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# Triple systems with tripoints

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## Abstract

We study the existence of triple systems in which there are  $i$  points such that any pair involving them must occur 3 times in the system. We call these  $\text{TS}(3, i, v)$  systems (Steiner triple systems could then be denoted as  $\text{TS}(1, v, v)$  systems). Those cases for which  $i < 6$  are discussed.

## 1 Introduction.

In analogy with the study of triple systems possessing bipoints [4], we consider triple systems on  $v$  points of which  $i < v$  are tripoints and the remaining  $v - i$  are regular points. We may denote such systems by the notation  $\text{TS}(3, i, v)$ . Any pair that involves a tripoint must occur three times in the system.

Clearly, the number of pairs that will appear in such a system is

$$3v(v-1)/2 - 2(v-i)(v-i-1)/2 = \{v^2 + (4i-1)v - 2i(i+1)\}/2.$$

Since these pairs must be formed into triples, the number of triples is a third of this amount, and we obtain:

**Lemma 1** *A necessary condition for the existence of a triple system with  $i$  tripoints is that*

$$v^2 + (4i-1)v - 2i(i+1) \equiv 0 \pmod{6}.$$

Also, any regular point occurs with  $3i + (v-1-i)$  other points in the triples. This requires that  $v-1$  be even, and so we have a second necessary condition:

**Lemma 2** *In a triple system with tripoints,  $v$  must be odd.*

We note that, in case  $i=0$ , we have a Steiner triple system  $\text{STS}(v)$ , and these two conditions reduce to the well known requirement that  $v$  should be either congruent to 1 or 3, modulo 6.

## 2 The Case of One Tripoint.

In this case, Lemmas 1 and 2 require that  $v^2 + 3v - 4 \equiv 0 \pmod{6}$  and that  $v$  be odd. Consequently  $v \equiv 1$  or  $5 \pmod{6}$ .

First take an  $\text{STS}(6t+3)$ . Take a triple  $abc$ , and identify the elements  $a, b, c$ : then we have a system on  $6t+1$  elements with one tripoint. The number of triples is  $(2t+1)(3t+1) - 1 = 6t^2 + 5t$ , and this is the required number  $\{(6t+1)^2 + 3(6t+1) - 4\}/6$ .

The same construction works for  $v = 6t+5$ . Take an  $\text{STS}(6t+7)$ , and identify three elements that occur in a triple  $abc$ . The number of triples remaining is  $(6t+7)(t+1) - 1 = 6t^2 + 13t + 6$ , and this is again the required number  $\{(6t+5)^2 + 3(6t+5) - 4\}/6$ .

Of course, other constructions are possible. In the case  $6t+1$ , it is easy to write down the 11 triples for  $v=7$  [take the triples 123, 456, along

with  $a_{14}, a_{25}, a_{36}, a_{15}, a_{26}, a_{34}, a_{16}, a_{24}, a_{35}$ ]. For  $v > 7$ , choose an STS( $6t + 1$ ) that contains a partial parallel class of  $2t$  triples. Let  $\infty$  be the element that does not appear in this partial parallel class. Expand the class by  $\infty$ , as in [4], and we obtain the required system with  $6t^2 + t + 2(2t)$  triples.

It is interesting to note that a triple system on  $6t + 1$  elements with a single tripoint can be viewed as an intermediate system between an ordinary STS on  $6t + 1$  elements and an ordinary STS on  $6t + 3$  elements. As we have seen, such a system can be obtained by moving up from an STS( $6t + 1$ ) or moving down from an STS( $6t + 3$ ). This fact gives a basis for an “adding 2” construction for Steiner triple systems. Take an STS( $6t + 1$ ) containing a partial parallel class  $P$ , and let  $\infty$  be the element that does not appear in this class. Define a graph  $G$  as follows. The vertex set of  $G$  is the set of all  $6t + 1$  elements of the STS( $6t + 1$ ) except for the element  $\infty$ . Two vertices  $x$  and  $y$  are joined by an edge if either  $\infty xy$  is a triple in the STS( $6t + 1$ ) or if there is a triple  $xyz$  in  $P$ . The graph  $G$  is a cubic graph whose chromatic index is, by Vizing’s Theorem, either 3 or 4. If the chromatic index is 3, then each colour class is a one-factor, and so we have three one-factors  $F_1, F_2$ , and  $F_3$ . Now delete the triples containing  $\infty$  and the parallel class  $P$ . Then introduce three new points  $a, b$ , and  $c$ , and form triples  $aF_1, bF_2, cF_3$ , to form an STS( $6t + 3$ ).

