Third-regular biembeddings of Latin squares

D. M. Donovan[1],
M. J. Grannell[2] and T. S. Griggs[2]

[1] Centre for Discrete Mathematics and Computing, University of Queensland, St Lucia 4072, Australia.

[2] Department of Mathematics and Statistics, The Open University, Walton Hall, Milton Keynes MK7 6AA, United Kingdom.

Abstract

For each positive integer $n \geq 2$, there is a well-known regular orientable Hamiltonian embedding of $K_{n,n}$, and this generates a regular face 2-colourable triangular embedding of $K_{n,n,n}$. In the case $n \equiv 0 \pmod 8$, and only in this case, there is a second regular orientable Hamiltonian embedding of $K_{n,n}$. The current paper presents an analysis of the face 2-colourable triangular embedding of $K_{n,n,n}$ that results from this. The corresponding Latin squares of side n are determined, together with the full automorphism group of the embedding.

Keywords: Biembedding, Latin square, Orientable surface, Regular embedding, Transversal design.
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1 Introduction

Topological graph theory is concerned with embedding graphs in surfaces in such a way that the edges of the graph intersect only at the vertices with which they are incident. Such an embedding is called a *map*; see [7] for precise definitions. The surface may be orientable or nonorientable. Amongst the embeddings of a graph G, particular interest arises in those embeddings which possess the greatest possible symmetry. An *automorphism* of an embedding M is a bijection ϕ on the vertices of M that preserves the edges and faces of M, and the *automorphism group* of M is the set of all automorphisms which map M to M. An embedding M of a graph G is said to be *regular* if and only if for every two *flags*, i.e. triples (v_1, e_1, f_1) and (v_2, e_2, f_2), where e_i is an edge incident with the vertex v_i and the face f_i, there exists an automorphism of M which maps v_1 to v_2, e_1 to e_2 and f_1 to f_2. Plainly G can have a regular embedding M only if G is both vertex and edge transitive. Furthermore, the regularity of an embedding M requires that all the face boundaries are of the same length, and the order of the full automorphism group of M, $|Aut(M)|$, is simply the number of flags.

We point out that the definition of regularity varies somewhat between authors; see [1] (p.36) for a discussion of the terminology. The definition given here requires the admission of automorphisms which reverse the orientation of the surface in the orientable case. However, many authors require that any global orientation of the surface is preserved and this means that their regular embeddings may be less symmetric. We also remark that when we speak of uniqueness of embeddings, we mean uniqueness up to isomorphism.

In [2, 9], Biggs and White describe a regular embedding B_n, with Hamiltonian face boundaries, of the complete bipartite graph $K_{n,n}$ in an orientable surface. Subsequently it was shown in [4] that for $n \not\equiv 0 \pmod 8$ there is no further regular Hamiltonian embedding of $K_{n,n}$ in an orientable surface. In the same paper it was also established that, for $n \equiv 0 \pmod 8$, there is precisely one other regular Hamiltonian embedding B_n^*, of $K_{n,n}$ in an orientable surface, nonisomorphic with B_n.

In [8], a regular triangular embedding T_n of the complete tripartite graph $K_{n,n,n}$ in an orientable surface was given for each positive integer n. In a further paper [6], the embedding T_n was shown to be the unique regular triangular embedding of $K_{n,n,n}$ in an orientable surface. Furthermore, as proved in [5], the orientability of such a triangular embedding is equivalent to face 2-colourability. By selecting one of the three sets of the tripartition of $K_{n,n,n}$, and by deleting these vertices and the edges incident with them, one may obtain from T_n a regular Hamiltonian embedding of $K_{n,n}$ in an orientable surface. This embedding is the B_n described by Biggs and White. The existence, for $n \equiv 0 \pmod 8$, of a second regular Hamiltonian embedding B_n^* of $K_{n,n}$ in an orientable surface shows that this process cannot, in general, be reversed. Nevertheless, it is still possible to take this second embedding, insert a vertex in the interior of each face, join it by non-intersecting edges to all the vertices on the face boundary, and thereby obtain (for $n \equiv 0 \pmod 8$) a triangular embedding T_n^* of $K_{n,n,n}$ in an orientable surface. The orientability of this embedding ensures face 2-colourability.

