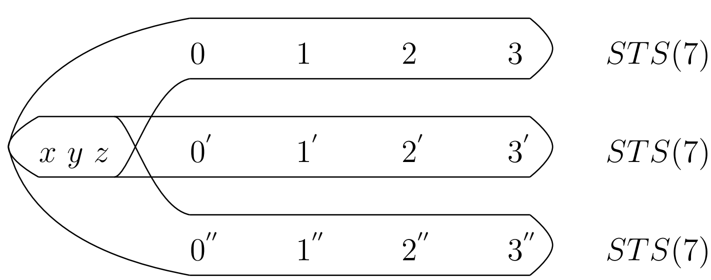
Design ♯5 (5 Pasch configurations)

1	2	3	A	2	4	6	C	3	5	12	D	5	6	10	E
1	4	5		2	8	10	A	3	6	13	E	5	8	11	D
1	8	9	A	2	9	12	C	3	9	10	AE	5	9	13	E
1	10	11	B	2	11	13		4	10	13	B	6	8	12	
1	12	13	B	3	4	8	D	4	11	12	BCD	6	9	11	C

It follows that neither $STS(13)$ can be coloured with 5 colours and so $\underline{\chi}''(13) = \overline{\chi}''(13) = 6$.

3.3 $\underline{v = 15}$

There are eighty non-isomorphic $STS(15)$s ([14]; see also [6]). In section 4 we determine the value of $\chi''(S)$ for each of these systems S. However, it is easy to prove that $\underline{\chi}''(15) = 6$. Suppose that an $STS(15), S$ say, could be coloured in 5 colours. Since S has 15 points, each of valency 7, a colour class cannot contain 8 blocks. Hence all 5 colour classes contain exactly 7 blocks. An $STS(15)$ cannot contain two block-disjoint $STS(7)$s. Consequently at least four of the colour classes are 7-stars, which is impossible. Thus no $STS(15)$ can be coloured in 5 colours. On the other hand there is an $STS(15)$ which can be coloured in 6 colours



19 distinct blocks are formed from the blocks of the three $STS(7)$s indicated. The remaining 16 blocks are formed as the triples $\{a, b', c''\}$ such that $a+b+c \equiv 0 \pmod 4$. A 6-colouring can be obtained by taking the 4 star-centres at 0, 1, 2 and 3, the $STS(7)$ on $\{x, y, z, 0', 1', 2', 3'\}$ and a semihead on $\{x, y, z, 0'', 1'', 2'', 3''\}$ (with $\{x, y, z\}$ as the missing block). From this it follows that $\underline{\chi}''(15) = 6$.

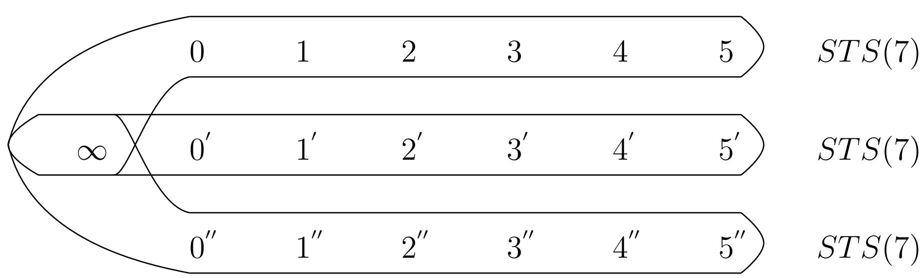
3.4 $\underline{v = 19}$

The number of non-isomorphic $STS(19)$s exceeds 2×10^6 [11]. The estimate of Theorem 2.1 gives $\underline{\chi}''(19) \leq 9$. Any improvement on this bound requires an $STS(19)$ in which some of the colour classes are non-stars. Using the approach employed in the proof of Theorem 2.2, we may take the number of blocks, l, in the non-star colour classes to be given by

$$l = \frac{(9-s)(19-s)}{3} + \frac{a}{3},$$

where s is the number of star colour classes and a is the number of blocks containing exactly two star centres. Note that $a = 0$ for $s = 0$ and $s = 1$, while $a = 1$ for $s = 2$.

It is easy to see that a 7-colouring is impossible because l exceeds $7(7-s)$ for $s = 0, 1, ..., 6$. However, 8-colourings might be possible for $s = 1, 2, 3, 4, 5$ or 6 because l might not exceed $7(8-s)$ in these cases. In fact an 8-colouring is possible for $s = 6$ as the following example shows



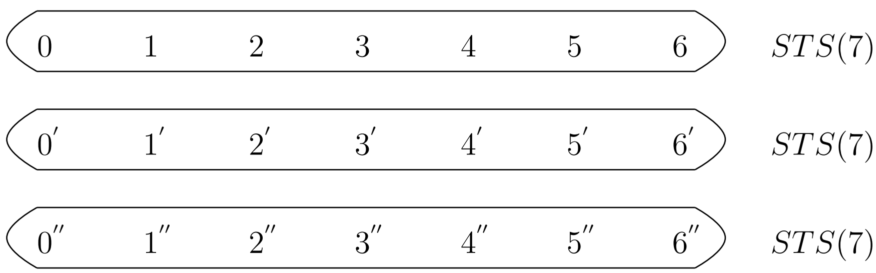
The $STS(19)$ is formed from the 21 blocks of the three $STS(7)$s indicated together with all triples of the form $\{a, b', c''\}$ such that $a + b + c \equiv 0 \pmod 6$. An 8-colouring can be obtained by taking 6 star-centres at 0,1,2,3,4,5, and the $STS(7)$s on $\{\infty, 0', 1', 2', 3', 4', 5'\}$ and on $\{\infty, 0'', 1'', 2'', 3'', 4'', 5''\}$. From this it follows that $\underline{\chi}''(19) = 8$.