We illustrate the above procedure by a numerical example. Take the cyclic STS(13) with 26 blocks. It is generated by the triples 0 1 4 and 0 2 7 under the action of the mapping  $f(i) = i + 1 \pmod{13}$ . From the formula given in [2], this contains precisely 13 resolution classes of four triples. Choose  $P$  to be the class consisting of the triples 78B, 249, 35A, C16 (where we use  $A, B, C$ , for 10, 11, 12). Delete these triples and those containing the point 0, that is, 014, 09A, 03C, 027, 068, 05B. Introduce three new points  $d, e, f$ , and triples  $d7B, e8B, f78, d24, e49, f29, d35, e5A, f3A, d1C, e16, f6C, f14, d9A, e3C, e27, d68, f5B, def$ . This gives  $26 - 10 + 19 = 35$  triples, as required. The STS(15) obtained is #75 in the standard listing [3]. It is particularly interesting that this STS(15), which otherwise appears to have no significant feature, is the unique STS(15) that can be obtained from the cyclic STS(13) in this way.

We have also performed the above calculation for the non-cyclic STS(13). This contains precisely 8 resolution classes of four triples, and these partition into sets of 1+1+3+3 classes under the action of the automorphism group.

The STS(15)s that are obtained are #6, #43, #15, and #42 in the standard listing [3]. It is interesting to note that systems #6 and #15 contain an STS(7) subsystem.

### 3 The Case of Two Tripoints.

The conditions of Lemmas 1 and 2 require that  $v$  be odd and that  $v^2 + 7v - 12$  be a multiple of 6. Consequently,  $v \equiv 3$  or  $5 \pmod{6}$ .

First, we construct the trivial system on 3 elements. It is merely  $ab1$ ,  $ab1$ ,  $ab1$  (henceforth, we shall consistently use letters  $a, b, c$ , for the tripoints and numbers 1, 2, 3, ... for the regular points). Also, if  $v = 9$ , we may form a system  $ab1$ ,  $ab2$ ,  $ab3$ , 123,  $a45$ ,  $a67$ ,  $b46$ ,  $b57$ ,  $a47$ ,  $b56$ , along with  $a14$ ,  $a17$ ,  $a25$ ,  $a26$ ,  $a35$ ,  $a36$ , and  $b15$ ,  $b16$ ,  $b24$ ,  $b27$ ,  $b34$ ,  $b37$ .

Now the unique STS(7) has the property that there exists in it a triple  $abc$  such that  $ade$ ,  $bdf$ , and  $cef$  are also triples [take the usual representation of the system obtained by cycling the triple 124, modulo 7, and use the block 124 with blocks 137, 267, 436]. The cyclic system on 13 elements also has this property [take the representation obtained by cycling 145 and 138, mod 13, and use the triples 145, 12e, 46e, 562, where we have written e for the element 11]. This configuration is, of course, the well known Pasch configuration. The Doyen-Wilson Theorem [1] states that an STS( $v$ ) can be embedded in an STS( $w$ ) for  $w \geq 2v + 1$ . Hence, provided that  $t \geq 3$ , there exists an STS( $6t + 1$ ) that contains an STS(7) and hence a Pasch configuration.