In this paper we will determine the full automorphism group of T_n^* and show that it has order $4n^2$, that is one-third of the order of the automorphism

phism group of the regular embedding T_n. For this reason, we will refer to this embedding as the *third-regular* face 2-colourable triangular embedding of $K_{n,n,n}$. It is perhaps appropriate at this point to note that in a recent paper [3], a new infinite family of face 2-colourable triangular embeddings of complete tripartite graphs $K_{n,n,n}$ was constructed having automorphism groups of order $3n^2$, and these were called *quarter-regular*. In a face 2-colourable triangular embedding of $K_{n,n,n}$, we will take the colour classes of the faces to be black and white. Each colour class of faces determines a transversal design $TD(3, n)$ and consequently a Latin square of side n in the manner described below.

A *transversal design* of order n and block size 3, $\mathrm{TD}(3, n)$, is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a $3n$-element set (the points), $\mathcal{G}$ is a partition of V into three parts (the groups) each of cardinality n, and $\mathcal{B}$ is a collection of 3-element subsets (the blocks) of V such that each 2-element subset of V is either contained in exactly one block of $\mathcal{B}$ or in exactly one group of $\mathcal{G}$, but not both. Two $\mathrm{TD}(3, n)$s $(V, \{G_1, G_2, G_3\}, \mathcal{B})$ and $(V', \{G'_1, G'_2, G'_3\}, \mathcal{B}')$ are said to be *isomorphic* if, for some permutation π of $\{1, 2, 3\}$, there exist bijections $\alpha_i : G_i \mapsto G'_{\pi(i)}$, $i = 1, 2, 3$ that map blocks of $\mathcal{B}$ to blocks of $\mathcal{B}'$. Automorphisms of a $\mathrm{TD}(3, n)$ are defined similarly.

A *Latin square* L of side n is an $n \times n$ array in which each element of a set of entries E, of cardinality n, occurs once in each row and once in each column. If the rows and columns of L are indexed respectively by sets R and C, an equivalent representation of L as a subset of $R \times C \times E$ is obtained by listing the n^2 (row, column, entry) triples. If entry e appears in row r and column c of L, we may write $e = L(r, c)$ or $(r, c, e) \in L$. Given a Latin square of side n, a $\mathrm{TD}(3, n)$ may be obtained by taking the three groups as the sets of row labels, column labels and entries, and the blocks as the n^2 triples of the Latin square. Conversely, given a $\mathrm{TD}(3, n)$ a Latin square of side n may be formed by taking the three groups as the row labels, column labels and entries in any one of six possible orders. Thinking of Latin squares as transversal designs, two Latin squares P and Q, of side n, are said to belong to the same *main class* if the corresponding transversal designs are isomorphic.

Starting with M, a face 2-colourable triangular embedding of the complete tripartite graph $K_{n,n,n}$, the faces in each colour class (black and white) determine the blocks of a $\mathrm{TD}(3, n)$ or, equivalently, a Latin square of side n. Thus a face 2-colourable triangular embedding of $K_{n,n,n}$ may be regarded as a *biembedding* of two Latin squares P and Q of side n. The triangles of the embedding will correspond to the triples of the Latin squares which we always list in (row, column, entry) order.

2 The embedding T_n^* and its automorphism group

The embedding B_n^* was given in [4] by means of a voltage graph; we refer the reader to [7] for details of such graphs. The particular graph employed in this case was a dipole embedded in a sphere as shown in Figure 1 with voltages $a_0 = 0$ and $a_i = 1 + d + d^2 + \ldots + d^{i-1}$ for $i = 1, 2, \ldots, n-1$, where $n = 8s$, $d = 4s + 1$ and all arithmetic in this section is in $\mathbb{Z}_n$. Note that $d^r = 1$ if r is even, and $d^r = d$ if r is odd.

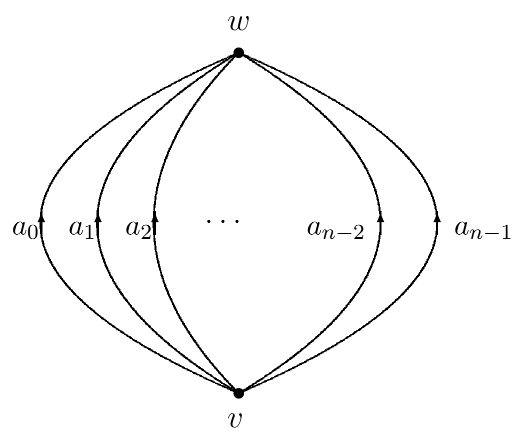

Figure 1: Voltage graph for the embedding B_n^*.