3.5 $\underline{v = 21}$

The estimate of Theorem 2.1 gives $\underline{\chi}''(21) \leq 11$. If there is a better colouring then Theorem 2.2 gives one with the number of blocks in the non-star colour classes as

$$l = \frac{(10-s)(21-s)}{3} + \frac{a}{3}$$

Since l exceeds $7(8-s)$ for $s = 0, 1, ..., 7$ we find that an 8-colouring is impossible. On the other hand a 9-colouring is possible as the following example shows



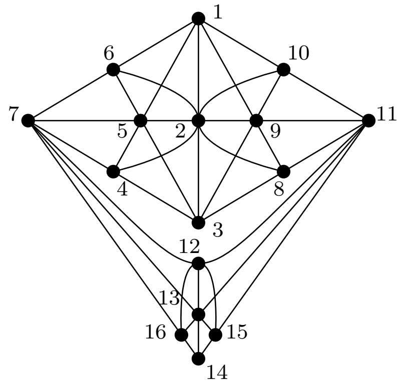
The $STS(21)$ is formed from the 21 blocks of the three $STS(7)$s indicated together with all triples of the form $\{a, b', c''\}$ such that $a + b + c \equiv 0 \pmod 7$. A 9-colouring can be obtained by taking 7 star-centres at 0,1,2,3,4,5,6 and the $STS(7)$s on $\{0', 1', 2', 3', 4', 5', 6'\}$ and on $\{0'', 1'', 2'', 3'', 4'', 5'', 6''\}$. From this it follows that $\underline{\chi}''(21) = 9$.

3.6 $\underline{v = 25}$

The estimate of Theorem 2.1 gives $\underline{\chi}''(25) \leq 13$. If there is a better colouring then Theorem 2.2 gives one with the number of blocks in the non-star colour classes as

$$l = \frac{(12-s)(25-s)}{3} + \frac{a}{3}$$

Since l exceeds $7(11-s)$ for $s = 0, 1, ..., 10$, we find that an 11-colouring is impossible. However there is an $STS(25)$ with a 12-colouring having three non-star colour classes forming the configuration described below.



Colour classes		
♯1	♯2	♯3
{1,2,3}	{1,8,9}	{7,11,12}
{1,4,5}	{1,10,11}	{7,13,15}
{1,6,7}	{2,8,10}	{7,14,16}
{2,4,6}	{2,9,11}	{11,13,16}
{2,5,7}	{3,8,11}	{11,14,15}
{3,4,7}	{3,9,10}	{12,13,14}
{3,5,6}		{12,15,16}

These colour classes form, respectively, an $STS(7)$, a semihead and a further $STS(7)$. In order to extend this configuration to an $STS(25)$ with a 12-colouring we introduce 9 new points $A, B, C, D, E, F, G, H, I$ and arrange matters so that all the missing pairs of elements from $\{1, 2, ..., 16\}$ each occur with one of these points. The remaining 9 colour classes are stars centred on the points $A - I$. This can be done as follows.

Point	Pairs						
A	$\{2, 16\}$	$\{3, 15\}$	$\{4, 8\}$	$\{5, 13\}$	$\{6, 12\}$	$\{7, 10\}$	$\{9, 14\}$
B	$\{3, 16\}$	$\{4, 13\}$	$\{5, 14\}$	$\{6, 10\}$	$\{8, 12\}$	$\{9, 15\}$	
C	$\{2, 15\}$	$\{3, 12\}$	$\{4, 14\}$	$\{5, 10\}$	$\{6, 11\}$	$\{8, 13\}$	$\{9, 16\}$
D	$\{1, 16\}$	$\{2, 12\}$	$\{4, 10\}$	$\{5, 15\}$	$\{6, 13\}$	$\{7, 9\}$	$\{8, 14\}$
E	$\{1, 14\}$	$\{2, 13\}$	$\{4, 11\}$	$\{5, 16\}$	$\{6, 9\}$	$\{8, 15\}$	$\{10, 12\}$
F	$\{2, 14\}$	$\{4, 12\}$	$\{5, 9\}$	$\{6, 15\}$	$\{8, 16\}$	$\{10, 13\}$	
G	$\{1, 15\}$	$\{3, 13\}$	$\{4, 9\}$	$\{5, 12\}$	$\{6, 16\}$	$\{7, 8\}$	$\{10, 14\}$
H	$\{1, 13\}$	$\{3, 14\}$	$\{4, 16\}$	$\{5, 11\}$	$\{6, 8\}$	$\{9, 12\}$	$\{10, 15\}$
I	$\{1, 12\}$	$\{4, 15\}$	$\{5, 8\}$	$\{6, 14\}$	$\{9, 13\}$	$\{10, 16\}$	

The system may then be completed with the blocks

$\{A, B, 11\}$	$\{A, C, 1\}$	$\{B, C, 7\}$	$\{D, E, 3\}$
$\{D, F, 11\}$	$\{E, F, 7\}$	$\{G, H, 2\}$	$\{G, I, 11\}$
$\{H, I, 7\}$	$\{B, F, 1\}$	$\{B, I, 2\}$	$\{F, I, 3\}$
$\{C, D, G\}$	$\{A, E, H\}$	$\{A, F, G\}$	$\{B, D, H\}$
$\{C, E, I\}$	$\{C, F, H\}$	$\{A, D, I\}$	$\{B, E, G\}$

These blocks can be adjoined to the stars above to complete the colour classes. It follows that $\underline{\chi}''(25) = 12$.