We could also deduce this result from the fact that a triple system on  $x$  elements can be embedded in a system on  $2x + 1$  elements, and also in a system on  $2x + 7$  elements (for example, cf. [5]), along with the result of Stinson and Seah that there are many (precisely, 284 457) triple systems on 19 points that contain both an STS(7) and an STS(9) (cf. [6]).

We thus know that, for  $t \geq 3$ , there exists an STS( $6t + 1$ ) that contains triples  $abc$ ,  $ade$ ,  $bdf$ ,  $cef$ . For these values, we identify  $a$  with  $b$  and  $c$ , and we identify  $d$  with  $e$  and  $f$ . The result is a system with two tripoints. The number of triples is  $(6t+1)t-4$  and the number of points is  $(6t+1)-4 = 6t-3$ . This checks, since

$$[(6t - 3)^2 + 7(6t - 3) - 12]/6 = 6t^2 + t - 4.$$

We now introduce a useful notation. Let there be  $p, q, r$ , and  $s$  triples of type nnn, nnx, nxx, xxx, where n represents a regular point, x represents a tripoint. Then  $2(q + r) = 3(2)(v - 2)$  and  $q + r$  can not exceed the number of triples. Hence

$$\begin{aligned} q + r &\leq (v^2 + 7v - 12)/6, \text{ whence} \\ 18(v - 2) &\leq v^2 + 7v - 12, \\ 0 &\leq v^2 - 11v + 24 = (v - 3)(v - 8). \end{aligned}$$

Hence  $v$  can not lie between 3 and 8. So the first case when  $v$  is congruent to 5 occurs for  $v = 11$  (31 blocks). Such a design can be constructed as follows. Let  $F_1, \dots, F_7$  be a one-factorization of the set  $\{2, 3, 4, 5, 6, 7, 8, 9\}$ . Take the blocks  $1F_1, ab1, ab1, ab1, aF_2, aF_3, aF_4, bF_5, bF_6, bF_7$ .

The general construction for  $v \equiv 5 \pmod{6}$ , with  $v > 11$ , is the same as for  $v \equiv 3 \pmod{6}$ ; one uses an STS( $6t + 3$ ) that contains a Pasch configuration. The number of triples is given by  $(3t + 1)(2t + 1) - 4$ , and the number of points is  $(6t + 3) - 4 = 6t - 1$ . This checks, since

$$[(6t - 1)^2 + 7(6t - 1) - 12]/6 = 6t^2 + 5t - 3 = (3t + 1)(2t + 1) - 4.$$

In this section, we have used the special case of  $v = 7$  in the Doyen-Wilson Theorem that an STS( $v$ ) can be embedded in an STS( $w$ ) provided that  $w \geq 2v + 1$  (cf. [1]). This theorem is a fundamental result and we shall employ it for the cases  $v = 9$  and  $v = 15$  in subsequent sections.

## 4 The Case of Three Tripoints.

The two necessary conditions for  $i = 3$  require that  $v$  be odd and that  $v^2 + 11v - 24 \equiv 0 \pmod{6}$ . Hence  $v \equiv 1$  or  $3 \pmod{6}$ .

Again, we suppose that there are  $p, q, r$ , and  $s$  triples of type nnn, nnx, nxx, xxx, where n represents a regular point, x represents a tripoint. Then  $2(q + r) = 3(3)(v - 3)$  and  $q + r$  can not exceed the number of triples. Hence

$$\begin{aligned} q + r &\leq (v^2 + 11v - 24)/6, \text{ whence} \\ 27(v - 3) &\leq v^2 + 11v - 24, \\ 0 &\leq v^2 - 16v + 57. \end{aligned}$$

Hence  $v$  can not assume the values 7 and 9; thus  $v \geq 13$ .

**Remark.** The  $q + r$  condition for  $i$  tripoints is simply

$$0 \leq v^2 - (5i + 1)v + i(7i - 2).$$

It ceases to place restrictions on  $v$  once we have  $3i^2 - 18i - 1 \geq 0$ , that is, for  $i > 6$ .