The face of B_n^* arising from the region in Figure 1 with voltage a_i on the left-hand edge has the face boundary (showing the neighbours of a typical vertex v_j)

$$\big(v_0 w_{(a_i)} v_{(-d^i)} w_{(a_i - d^i)} v_{(-2d^i)} w_{(a_i - 2d^i)} \cdots w_{(a_i + j + d^i)} v_j w_{(a_i + j)} \cdots w_{(a_{(i+1)})}\big).$$

If a vertex labelled u_{a_i} is placed in the interior of this face and is joined by nonintersecting edges to all the vertices on the boundary, then this face boundary becomes the rotation about the vertex u_{a_i}. We may then take the two Latin squares in the resulting embedding of $K_{n,n,n}$ to be given respectively by triples $(u_{a_i}, v_j, w_{(a_i+j)})$ and $(u_{a_i}, v_j, w_{(a_i+j+d^i)})$, where $i, j \in \mathbb{Z}_n$. Note that if i is even (respectively, odd) then a_i is even (respectively, odd). So, equivalently, and somewhat more conveniently, by renaming the row, column and entry labels, the Latin squares forming the embedding T_n^* may be taken as

$$L(i,j) = i + j \quad \text{and} \quad L'(i,j) = i + j + d^i = \begin{cases} i+j+1 & \text{if } i \text{ is even,} \\ i+j+d & \text{if } i \text{ is odd.} \end{cases}$$

The square L is just the cyclic Latin square of side n and the square L' lies in the same main class as L, being obtained from it by an obvious permutation of the rows.

In order to identify all bijections θ which form an automorphism of the biembedding of L with L', that is of the embedding T_n^*, we will think of θ as consisting of three bijections α, β and γ acting respectively on the row labels, column labels and entries of L and L'. We divide the analysis into a number of cases.

A1. Automorphisms that preserve the colour classes, the tripartition and the orientation.

A2. Automorphisms that preserve the colour classes, but cyclically (and non-trivially) permute the tripartition, thereby preserving the orientation.

A3. Automorphisms that preserve the colour classes but reverse the orientation by non-cyclically permuting the tripartition.

A4. Automorphisms that exchange the colour classes.

It is easy to see that any automorphism of the biembedding must be a composition of automorphisms of these types. For $i, j \in \mathbb{Z}_n$, it will be convenient to define

$$i * j = \begin{cases} i+j+1 & \text{if } i \text{ is even,} \\ i+j+d & \text{if } i \text{ is odd.} \end{cases}$$

Case A1.
Assume that there exists bijections $\alpha,\ \beta,\ \gamma$ of $\mathbb{Z}_n$ such that

$$(\alpha(i), \beta(j), \gamma(i+j)) \in L \quad \text{and} \quad (\alpha(i), \beta(j), \gamma(i*j)) \in L'.$$

Then

$$\begin{aligned} \alpha(i) + \beta(j) &= \gamma(i+j), & (1)\\ \alpha(i) * \beta(j) &= \gamma(i*j). & (2) \end{aligned}$$

Taking $i = 0$ in (1) implies that for all $j \in \mathbb{Z}_n$, $\beta(j) = \gamma(j) - \alpha(0)$. Similarly $j = 0$ implies that for all $i \in \mathbb{Z}_n$, $\alpha(i) = \gamma(i) - \beta(0)$. Thus there exist constants b and c such that for all $i \in \mathbb{Z}_n$,

$$\begin{aligned} \beta(i) &= \alpha(i) + b, & (3)\\ \gamma(i) &= \alpha(i) + c. & (4) \end{aligned}$$

Consequently (1) can be rewritten as

$$\alpha(i+j) = \alpha(i) + \alpha(j) + a, \qquad (5)$$

for some constant a. Taking $j = 1$ in (5) gives $\alpha(i+1) = \alpha(i) + e$, where $e = \alpha(1) + a$. Consequently, for all $i \in \mathbb{Z}_n$, $\alpha(i) = \alpha(0) + ie$. This implies that e is coprime with n, since otherwise α is not a bijection, and hence e must be odd. Substituting in (3) gives $\beta(i) = \alpha(i) + b = \alpha(0) + ie + b = \beta(0) + ie$, and similarly (4) gives $\gamma(i) = \gamma(0) + ie$.