3.7 $\underline{v = 27}$

Theorem 2.1 gives $\underline{\chi}''(27) \leq 13$. If there is a better colouring then Theorem 2.2 gives one with the number of blocks in the non-star colour classes as

$$l = \frac{(13-s)(27-s)}{3} + \frac{a}{3}$$

Since l exceeds $7(12-s)$ for $s = 0, 1, \ldots, 11$ we find that a 12-colouring is impossible. Hence $\underline{\chi}''(27) = 13$.

3.8 $\underline{v = 31}$

Theorem 2.1 gives $\underline{\chi}''(31) \leq 15$. If there is a better colouring then Theorem 2.2 gives one with the number of blocks in the non-star colour classes as

$$l = \frac{(15-s)(31-s)}{3} + \frac{a}{3}$$

Since l exceeds $7(14-s)$ for $s = 0,1,\ldots,13$ we find that a 14-colouring is impossible. Hence $\underline{\chi}''(31) = 15$.

3.9 $\underline{v = 33}$

Theorem 2.1 gives $\underline{\chi}''(33) \leq 17$. If there is a better colouring then Theorem 2.2 gives one with the number of blocks in the non-star colour classes as

$$l = \frac{(16-s)(33-s)}{3} + \frac{a}{3}$$

We find that $l > 7(16-s)$ for $s = 0,1,\ldots,12$ (in the case $s = 12$, using the fact that $a > 0$) but that $l \leq 7(16-s)$ for $s = 13$ ($a = 0$ or 3), for $s = 14$ ($a = 1$ or 4), and for $s = 15$ ($a = 0$ or 3). We shall prove that, in fact, none of these corresponds to an $STS(33)$ which is 16-colourable. The case $s = 14$ is easily eliminated because $s = 14$ implies that $a \geq 7$. The other cases are dealt with individually below.

(a) $s = 13$, $a = 0$, $l = 20$. In this case the star-centres form an $STS(13)$. Moreover, each star-centre is associated with a 1-factor on the remaining 20 points, and these 1-factors are disjoint. What remains aside from these 1-factors is a set of pairs of these 20 points, every point appearing in precisely 6 pairs. Because $l = 20$, these pairs form 20 blocks and consequently we require them to form two $STS(7)$s and one semihead. Such a combination would appear to have 21 points and hence two of the three components must overlap at a single point. Now if two $STS(7)$s overlap at a single point then that point lies in 12 pairs, which is not possible. If an $STS(7)$ and a semihead have a point in common then that point lies in either 12 pairs or 10 pairs, again a contradiction.

(b) $s = 13$, $a = 3$, $l = 21$. In this case the star-centres form an $STS(13)$ with a block deleted. As before we obtain a set of pairs on the remaining 20 points, every point appearing in 6 pairs and a further three of these points each appearing in a further 2 pairs. Moreover, these 63 pairs must be packed into 21 blocks forming three $STS(7)$s. As before, two of the $STS(7)$s must overlap in a single point and this overlapping point will then appear in 12 pairs, a contradiction.

(c) $s = 15, a = 0, l = 6$. In this case the star-centres form an $STS(15)$. Each star-centre accounts for a 1-factor on the remaining 18 points and these 1-factors are disjoint. What remains is a set of pairs of these 18 points, every point appearing in exactly 2 pairs. These pairs cannot appear in the single semihead required by the condition $l = 6$ because there are too many points.

(d) $s = 15$, $a = 3$, $l = 7$. The star-centres form an $STS(15)$ with a block deleted. As before we obtain a set of pairs on the remaining 18 points, every point appearing in 2 pairs and a further three of these points each appearing in a further 2 pairs. These pairs cannot appear in the single $STS(7)$ required by the condition $l = 7$ because, again, there are too many points.

We can therefore conclude that $\underline{\chi}''(33) = 17$.

3.10 $\underline{v = 37}$

Theorem 2.1 gives $\underline{\chi}''(37) \leq 19$. If there is a better colouring then Theorem 2.2 gives one with the number of blocks in the non-star colour classes as

$$l = \frac{(18 - s)(37 - s)}{3} + \frac{a}{3}$$

Now l clearly exceeds $7(18 - s)$ for $s = 0,1,...,15$.
For $s = 16$, $a > 0$ and so $l > 14 = 7(18 - s)$.
For $s = 17$, $a \geq 4$ and so $l \geq 8 > 7(18-s)$. It follows that $\underline{\chi}''(37) = 19$.

4. An analysis of the $STS(15)$s

We use the standard numbering of the eighty non-isomorphic $STS(15)$s as given in [6] from which other information used in this section may also be obtained. Denote the 2-parallel chromatic index of $STS(15)$, $\sharp n$ by $\chi''(\sharp n)$. In section 3.3 it was shown that $\chi''(\sharp n) \geq 6$, $1 \leq n \leq 80$. On the other hand it is straightforward to show that $\chi''(\sharp n) \leq 7$, $1 \leq n \leq 80$. Because this is easy we do not give a complete formal proof but outline an effective strategy from which a reader will readily be able to construct such colourings.

Strategy

1. For any system containing an $STS(7)$ as a subsystem, a 7-colouring can be obtained by taking star-centres at the points of the $STS(7)$. This accounts for systems $\sharp 1 - 22$ and $\sharp 61$.

