The smallest defining set of a Steiner triple system

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This note represents some thoughts about a recent paper by Ramsay [11]. Using similar techniques we are able to make extensions to his result in three different directions. The reader is referred to [11] for basic details, background and further references, but for the sake of completeness we recap on the definitions.

Given a t-(v, k, λ) design D, a *trade* T^+ is a collection of blocks of D which may be removed and replaced by a different collection of blocks T^- which collectively cover exactly the same t-subsets to the same multiplicities as T^+. A *defining set* dD is a collection of blocks of D which occurs in no other t-(v, k, λ) design on the same set of points. A defining set, no proper subset of which is also a defining set, is said to be *minimal* and is denoted by d_mD. A defining set for which no other has smaller cardinality is called a *smallest* defining set d_sD.

Ramsay focuses attention on the quantity $f = |d_sD|/b$, where b is the number of blocks in the design D. Thus f gives the proportion of blocks of a design in a smallest defining set. Quoting a result of Gray [7] that for every 2-$(v, k, 1)$ design D with $k > 2$, $|dD| \geq 2(v-1)/(k+1)$, it follows that for a Steiner triple system STS(v), $f \geq 3/v$. But $\lim_{v\to\infty} 3/v = 0$. Using the result of Adams, Billington and Rodger [3], see also [8] and [9], that for all $v \equiv 1, 9 \pmod{24}$, $v \geq 25$, there exists an STS(v) decomposable into Pasch configurations, the following result is obtained.

Theorem 1 *(Ramsay [11])*
For all $v \equiv 1, 9 \pmod{24}$, $v \geq 25$, there exists an STS(v) with $f \geq 1/4$. $\square$

The relationship between trades and defining sets is close. Every defining set of a design contains a block of every possible trade. Ramsay then makes the key observation, which he acknowledges is immediate, that if a t-(v, k, λ) design D contains m mutually disjoint trades then $|d_s D| \geq m$. It is this observation which gives Theorem 1. However, the idea can be pushed further. Suppose that D contains m mutually disjoint, but not necessarily isomorphic, trades $T_1^+, T_2^+, \ldots, T_m^+$. Suppose further that a smallest defining set for each trade T_i^+ (defined as a smallest set of blocks which occurs in no other trade covering the same pairs of points) contains d_i blocks. Then $f \geq \sum_{i=1}^{m} d_i/b$, where b is the number of blocks in the design. Using this observation we can show that Theorem 1 holds for all admissible v.

Theorem 2
For all $v \equiv 1, 3 \pmod 6$, $v \geq 7$, there exists an STS(v) with $f > 1/4$.
Proof
Let STS(u)= $(U, \mathcal{B}_u)$ be a Steiner triple system where U is the set of points and $\mathcal{B}_u$ is the set of blocks. Let $U' = \{x' : x \in U\}$. The proof is divided into four cases which are similar.
(i) $v = 12s + 7, \quad s \geq 0$.
Put $u = 6s + 3$ and take an STS(u) with a parallel class P (for example, a Kirkman triple system). Construct an STS(v)=$(V, \mathcal{B}_v)$ as follows. Let $V = U \cup U' \cup \{\infty\}$. If $\{a, b, c\} \in P$ then let $\{a, b, c\}, \{a, b', c'\}, \{a', b, c'\}, \{a', b', c\}$, $\{\infty, a, a'\}, \{\infty, b, b'\}, \{\infty, c, c'\} \in \mathcal{B}_v$. If $\{x, y, z\} \in \mathcal{B}_u \setminus P$ then let $\{x, y, z\}$, $\{x, y', z'\}, \{x', y, z'\}, \{x', y', z\} \in \mathcal{B}_v$. It is easily seen that we have a Steiner triple system STS(v) which is partitioned into Fano planes and Pasch configurations. Since the smallest defining set of a Fano plane is a *triangle* (three blocks isomorphic to $\{a, z, b\}, \{b, x, c\}, \{c, y, a\}$), it follows that

$$f \geq \frac{3u/3 + (u(u-1)/6 - u/3)}{(2u+1)2u/6} = \frac{u+3}{2(2u+1)} > \frac{1}{4}.$$