We now take an STS( $v$ ) with an embedded STS(9); this is available, in virtue of the Doyen-Wilson Theorem, whenever  $v \geq 19$ . We may take this embedded system in the usual form as 123, 456, 789, 147, 258, 369, 159, 267, 358, 168, 249, 357. We identify elements 1,2,3, as  $a$ ; 4,5,6, as  $b$ ; 7,8,9, as  $c$ . We then remove the triples of the STS(9) and add three triples  $abc$ ,  $abc$ ,  $abc$ . The result is our required tripoint triple system. The number of points is  $v - 6$  and the number of triples is  $v(v - 1)/6 - 9$ . This checks, since

$$[(v - 6)^2 + 11(v - 6) - 24]/6 = [v^2 - v - 54]/6.$$

We thus see that our system on  $6t + 1$  or  $6t + 3$  points can be derived from an STS on  $6t + 7$  or  $6t + 9$  points by this construction. We merely need an embedded STS(9), and this is available, by the Doyen-Wilson Theorem, when we have an STS( $w$ ) with  $w \geq 19$ .

We now give the constructions for the two small values  $v = 13$  and  $v = 15$ .

For  $v = 13$ , the above construction is equivalent to the following. The number of triples is 48, and  $q + r = 45$ . We take triples  $abc$ ,  $abc$ ,  $abc$ , and then require 15 number pairs with each of  $a, b, c$ . Since we can form 9 one-factors on the ten numbers in the set  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ , this is easily arranged (put 3 one-factors with each of  $a, b, c$ ).

For  $v = 15$ , take  $abc$ ,  $abc$ ,  $abc$ , 159, 26A, 37B, 48C (where we write A, B, C, for 10,11,12). Let  $C(x)$  be the cycle union formed by adding  $x \pmod{12}$ ; thus

$$\begin{aligned} C(1) &= (123456789ABC), & C(2) &= (13579B)(2468AC), \\ C(3) &= (147A)(258B)(369C), & C(5) &= (16B4927C5A38), \\ C(6) &= (17)(28)(39)(4A)(5B)(6C). \end{aligned}$$

So  $C(6)$  is a one-factor, and  $C(x)$  is the union of two one-factors for  $x = 1, 2, 3, 5$  [we use the well known fact that a cycle union made up only of even cycles can be decomposed into two one-factors]. Then we take the triples  $aF_1, aF_2, aF_3, bF_4, bF_5, bF_6, cF_7, cF_8, cF_9$ , as the remaining triples in the system, where the  $F_i$  are the 9 one-factors just constructed from the cycle unions  $C(x)$ .

## 5 The Case of Four Tripoints.

The two necessary conditions for  $i = 4$  require that  $v$  be odd and that  $v^2 + 15v - 40$  be a multiple of 6. Hence  $v \equiv 1$  or  $5 \pmod{6}$ . Again, let there be  $p, q, r$ , and  $s$  triples of type nnn, nnx, nxx, xxx, where n represents a regular point, x represents a tripoint. Then  $2(q+r) = 3(4)(v-4)$  and  $q+r$  must be no more than the number of triples. Hence

$$\begin{aligned} q+r &\leq (v^2 + 15v - 40)/6, \text{ whence} \\ 36(v-4) &\leq v^2 + 15v - 40, \\ 0 &\leq v^2 - 21v + 104 = (v-8)(v-13). \end{aligned}$$

Hence  $v$  can not be 11, and the case 5 is trivial since the one regular point behaves like a tripoint and the system is just the BIBD(5, 10, 6, 3, 3). Consequently, the first non-trivial case with  $v$  congruent to 5 occurs for  $v = 17$ . We first construct the small cases.