2. For all other systems except $\sharp 80$, which is the unique anti-Pasch $STS(15)$, choose the first colour class to be a mia (5 blocks).
Next choose a block which is point-disjoint from the mia and take star-centres at each point of the block (7+6+6=19 blocks). This leaves 11 blocks to be assigned to three further colour classes. The flexibility inherent in the choice of the mia and point-disjoint block enables this to be an effective method.

3. A 7-colouring of system $\sharp 80$ is as follows. Take three star-centres at the points 1, 2 and 3. The remaining 16 blocks can be assigned to four further colour classes as indicated below.

4-Star	Sail	Sail	Sail
$\{4,7,12\}$	$\{5,6,14\}$	$\{7,8,11\}$	$\{5,12,15\}$
$\{4,8,13\}$	$\{5,9,10\}$	$\{7,10,15\}$	$\{6,8,12\}$
$\{4,9,14\}$	$\{5,11,13\}$	$\{7,13,14\}$	$\{6,9,15\}$
$\{4,11,15\}$	$\{6,10,13\}$	$\{8,10,14\}$	$\{9,11,12\}$

We now prove a number of theorems which enable us to determine the 2-parallel chromatic index of all eighty non-isomorphic $STS(15)$s.

Theorem 4.1
Let S by an STS(15). If S contains both an $STS(7)$ and a semihead which is block-disjoint from the STS(7), then $\chi''(S) = 6$.

Proof
Let the base set of the STS(15) be $V = L \cup N$, where $L = \{A,B,C,D,E,F,G\}$ and $N = \{0,1,2,3,4,5,6,7\}$. Further let the STS(7) be on L. The structure of the remaining blocks is well known. Each contains precisely one element of L and two elements of N. Denote by $F_x, x \in L$, the collection of all these blocks which contain x. Then for each $x \in L$, the pairs of elements of N from F_x form a one-factor of K_8 on the set N and the set of all such one-factors are a one-factorization. WLOG the block-disjoint semihead can be assumed to be

$$\{A,0,1\}, \{A,2,3\}, \{B,0,2\}, \{B,1,3\}, \{C,0,3\}, \{C,1,2\}.$$

(It is immaterial whether $\{A,B,C\}$ is a block of the STS(15)). A 6-colouring can be obtained by choosing two non-star colour classes to be the STS(7) and block-disjoint semihead, and star-centres at the points 4,5,6 and 7.

Corollary 4.1.1
For $n \in \{m : 1 \leq m \leq 18, m \neq 11, 12\}, \chi''(\sharp n) = 6$.

Proof
All systems referred to in the corollary contain an $STS(7)$, [6]. Computer analysis further shows that they all contain a block-disjoint semihead. (Indeed systems $\sharp$ 1-7 contain at least two $STS(7)$s intersecting in a common block).

Alternatively the proof can be framed in terms of the one-factorization(s) induced by the remaining blocks after the $STS(7)$ has been assigned to the first non-star colour class. There are precisely six non-isomorphic one-factorizations of K_8 and these are tabulated on page 93 of [13]. It is immediately clear that one-factorizations F_1, F_2, F_3 and F_4 generate the required block-disjoint semihead whereas F_5 and F_6 do not.
Again using a computer, the $STS(15)$s containing an $STS(7)$ which

give F_1, F_2, F_3 and F_4 are exactly those stated in the corollary.

Theorem 4.2
Any 6-colouring of an $STS(15)$ must contain as colour classes either

(a) a semihead and a Fano configuration, or
(b) two semiheads, or
(c) a Fano configuration and three mias.

Proof
Firstly suppose that there is a 6-colouring in which every non-star colour class contains at most 5 blocks. The maximum number of blocks which can be contained in six colour classes is achieved by firstly taking star-centres at each point of a block (7+6+6=19 blocks) and then three further colour classes each containing 5 blocks. This totals to 34 blocks, one less than the number of blocks in an $STS(15)$. This is a contradiction and hence there exists at least one non-star colour class containing at least 6 blocks. Distinguish the two cases in which the largest such class is (i) a Fano configuration and (ii) a semihead. As in the proof of Theorem 4.1 let the base set of the $STS(15)$ be $V = L \cup N$ where $L = \{A, B, C, D, E, F, G\}$ and $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

Case(i) Fano configuration.
WLOG it can be assumed that this is on L and the structure of the $STS(15)$ is as in Theorem 4.1. Again denote by $F_x, x \in L$, the collection of all blocks not in the Fano configuration which contain x. If there is another non-star colour class consisting of a semihead (a Fano configuration is impossible since no $STS(15)$ contains two block-disjoint $STS(7)$s) this is condition (a). Otherwise the remaining 28 blocks can only be partitioned into five colour classes of 7+6+5+5+5 blocks, the first two being stars. So choose any $x \in L$, then any block $B \in F_x$, and take star-centres at each of the two elements of N contained in B. Of the 15 blocks which remain, F_x contains precisely 3 of them and each $F_y, y \neq x$, precisely two. The colour classes now can only be completed either as a 5-star and two mias or as three mias. In the former case the removal of a 5-star with a star-centre which must be an element of N will reduce four of the classes F_x to just one block and hence, in what

remains, four elements of L have valency one. It is thus impossible for the remaining 10 blocks to be partitioned into two mias. So we have condition (c).