(ii) $v = 12s + 3, \quad s \geq 1$.
It is known [10] that the STS(15)=PG(3,2) has $f = 16/35$. Hence the result is true for $s = 1$. If $s \geq 2$ put $u = 6s + 1$ and take an STS(u) with an almost parallel class P (if $s = 2$, use the cyclic STS(13) and if $s \geq 3$ use an Hanani triple system of order $6s + 1$). Construct an STS(v) exactly as in case (i) but with an additional block $\{\infty, x, x'\}$ where $x \in U$ is the missing point from the almost parallel class P. Then

$$f \geq \frac{3(u-1)/3 + (u(u-1)/6 - (u-1)/3)}{(2u+1)2u/6} = \frac{(u-1)(u+4)}{2u(2u+1)} > \frac{1}{4}.$$

(iii)$v = 12s + 9, \quad s \geq 0$
Put $u = 6s + 3$ and take an STS(u) with a parallel class P. Construct an STS(v)= $(V, \mathcal{B}_v)$ as follows. Let $V = U \cup U' \cup \{A, B, C\}$. If $\{a, b, c\} \in P$ then let $\{a, b, c\}, \{A, a, a'\}, \{A, b, c'\}, \{A, b', c\}, \{B, b, b'\}, \{B, c, a'\}, \{B, c', a\}$, $\{C, c, c'\}, \{C, a, b'\}, \{C, a', b\}, \{a', b', c'\} \in \mathcal{B}_v$. If $\{x, y, z\} \in \mathcal{B}_u \setminus P$ then let $\{x, y, z\}, \{x, y', z'\}, \{x', y, z'\}, \{x', y', z\} \in \mathcal{B}_v$. Finally let $\{A, B, C\} \in \mathcal{B}_v$ and we have a Steiner triple system STS(v) which is partitioned into STS(9) configurations with a block removed, Pasch configurations, and the block $\{A, B, C\}$. Since the smallest defining set of the unique STS(9) consists of four blocks, it follows that

$$f \geq \frac{(4-1)u/3 + (u(u-1)/6 - u/3)}{(2u+3)(2u+2)/6} = \frac{u(u+3)}{2(u+1)(2u+3)} > \frac{1}{4},$$

provided that $u > 3$, i.e. $s > 0$. By considering the STS(9) it is seen that the result is also true for $s = 0$.
(iv)$v = 12s + 13, \quad s \geq 0$
It is known [12] that the cyclic STS(13) has $f = 9/26$. Hence the result is true for $s = 0$. If $s \geq 1$ put $u = 6s + 3$ and take an STS(u) with three disjoint parallel classes, P_1, P_2, P_3 (again, for example, a Kirkman triple system). Construct an STS(v)= $(V, \mathcal{B}_v)$ as follows. Let $V = U \cup U' \cup \{A_1, B_1, A_2, B_2, A_3, B_3, \infty\}$. If $\{a, b, c\} \in P_1$ then let $\{a, b, c\}, \{A_1, a, b'\}$, $\{A_1, b, c'\}, \{A_1, c, a'\}, \{B_1, a', b\}, \{B_1, b', c\}, \{B_1, c', a\}, \{\infty, a, a'\}, \{\infty, b, b'\}$, $\{\infty, c, c'\} \in \mathcal{B}_v$. Observe that again this is an STS(9) configuration with a block removed. For $i = 2, 3$ if $\{a, b, c\} \in P_i$ then let $\{a, b, c\}, \{A_i, a, b'\}$, $\{A_i, b, c'\}, \{A_i, c, a'\}, \{B_i, a', b\}, \{B_i, b', c\}, \{B_i, c', a\}, \{a', b', c'\} \in \mathcal{B}_v$. This is an STS(9) configuration with all blocks through one point removed and its smallest defining set consists of two blocks. If $\{x, y, z\} \in \mathcal{B}_u \setminus (P_1 \cup P_2 \cup P_3)$ then let $\{x, y, z\}, \{x, y', z'\}, \{x', y, z'\}, \{x', y', z\} \in \mathcal{B}_v$. Finally, let $\mathcal{B}_v$ contain the blocks of an STS(7) on the set $\{A_1, B_1, A_2, B_2, A_3, B_3, \infty\}$. The STS($v$) so constructed is partitioned into block-deleted STS(9)s, point-deleted STS(9)s, Pasch configurations, and a Fano configuration. We have therefore that

$$f \geq \frac{3u/3 + 2.2u/3 + (u(u-1)/6 - u) + 3}{(2u+7)(2u+6)/6} = \frac{u^2 + 7u + 18}{2(u+3)(2u+7)} > \frac{1}{4}.$$