For  $v = 7$ , there are 19 triples, and  $q+r = 18$ . Take a triple 123; then there are 3 triples each that contain  $ab, ac, ad, bc, bd, cd$ . We merely use  $ab1$  and  $cd1$  three times each;  $ac2$  and  $bd2$  three times each;  $ad3$  and  $bc3$  three times each. Or, if we wish a design without repeated triples, match each of 1, 2, 3, with each of  $ab, ac, ad, bc, bd, cd$ .

For  $v = 13$ ,  $q+r$  is the total number of triples, that is, 54. We have 18 blocks containing letter pairs and 36 blocks containing number pairs. One construction is as follows. Take a one-factorization of  $K_{10}$ , on the set  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, A\}$ , and delete the symbol  $A$ . This leaves the usual general one-factorization of  $K_9$  in which there are nine generalized one-factors, each consisting of 4 pairs of elements and a singleton, with the singleton ranging through the values 1, 2, 3, 4, 5, 6, 7, 8, 9; name these one-factors

$F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9$  in the obvious way. There is no loss of generality in our taking  $F_9$  as  $\{12, 34, 56, 78, 9\}$ . Now construct the following triples:  $ab$  and  $cd$  with each of 1, 2, 3 (6 triples);  $ac$  and  $bd$  with each of 4, 5, 6 (6 triples);  $ad$  and  $bc$  with each of 7, 8, 9 (6 triples); then take  $a12, b34, c56, d78$  (this uses up the pairs in  $F_9$ ), along with  $aF_1, aF_2, bF_3, bF_4, cF_5, cF_6, dF_7, dF_8$ .

For  $v = 17$ , we have  $3p + q = 78, q + r = 78, r + 3s = 18$ . There are many possible solutions, of which we present two examples.

First take the 4 tripoints as  $a, b, c, d$ ; the 13 regular points as 1, 2, 3, 4, 5, 6, 7, 8, 9,  $A, B, C, D$ , and use the assignment  $p = 1, q = 75, r = 3, s = 5$ . Start with the blocks:  $abc, abc, abd, acd, bcd, 1ad, 2bd, 3cd, 123$ . Add the following blocks.

$$\begin{array}{cccccc}
14d & 24d & 34d & 15d & 25d & 35d \\
16a & 26b & 36c & 17a & 27b & 37c \\
18b & 28c & 38a & 19b & 29c & 39a \\
1Ab & 2Ac & 3Aa & 1Bc & 2Ba & 3Bb \\
1Cc & 2Ca & 3Cb & 1Dc & 2Da & 3Db
\end{array}$$

Then take a one-factorization  $F_1, F_2, F_3, F_4, F_5, F_6, F_7$  on the  $K_8$  formed on the set  $\{6, 7, 8, 9, A, B, C, D\}$ , where we take  $F_7$  to be  $\{67, 89, AB, CD\}$ . Take as triples  $F_1, F_2, F_3$  with  $d$ ;  $F_4$  with  $a$ ,  $F_5$  with  $b$ ,  $F_6$  with  $c$ ; and  $F_7$  as follows:  $67a, 89b, ABc, CDa$ . Complete by adjoining the following triples:

$$\begin{array}{cccccccc}
46b & 56c & 47b & 57c & 48c & 58a & 49c & 59a \\
4Aa & 5Ab & 4Ba & 5Bb & 4Cb & 5Cc & 4Dc & 5Db \\
45a
\end{array}$$

As another example of a system with  $v = 17$ , consider  $p = 6, q = 60, r = 18, s = 0$ . Use a one-factorization of  $K_{12}$  on  $\{1, 2, \dots, A, B, C\}$  with factors  $F_i$  ( $i$  ranging from 1 to 11). Then take the triples  $DF_1, aF_2, aF_3, bF_4, bF_5, cF_6, cF_7, dF_8, dF_9$ . We assume that we have chosen the one-factors  $F_{10}$  and  $F_{11}$  so that their union is the 12-cycle  $(1, 2, 3, \dots, A, B, C)$ : this is always possible, for example, by selecting a perfect one-factorization of  $K_{12}$ . Then we form the triples  $ac1, ac2, ac3, ad4, ad5, ad6, bd7, bd8, bd9, bcA, bcB, bcC$ , along with  $abD, abD, abD, cdD, cdD, cdD$ . We then form triples  $a78, a9A, aBC$ ;  $b12, b34, b56$ ;  $c45, c67, c89$ ;  $dAB, dC1, d23$ . This completes the required system.