Case(ii) Semihead.
WLOG the blocks of the $STS(15)$ must be

ADE,	AFG,	BDF,	BEG,	CDG,	CEF,	$AB0$,	$AC1$,	$BC2$
$A2\cdot$,	$A\cdot\cdot$,	$A\cdot\cdot$,	$B1\cdot$,	$B\cdot\cdot$,	$B\cdot\cdot$,	$C0\cdot$,	$C\cdot\cdot$,	$C\cdot\cdot$,
$D0\cdot$,	$D1\cdot$,	$D2\cdot$,	$D\cdot\cdot$,	$E0\cdot$,	$E1\cdot$,	$E2\cdot$,	$E\cdot\cdot$	
$F0\cdot$,	$F1\cdot$,	$F2\cdot$,	$F\cdot\cdot$,	$G0\cdot$,	$G1\cdot$,	$G2\cdot$,	$G\cdot\cdot$,	
012.								

(Set brackets and commas are omitted for clarity; a dot represents an element of N).

After removing the semihead on L as a non-star colour class, assume that no non-star colour classes containing more than 5 blocks can be chosen. The only possibility for partitioning the remaining 29 blocks into five colour classes is then 7+6+6+5+5 blocks, the first three being stars. Moreover the star-centres must be at the points 0, 1 and 2. This leaves the 10 blocks above containing 'two dots' to be partitioned into two colour classes of 5 blocks. But no element has valency ≥ 5 so there is no 5-star. Furthermore elements D, E, F and G have valency one, so again it is impossible for the 10 blocks to be partitioned into two mias. We conclude that the assumption is incorrect and that there exists a further non-star colour class containing at least 6 blocks. This is condition (b).

Corollary 4.2.1
For $n \in \{19, 20, 21, 22, 61\}, \chi''(\sharp n) = 7$.

Proof
Systems $\sharp 21, \sharp 22$ and $\sharp 61$ contain only one Fano configuration and no semiheads other than those included in the Fano. So conditions (a) and (b) of Theorem 4.2 are not fulfilled. Moreover the one-factorization induced by the non-$STS(7)$ blocks is F_6 which is the perfect one-factorization of K_8 and hence, after the removal of the Fano configuration colour class, there are no Pasch configurations remaining in

the STS(15). Therefore condition (c) is also not fulfilled.

Systems ♯19 and ♯20 also contain only one Fano configuration and no semiheads other than those included in the Fano. However the induced one-factorization in these two cases is F_5 which we give below in the same format as in [13], but with the typographical error corrected.

$$\begin{array}{rcccc} A = & 01 & 23 & 45 & 67 \\ B = & 02 & 13 & 46 & 57 \\ C = & 03 & 14 & 27 & 56 \\ D = & 04 & 16 & 25 & 37 \\ E = & 05 & 17 & 26 & 34 \\ F = & 06 & 12 & 35 & 47 \\ G = & 07 & 15 & 24 & 36 \end{array}$$

Altogether there are six Pasch configurations which are block-disjoint from the Fano configuration:

$$\begin{array}{cc} A01, A23, B02, B13; & A45, A67, B46, B57; \\ C03, C56, F06, F35; & C14, C27, F12, F47; \\ E05, E17, G07, G15; & E26, E34, G24, G36. \end{array}$$

Choosing any three of these and extending them to mias leaves 13 blocks which must be assigned to two colour classes as a 7-star and a 6-star. However as is easily verified, no matter which three of the Pasch configurations are chosen, this is not possible.

Corollary 4.2.2
For $n \in \{m : 33 \leq m \leq 80, m \neq 61\}, \chi''(\sharp n) = 7$.

Proof
None of the systems contain an STS(7). Computer analysis further shows that systems $\sharp 36 - 38$, $\sharp 42 - 46$, $\sharp 48 - 52$, $\sharp 55 - 57$, ♯60 and $\sharp 65 - 80$ contain no semiheads; see also [4] for this information. In addition systems $\sharp 33 - 35$, $\sharp 39 - 41$, ♯47, ♯53 and 54, ♯58 and 59 and $\sharp 62 - 64$ contain only one semihead.

We have thus determined the 2-parallel chromatic index of 68 of the eighty non-isomorphic STS(15)s. Remaining are systems ♯11 and ♯12 which both contain an $STS(7)$, and ♯23-32 which have no subsystems but multiple (more than one) semiheads. We deal firstly with the two systems which contain a subsystem.

<u>Theorem 4.3</u>
$\chi''(\sharp 11) = 7$; $\chi''(\sharp 12) = 6$.

<u>Proof</u>
System ♯11 contains one Fano configuration

$$\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\},$$

and two semiheads not contained within the Fano configuration:

$$\{1,2,3\},\{1,8,9\},\{1,10,11\},\{2,8,10\},\{2,9,11\},\{3,8,11\} \text{ and}$$

$$\{1,2,3\},\{1,12,13\},\{1,14,15\},\{2,12,14\},\{2,13,15\},\{3,13,14\}.$$

Note that all three configurations intersect in a common block $\{1,2,3\}$. This is crucial to the argument below. Assume that the system has a 6-colouring. Then one of the conditions (b) or (c) of Theorem 4.2 must be satisfied. If it is condition (c) then the Fano configuration above must comprise one colour class and in addition three further classes must be mias. But the induced one-factorization is F_5 and by the same argument as was applied to systems ♯19 and ♯20 in Corollary 4.2.2 this is not possible. So condition (c) is not fulfilled.

Now consider condition (b). For it to be satisfied one colour class must be the Fano configuration with the block $\{1,2,3\}$ removed and another class must be one of the two semiheads. Now if a 6-colouring existed based on these conditions, then another 6-colouring could be obtained by transferring the block $\{1,2,3\}$ to the Fano configuration. But this colouring does not satisfy condition (a) and so must satisfy condition (c). However we have previously shown that there is no 6-colouring fulfilling the latter and so must conclude that $\chi''(\sharp 11) = 7$.