Applying these results to (2), we see that for all $i, j \in \mathbb{Z}_n$,

$$(\alpha(0) + ie) * (\beta(0) + je) = \gamma(0) + (i*j)e. \qquad (6)$$

Hence there exists integers $r, s \in \{1, d\}$ such that

$$(\alpha(0) + ie) + (\beta(0) + je) + r = \gamma(0) + (i + j + s)e.$$

Since $\alpha(0) + \beta(0) = \gamma(0)$, it follows that $r = se$. The table given in Figure 2 gives the possible parities of relevant terms in (6) and calculates the corresponding values for r and s (E represents even parity and O represents odd parity). There are two cases depending on the parity of $\alpha(0)$. From the table it will be seen that if $\alpha(0)$ is even we have $e = 1$, while if $\alpha(0)$ is odd we have $e = d$.

$\alpha(0)$	i	$\alpha(0)+ie$	r	s	$r=se$
E	E	E	1	1	$1=e$
E	O	O	d	d	$d=de$
O	E	O	d	1	$d=e$
O	O	E	1	d	$1=de$

Figure 2: Summary of possible parities for $\alpha(0)$ and i in (6).

For $a, b \in \mathbb{Z}_n$ we now define

$$\phi_{a,b} : (i,j,k) \longrightarrow \begin{cases} (i+a, j+b, k+(a+b)) & \text{if } a \text{ is even,} \\ (di+a, dj+b, dk+(a+b)) & \text{if } a \text{ is odd.} \end{cases}$$

It is easy to verify that $\phi_{a,b}$ is simultaneously an automorphism of both L and L' and hence defines an automorphism of T_n^* that preserves the colour classes, the tripartition and the orientation. It follows that there are precisely n^2 automorphisms of type A1, and that these are given by $\{\phi_{a,b} : a, b \in \mathbb{Z}_n\}$.

Case A2.
Assume that there exists bijections $\alpha,\ \beta,\ \gamma$ of $\mathbb{Z}_n$ such that

$$(\gamma(i+j), \alpha(i), \beta(j)) \in L \quad \text{and} \quad (\gamma(i*j), \alpha(i), \beta(j)) \in L'.$$

Then

$$\beta(j) = \gamma(i+j) + \alpha(i), \tag{7}$$
$$\beta(j) = \gamma(i*j) * \alpha(i). \tag{8}$$

For $i = 0$, (7) implies that for all $j \in \mathbb{Z}_n$ $\beta(j) = \gamma(j) + \alpha(0)$. Similarly $j = 0$ implies that for all $i \in \mathbb{Z}_n$, $\gamma(i) = -\alpha(i) + \beta(0)$. Thus there exist constants b and c such that for all $i \in \mathbb{Z}_n$,

$$\beta(i) = -\alpha(i) + b,$$
$$\gamma(i) = -\alpha(i) + c.$$

Arguing as in case A1, we find that there exists a constant e coprime with n such that

$$\alpha(i) = \alpha(0) + ie, \quad \beta(i) = \beta(0) - ie, \quad \gamma(i) = \gamma(0) - ie.$$

From (8) we then deduce that for all $i, j \in \mathbb{Z}_n$,

$$(\gamma(0) - (i * j)e) * (\alpha(0) + ie) \quad = \quad \beta(0) - je. \tag{9}$$

Hence there exists integers $r, s \in \{1, d\}$ such that

$$(\gamma(0) - (i + j + s)e) + (\alpha(0) + ie) + r \quad = \quad \beta(0) - je.$$

Since $\gamma(0) + \alpha(0) = \beta(0)$, it follows that $r = se$.

We now examine the consequences of (9) when $i = 0$. In this case $s = 1$. If $j = 0$, then $r = 1$ if $\gamma(0) - se$ is even, and $r = d$ if $\gamma(0) - se$ is odd. Since e is odd, we have that $r = 1$ if $\gamma(0)$ is odd, and $r = d$ if $\gamma(0)$ is even. If $j = 1$, then $r = 1$ if $\gamma(0) - (1 + s)e$ is even, and $r = d$ if $\gamma(0) - (1 + s)e$ is odd. Hence $r = 1$ if $\gamma(0)$ is even, and $r = d$ if $\gamma(0)$ is odd. From this contradiction, we deduce that there are no automorphisms of the form $\theta : (i, j, k) \longrightarrow (\gamma(k), \alpha(i), \beta(j))$. If there were an automorphism of the form $\psi : (i, j, k) \longrightarrow (\beta(j), \gamma(k), \alpha(i))$, then ψ^2 would give an automorphism of the form θ just eliminated. We may therefore conclude that the embedding T_n^* has no automorphisms that preserve the colour classes, but cyclically (and non-trivially) permute the tripartition.