The last special case required is the case  $v = 19$ , with 4 tripoints (101 blocks). Consider the case  $p = 7, q = 84, r = 6, s = 4$ . Take blocks  $abc, abd, acd, bcd$ , as well as the blocks  $ab1, ac1, ad1, bc1, bd1, cd1$ . Take a one-factorization of  $K_{14}$  on the set  $\{2, 3, \dots, D, E\}$ ; then our remaining triples are simply  $1F_1, aF_2, aF_3, aF_4; bF_5, bF_6, bF_7; cF_8, cF_9, cF_{10}; dF_{11}, dF_{12}, dF_{13}$ .

In the next cases ( $v = 23$  and  $v = 25$ ), the general construction is applicable, since  $23 + 8$  and  $25 + 8$  are values for which the Steiner triple system involved contains an embedded STS(15).

For the general construction, we proceed in the following manner (a similar procedure will be used later in the case of 5 tripoints). Take an STS( $w$ ) which has an embedded STS(15). The latter contains a partial parallel class of 4 triples  $123, 456, 789, ABC$ . Put  $1, 2, 3 = a; 4, 5, 6 = b; 7, 8, 9 = c; A, B, C = d$ ; and retain the other 3 points. Then replace the STS(15) by the system on 7 points just constructed (the 7 points are now  $a, b, c, d, D, E, F$ ). Then we have  $v = w - 8$  points in  $w(w - 1)/6 - (35 - 19)$  triples. This checks since

$$[(w - 8)^2 + 15(w - 8) - 40]/6 = w(w - 1)/6 - 16.$$

Application of the Doyen-Wilson Theorem then shows that this procedure works for any  $w \geq 31$ ,  $w \equiv 1$  or  $3 \pmod{6}$ , that is, for any  $v \geq 23$ ,  $v \equiv 1$  or  $5 \pmod{6}$ .

## 6 The Case of Five Tripoints.

The two necessary conditions for  $i = 5$  require that  $v$  be odd and that  $v^2 + 19v - 60 \equiv 0 \pmod{6}$ . Hence  $v \equiv 3$  or  $5 \pmod{6}$ .

As before, let there be  $p, q, r$ , and  $s$  triples of type nnn, nnx, nxx, xxx, where n represents a regular point, and x represents a tripoint. Then  $2(q + r) = 3(5)(v - 5)$  and  $q + r$  must be no more than the number of triples. Hence

$$\begin{aligned} q + r &\leq (v^2 + 19v - 60)/6, \text{ whence} \\ 45(v - 5) &\leq v^2 + 19v - 60, \\ 0 &\leq v^2 - 26v + 165 = (v - 11)(v - 15). \end{aligned}$$

It follows that  $v$  can not lie between 11 and 15. This does not exclude any values. We first construct the small cases.

For  $v = 9$ , there are 4 regular points and 32 triples; also,  $q + r = 30$ . Take triples 123 and  $abc$ , along with  $14a, 24b, 34c$ . Then we need  $ab3$  and  $ab4$ ;  $ac2$  and  $ac4$ ;  $bc1$  and  $bc4$ ; each of  $ad, bd, cd, ae, be, ce$ , with each of 1, 2, 3;  $de4, de4$ .