System ♯12 contains one Fano configuration and six semiheads not contained within the Fano configuration. Using the Fano configuration as a non-star colour class does not lead to a 6-colouring because there are no block-disjoint semiheads and, as once again, the induced one-factorization is F_5. A 6-colouring based on the Fano configuration and 3 mias cannot be achieved, as may be seen by a similar argument to that given for systems ♯19 and ♯20. However, by choosing the colour classes in a different manner (three semiheads, a 7-star, a 6-star and a Pasch configuration) a 6-colouring can indeed be found. This is given below

Semihead	Semihead	Semihead	7-star	6-star	Pasch
$\{1,2,3\}$	$\{3,4,7\}$	$\{2,4,6\}$	$\{5,8,15\}$	$\{8,1,9\}$	$\{1,6,7\}$
$\{1,12,13\}$	$\{3,9,12\}$	$\{4,9,14\}$	$\{5,1,4\}$	$\{8,2,10\}$	$\{1,10,11\}$
$\{1,14,15\}$	$\{3,10,15\}$	$\{4,11,15\}$	$\{5,2,7\}$	$\{8,3,11\}$	$\{6,10,13\}$
$\{2,12,14\}$	$\{4,10,12\}$	$\{2,9,11\}$	$\{5,3,6\}$	$\{8,4,13\}$	$\{7,11,13\}$
$\{2,13,15\}$	$\{7,9,10\}$	$\{6,9,15\}$	$\{5,9,13\}$	$\{8,6,12\}$	
$\{3,13,14\}$	$\{7,12,15\}$	$\{6,11,14\}$	$\{5,10,14\}$	$\{8,7,14\}$	
			$\{5,11,12\}$		

To our mind this is probably the most interesting of the eighty non-isomorphic $STS(15)$s. It contains an $STS(7)$ and has the lower 2-parallel chromatic index. But unlike the other $STS(15)$s having the same properties, it is not possible for the subsystem to be one of the colour classes.

We consider next the ten systems, $\sharp 23 - 32$ containing more than one semihead.

Theorem 4.4
For $n \in \{m : 23 \leq m \leq 32, m \neq 25, 26, 29\}, \chi''(\sharp n) = 7$;
for $n \in \{25, 26, 29\}, \chi''(\sharp n) = 6$.

Proof
Since none of the above systems contain a Fano configuration, if they have a 6-colouring then condition (b) of Theorem 4.2 must be satisfied. The first step therefore is to identify the semiheads in the systems. The numbers of these is given in the table below.

System ♯:	23	24	25	26	27	28	29	30	31	32
Semiheads:	4	4	4	5	2	2	4	2	4	2

Furthermore in systems $\sharp30$ and $\sharp32$ the two semiheads intersect in a common block. Hence $\chi''(\sharp30) = \chi''(\sharp32) = 7$. Next consider system $\sharp27$. The two semiheads are

$$\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,4,6\}, \{2,5,7\}, \{3,4,7\}, \text{ and}$$

$$\{2,8,10\}, \{6,8,13\}, \{7,8,15\}, \{2,13,15\}, \{6,10,15\}, \{7,10,13\}.$$

For a 6-colouring, both of these must comprise colour classes. The remaining 23 blocks must then be assigned to four further colour classes and this can only be as 7+6+5+5 blocks or as 7+6+6+4 blocks with the 7 and 6 block classes as stars. In addition the star-centres have to be at points not contained in either of the semiheads viz. 9,11,12 and 14. The latter possibility is easily ruled out because there is no block formed entirely from these points. As for the former possibility, choosing each pair of points in turn and removing the stars leaves 10 blocks which must be assigned to two colour classes. It is simple to check whether this is possible and in each case it is not. The same process is used for system $\sharp28$ and hence $\chi''(\sharp27) = \chi''(\sharp28) = 7$.

Next consider system $\sharp23$. Although this contains four semiheads, the same strategy as outlined above can be adopted. Two of the four semiheads (apparently 6 possibilities but, in reality, only 3 because not all pairs of semiheads are block-disjoint) comprise two of the colour classes. Again the remaining 23 blocks must be partitioned into four further classes. But no three of the four semiheads are mutually block-disjoint, so the only possibilities are as above, 7+6+5+5 blocks, or 7+6+6+4 blocks with the 7 and 6 block classes as stars centred at points not contained in either of the two semihead colour classes. As before, the latter structure is easily eliminated and, in the former case, working through all possibilities for pairs of block-disjoint semiheads and star-centres at pairs of points not contained in either semihead colour class results in the 10 blocks remaining not being able to be partitioned into two colour classes in any of the cases. Thus $\chi''(\sharp23) = 7$. The same process is used for systems $\sharp24$ and $\sharp31$, again giving $\chi''(\sharp24) = \chi''(\sharp31) = 7$.

Finally systems $\sharp25$, $\sharp26$ and $\sharp29$ do contain three block-disjoint semiheads. In each of these three cases this structure can be used to find

6-colourings. These are given below.