Case A3. We start by considering the mapping

$$\mu_0 : (i, j, k) \longrightarrow (i, -k, -j).$$

If $(i, j, k) \in L$ then $k = i + j$ and so $(i, -k, -j) = (i, -k, i + (-k)) \in L$. If $(i, j, k) \in L'$ then $k = i * j$, so if i is even we have $(i, -k, -j) = (i, -k, i - k + 1) = (i, -k, i * (-k)) \in L'$, while if i is odd we have $(i, -k, -j) = (i, -k, i - k + d) = (i, -k, i * (-k)) \in L'$. It follows that μ_0 defines an automorphism of T_n^* that preserves the colour classes but reverses the orientation by exchanging the second and third vertex parts. Note also that μ_0 has order 2.

If there were an automorphism that preserved the colour classes but exchanged the first and second or first and third vertex parts, then it could be combined with μ_0 to produce an automorphism of type A2, so no such automorphism can exist. On the other hand, μ_0 may be combined with each automorphism $\phi_{a,b}$ of type A1 to give n^2 distinct automorphisms of type A3. Now assume that θ is any automorphism of type A3. Then $\mu_0\theta$ is an automorphism of type A1, say $\phi_{a,b}$, so that $\theta = \mu_0\phi_{a,b}$. Hence there are precisely n^2 automorphisms of type A3.

Case A4. We start by considering the mapping

$$\nu_0 : (i,j,k) \longrightarrow \begin{cases} (4s-i,-j,d-k) & \text{if } i \text{ is even,} \\ (-i,-j,d-k) & \text{if } i \text{ is odd.} \end{cases}$$

Firstly assume that $(i,j,k) \in L$ so that $k = i + j$. Then, if i is even, $(4s-i,-j,d-k) = (4s-i,-j,(4s-i)+(-j)+1) = (4s-i,-j,(4s-i)*(-j)) \in L'$. On the other hand, if i is odd, $(-i,-j,d-k) = (-i,-j,(-i)+(-j)+d) = (-i,-j,(-i)*(-j)) \in L'$. Secondly assume that $(i,j,k) \in L'$ so that $k = i * j$. Then, if i is even, $(4s-i,-j,d-k) = (4s-i,-j,d-i-j-1) = (4s-i,-j,(4s-i)+(-j)) \in L$. On the other hand, if i is odd, $(-i,-j,d-k) = (-i,-j,d-i-j-d) = (-i,-j,(-i)+(-j)) \in L$. It follows that ν_0 defines an automorphism of T_n^* that exchanges the colour classes, but preserves the tripartition and consequently reverses the orientation. Note also that ν_0 has order 2.

By combining ν_0 with the n^2 automorphisms of type A1, we obtain n^2 distinct automorphisms of type A4 and, arguing as in case A3, we see that there are no further automorphisms that exchange the colour classes but preserve the tripartition. Similarly by combining ν_0 with the n^2 automorphisms of type A3, we obtain a further n^2 distinct automorphisms of type A4 and, again arguing as in case A3, we see that there are no further automorphisms that exchange the colour classes and the second and third vertex parts of the tripartition.

We now argue that there are no further automorphisms of type A4. The possibilities are for automorphisms that exchange the colour classes but also exchange the first and second, or first and third vertex parts, or cyclically permute the tripartition. If there were an automorphism θ of any one of these types, then it could be combined with either μ_0 or ν_0 to give an automorphism of a form already eliminated. Thus there are precisely $2n^2$ automorphisms of type A4.

We now state our conclusions in the following theorem.

Theorem 2.1 *The full automorphism group of the embedding T_n^* is generated by the mappings $\phi_{a,b}$ $(a, b \in \mathbb{Z}_n)$, μ_0 and ν_0, and the order of this group is $4n^2$.*

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