For  $v = 11$ , take a one-factorization of  $K_6$  on the set  $\{1, 2, 3, 4, 5, 6\}$ ; name the one-factors  $F_1, F_2, F_3, F_4, F_5$ . Take as triples  $aF_1, bF_2, cF_3, dF_4$ , and  $eF_5$ . Then match each of the pairs  $ab, bc, cd, de, ea$  with each of 1, 3, 5, and match each of the pairs  $ac, ce, eb, bd, da$ . with each of 2, 4, 6.

For  $v = 15$ , we match  $a$  with 13, 58, 67, 49,  $2A$ ; 27, 36, 45,  $9A$  (omitting the pair 18). We match  $b$  with 15, 24, 78, 69,  $3A$ ; 38, 47, 56,  $1A$  (omitting the pair 29). We match  $c$  with 12, 48, 57, 39,  $6A$ ; 17, 26, 89,  $4A$  (omitting the pair 35). We match  $d$  with 14, 23, 68, 59,  $7A$ ; 28, 37, 19,  $5A$  (omitting the pair 46). We match  $e$  with 16, 25, 34, 79,  $8A$ ; 18, 29, 35, 46 (omitting the pair  $7A$ ). Finally we use  $ab$  with 1, 3, 9;  $bc$  with 2, 4, 5;  $cd$  with 3, 6, 7;  $de$  with 1, 4,  $A$ ;  $ea$  with 2, 7, 8;  $ac$  with 1, 8,  $A$ ;  $ce$  with 3, 5, 9;  $eb$  with 6, 7,  $A$ ;  $bd$  with 2, 8, 9;  $da$  with 4, 5, 6.

For  $v = 17$ , we have the usual 5 tripoints  $a, b, c, d, e$ , along with the 12 regular points 1, 2, 3, 4, 5, 6, 7, 8, 9,  $A, B, C$ . We consider the case  $p = 0, q = 66, r = 24, s = 2$ .

Take a one-factorization of a  $K_{12}$  on the regular points, and name the 1-factors  $F_1, \dots, F_{11}$ , where  $F_{11} = \{12, 34, 56, 78, 9A, BC\}$ .

Take the blocks:  $aF_1, aF_2; bF_3, bF_4; cF_5, cF_6; dF_7, dF_8; eF_9, eF_{10}$ ; along with  $12a, 56b, 9Ac, 34a, 78b, BCc$ . Complete the system with the blocks

$$\begin{array}{cccccccc}
 ad6 & aeA & bdA & be2 & cd2 & ce6 & ab9 & de1 & abc \\
 ad7 & aeB & bdB & be3 & cd3 & ce7 & ac5 & de5 & abc \\
 ad8 & aeC & bdC & be4 & cd4 & ce8 & bc1 & de9 & 
 \end{array}$$

For the construction of a tripoint triple system for values greater than 17, take an  $STS(w)$  with an embedded  $STS(15)$  containing a parallel class (for

example, a Kirkman triple system would suffice). We let the first parallel class be  $(123)$ ,  $(456)$ ,  $(789)$ ,  $(ABC)$ ,  $(DEF)$ , and we identify all elements in the same triple so that the 15 elements become 5 elements:  $a, b, c, d, e$ . We then adjoin the BIBD(5, 10, 6, 3, 3) on the elements  $a, b, c, d, e$ , to replace the original STS(15). The result is that the number of elements drops by 10 (from 15 to 5) and the number of triples drops by 25 (from 35 triples to 10). So we have  $w - 10$  elements in a tripoint triple system containing  $w(w - 1)/6 - 25$  triples. This checks, since

$$[(w - 10)^2 + 19(w - 10) - 60]/6 = w(w - 1)/6 - 25.$$

So we merely need to note that a suitable STS( $6t + 1$ ) gives a tripoint triple system on  $6t - 9$  elements and a suitable STS( $6t + 3$ ) gives a tripoint triple system on  $6t - 7$  elements. Since we have constructed all the required systems for the cases in which  $v + 10 < 31$ , the Doyen- Wilson theorem comes into play and handles all higher cases.

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