System ♯25

Semihead	Semihead	Semihead	7-star	6-star	Pasch
$\{1,2,3\}$	$\{3,4,7\}$	$\{1,4,5\}$	$\{6,8,12\}$	$\{8,1,9\}$	$\{2,5,7\}$
$\{1,12,13\}$	$\{3,9,12\}$	$\{4,10,14\}$	$\{6,1,7\}$	$\{8,2,10\}$	$\{2,9,11\}$
$\{1,14,15\}$	$\{3,10,15\}$	$\{4,11,12\}$	$\{6,2,4\}$	$\{8,3,5\}$	$\{5,9,13\}$
$\{2,12,14\}$	$\{4,9,15\}$	$\{1,10,11\}$	$\{6,3,11\}$	$\{8,4,13\}$	$\{7,11,13\}$
$\{2,13,15\}$	$\{7,9,10\}$	$\{5,10,12\}$	$\{6,5,15\}$	$\{8,7,14\}$	
$\{3,13,14\}$	$\{7,12,15\}$	$\{5,11,14\}$	$\{6,9,14\}$	$\{8,11,15\}$	
			$\{6,10,13\}$		

System ♯26

Semihead	Semihead	Semihead	7-star	6-star	Pasch
$\{1,2,3\}$	$\{3,5,8\}$	$\{1,10,11\}$	$\{9,14,5\}$	$\{14,1,15\}$	$\{2,8,10\}$
$\{1,4,5\}$	$\{3,6,12\}$	$\{5,10,12\}$	$\{9,1,8\}$	$\{14,2,12\}$	$\{2,13,15\}$
$\{1,6,7\}$	$\{3,11,15\}$	$\{7,10,13\}$	$\{9,2,11\}$	$\{14,3,13\}$	$\{4,8,13\}$
$\{2,4,6\}$	$\{5,6,15\}$	$\{1,12,13\}$	$\{9,3,10\}$	$\{14,4,11\}$	$\{4,10,15\}$
$\{2,5,7\}$	$\{6,8,11\}$	$\{5,11,13\}$	$\{9,4,12\}$	$\{14,6,10\}$	
$\{3,4,7\}$	$\{8,12,15\}$	$\{7,11,12\}$	$\{9,6,13\}$	$\{14,7,8\}$	
			$\{9,7,15\}$		

System ♯29

Semihead	Semihead	Semihead	7-star	6-star	Pasch
$\{1,2,3\}$	$\{3,5,8\}$	$\{1,12,13\}$	$\{9,10,5\}$	$\{10,1,11\}$	$\{4,8,15\}$
$\{1,4,5\}$	$\{3,6,12\}$	$\{2,12,14\}$	$\{9,1,8\}$	$\{10,2,8\}$	$\{4,11,13\}$
$\{1,6,7\}$	$\{3,11,14\}$	$\{5,12,15\}$	$\{9,2,11\}$	$\{10,3,15\}$	$\{7,8,13\}$
$\{2,4,6\}$	$\{5,6,11\}$	$\{1,14,15\}$	$\{9,3,13\}$	$\{10,4,14\}$	$\{7,11,15\}$
$\{2,5,7\}$	$\{6,8,14\}$	$\{2,13,15\}$	$\{9,4,12\}$	$\{10,6,13\}$	
$\{3,4,7\}$	$\{8,11,12\}$	$\{5,13,14\}$	$\{9,6,15\}$	$\{10,7,12\}$	
			$\{9,7,14\}$		

Finally in this section it may be appropriate to summarize the results. Consider first systems ♯1-22 and ♯61 which contain $STS(7)$s. Systems ♯1-7 contain three (or more in the cases of ♯1 and ♯2) $STS(7)$s intersecting in a common block. A 6-colouring is obtained by choosing as the colour classes (i) any of the STS(7)s as the first class, (ii) either of the other $STS(7)$s except the common block as the second class, and (iii) four star-centres at the points of the third $STS(7)$ not in the common block. For the other systems which contain only one STS(7), the 2-parallel chromatic index depends on the particular one-factorization of K_8 inherent in the non-$STS(7)$ blocks. Using the listing in [13], if the one-factorization is F_1, F_2, F_3 or F_4 then $\chi''(\sharp n)$=6. If it is F_6 then $\chi''(\sharp n) = 7$. If it is F_5 either value can be obtained. For systems ♯11, ♯19 and ♯20, $\chi''(\sharp n) = 7$ but system ♯12 is exceptional in this respect in that it has a 6-colouring although this can not be obtained using the $STS(7)$ as one of the colour classes. For the remaining $STS(15)$s which do not contain an $STS(7)$ subsystem, all but six of them contain two or less semiheads and these have $\chi''(\sharp n) = 7$. In addition systems ♯23, ♯24 and ♯31 containing four semiheads have $\chi''(\sharp n) = 7$. Systems ♯25, ♯26 and ♯29 containing 4, 5 and 4 semiheads respectively have $\chi''(\sharp n) = 6$.

We note that the proof of these results is, essentially, mathematical rather than computational. Although a computer is used, its only function is to identify structures, particularly semiheads, within the systems. Rudi Mathon [15] observed that the 2-parallel chromatic index of an STS is the chromatic number of the complement of the block-intersection graph, and has verified the results of this section using this.

5. The case v =19

For larger values of v no such analysis as that given in the previous section for $v = 15$ is possible, if only because the number of systems is unknown (and large). But it may be possible to determine the spectrum of the 2-parallel chromatic index for particular values of v. We do not underestimate the difficulty of this. As was shown in section 2, the 2-parallel chromatic index is related to the independence number and the spectrum of the latter has only been determined for admissible $v \leq 19$, [1], [9]. In this section we offer some first thoughts on the case v=19.

Denote by $\beta(S)$, the independence number of a Steiner triple system S (i.e. the maximum cardinality of an independent set for S).

Further define

(a) $B(v) = \{\beta : \beta(S) = \beta$ for some STS(v), S$\}$

(b) $P(v) = \{\chi'' : \chi''(S) = \chi''$ for some STS(v), $S\}$

It is known, [1], [9] that $B(19) = \{7, 8, 9, 10\}$. Thus using only star colour classes $\overline{\chi}''(19) \leq 19 - 7 = 12$. We have shown that $\underline{\chi}''(19) = 8$ and thus $P(19) \subseteq \{8, 9, 10, 11, 12\}$. Consider each of these values in turn.

An $STS(19)$ having $\chi''(S) = 8$ is exhibited in section 3.4. Indeed any $STS(19)$ containing three $STS(7)$s intersecting in a common point will have 2-parallel chromatic index equal to 8 by choosing two of the three $STS(7)$s as colour classes and star centres on the six non-common points of the third $STS(7)$.

To construct an $STS(19)$ having $\chi''(S) = 9$, firstly observe that any $STS(19)$ containing an $STS(9)$ subsystem has $\chi''(S) \leq 9$ by choosing star centres at the points of the $STS(9)$. If there is a colouring with s star colour classes then Theorem 2.2 gives the number of blocks in the non-star colour classes as

$$l = \frac{(9-s)(19-s)}{3} + \frac{a}{3}$$

Since l exceeds $6(8-s)$ for $s = 0, 1, ..., 8$, it follows that if the $STS(19)$ does not contain an $STS(7)$ then no 8-colouring is possible. Thus any $STS(19)$ containing an $STS(9)$ but not an $STS(7)$ subsystem has $\chi''(S) = 9$.

To construct an $STS(19)$ having $\chi''(S) = 10$ we first prove the following result.

<u>Theorem 5.1</u>
Let S be an anti-Pasch STS(19). Then $\chi''(S) \geq 10$.

Proof

As above in any colouring with s star colour classes, the number of blocks in the non-star colour classes is

$$l = \frac{(9-s)(19-s)}{3} + \frac{a}{3}$$

Now $l > 4(8-s)$ for s=0,1,...,8 so no 8-colouring is possible and thus $\chi''(S) \geq 9$.

Further $l > 4(9-s)$ for s=0,1,...,6 so no 9-colouring using these number of star colour classes is possible. This leaves s=7,8,9 as the only remaining possibilities.

(a) s=7. Here $l = 8+a/3 > 4(9-s)$ unless $a = 0$. Then b (the number of blocks containing three star centres)=7. Thus the $STS(19)$ contains an $STS(7)$ and so is not anti-Pasch.

(b) s=8. Here $l = (11+a)/3 > 4(9-s)$ unless $a = 1$. But then b=9, contradicting the fact that the maximum number of blocks in a $PSTS(8)$ is 8.

(c) s=9. Here $l = 0$ and the star centres form an $STS(9)$. But by a theorem of Griggs and Murphy [5], no anti-Pasch $STS(19)$ contains an $STS(9)$.

We can therefore conclude that $\chi''(S) \geq 10$.

It remains to show that there does indeed exist an anti-Pasch $STS(19)$ having a 10-colouring. A sensible strategy to achieve this is firstly to choose as star centres the 7 points of a mitre configuration (a 5-block configuration isomorphic to one with blocks $\{A, B, C\}$, $\{A, D, E\}$, $\{A, F, G\}$, $\{B, D, F\}$, $\{C, E, G\}$). This accounts for 47 blocks and leaves 10 blocks which must be assigned to three further colour classes. Two of the four pairwise non-isomorphic cyclic $STS(19)$s are anti-Pasch [5]. One of these is system A2 which is generated under the action of $i \mapsto i+1 \pmod{19}$ from the starter blocks $\{0, 1, 4\}$, $\{0, 2, 12\}$, $\{0, 5, 13\}$. Choosing star centres at the points 0,2,3,10,12,17 and 18 (which are the points of a mitre) leaves the following 10 blocks which are assigned to the three colour classes below.

Sail	Triangle	3-star
$\{5,6,9\}$	$\{7,8,11\}$	$\{7,14,16\}$
$\{6,13,15\}$	$\{4,11,13\}$	$\{4,6,16\}$
$\{1,6,14\}$	$\{4,5,8\}$	$\{5,11,16\}$
$\{1,9,15\}$		

This strategy cannot be applied to system A4 which is generated under the action of the same mapping as above from the starter blocks $\{0,1,8\}$, $\{0,2,5\}$, $\{0,4,13\}$. This is the Netto system and is anti-mitre as well as anti-Pasch [3]. It would seem to be a good candidate for an $STS(19)$ having 2-parallel chromatic index equal to 11. However it also has index 10 as the following colouring shows. Choose star centres at the points 0,1,5,8,9,14 and 18. The remaining 11 blocks can then be assigned to the three colour classes below.

Sail	Sail	Triangle
$\{2,3,10\}$	$\{4,15,16\}$	$\{11,13,16\}$
$\{3,4,11\}$	$\{10,12,15\}$	$\{7,13,17\}$
$\{3,7,16\}$	$\{2,6,15\}$	$\{2,11,17\}$
$\{2,4,7\}$	$\{6,12,16\}$	

So we have proved that $\{8,9,10\} \subseteq P(19) \subseteq \{8,9,10,11,12\}$. Since the Netto system, being both anti-Pasch and anti-mitre, is in many ways an "extreme" system (and at this time is the only known $STS(19)$ with the stated properties) this suggests that there may be no $STS(19)$ with 2-parallel chromatic index equal to 11 or 12. But whether there are $STS(19)$s with 2-parallel chromatic index equal to these values remains an open question.

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