□

The second extension is to show that there exist classes of STS(v) for which the inequality for f can be strengthened. Let D be a 2-$(v, 7, 1)$ design. Such designs are known to exist for all $v \equiv 1, 7 \pmod{42}$, $v \geq 3493$ (and many values $v < 3493$) [2]. Now replace each block by an STS(7) on the seven points of the block. We obtain an STS(v) which is decomposable into Fano planes. As was used in the proof of Theorem 2, the smallest defining set for a Fano plane consists of three blocks. It immediately follows therefore that for Steiner triple systems constructed in this manner $f \geq 3/7$. We state this formally.

Theorem 3
For all $v \equiv 1, 7 \pmod{42}$, $v \geq 3493$, there exists an STS(v) with $f \geq 3/7$.
□

Our next results come from the following Theorem which relates the proportion of blocks in a smallest defining set of a product Steiner triple system STS(uv) to the numbers of blocks in smallest defining sets of the component systems STS(u) and STS(v).

Theorem 4
Let STS(u) and STS(v) be Steiner triple systems having smallest defining sets d_sD_u and d_sD_v respectively. Let $s_u = |d_sD_u|$, $s_v = |d_sD_v|$ and $\sigma_{uv} = us_v + vs_u + \max\{u(u-1)s_v, v(v-1)s_u\}$. Then the Steiner triple system STS(uv) formed as the product of the systems STS(u) and STS(v) has $f \geq 6\sigma_{uv}/uv(uv-1)$.

Proof
The blocks of the STS(uv) can be partitioned as follows.

(a) $uv(v-1)/6$ blocks which comprise u copies of the STS(v).
The number of these blocks in a smallest defining set is at least us_v.

(b) $vu(u-1)/6$ blocks which comprise v copies of the STS(u).
The number of these blocks in a smallest defining set is at least vs_u.

(c) $uv(u-1)(v-1)/6$ blocks which can be resolved as either

  (i) $u(u-1)$ copies of the STS(v), with the number of blocks in a smallest defining set at least $u(u-1)s_v$, or

  (ii) $v(v-1)$ copies of the STS(u), with the number of blocks in a smallest defining set at least $v(v-1)s_v$.

Thus the number of blocks in a smallest defining set of the STS(uv) is at least $us_v + vs_u + \max\{u(u-1)s_v, v(v-1)s_u\} = \sigma_{uv}$. Since the number of blocks in an STS(uv) is $uv(uv-1)/6$, the result follows. □

If we now define $\overline{f_v}$ to be the maximum value of f for all STS(v), we may state the following corollaries.

Corollary 4.1 *For each $n \geq 1$, $\overline{f_{v^n}} \geq \overline{f_v}$.*
Proof The result follows from the preceding Theorem by an easy induction argument. □

Corollary 4.2 *For admissible u and v, $\overline{f_{uv}} \geq (1 - 1/v)\overline{f_v}$.*
Proof Using the notation of the Theorem, for any STS(u) and STS(v), $\sigma_{uv} \geq us_v + u(u-1)s_v = u^2 s_v$. Therefore,

$$\overline{f_{uv}} \geq \frac{6u^2 s_v}{uv(uv-1)}.$$

By taking an STS(v) with the maximum value of s_v (so that $\overline{f_v} = 6s_v/v(v-1)$), we obtain

$$\overline{f_{uv}} \geq \left(\frac{u(v-1)}{uv-1}\right)\overline{f_v}.$$

However, it is easily seen that $u(v-1)/(uv-1) \geq (1 - 1/v)$ and the result follows.
□

These two Corollaries provide further infinite classes of Steiner triple systems with $f > 1/4$. It is known [12] that $\overline{f_7} = 3/7$, $\overline{f_9} = 4/12 = 1/3$, $\overline{f_{13}} = 9/26$ and $\overline{f_{15}} = 16/35$. Thus, for example, the first Corollary gives $\overline{f_v} \geq 16/35$ for $v = 15^n$. Then applying the second Corollary with $v = 15^n$ gives $\overline{f_{15^n u}} \geq \frac{16}{35}(1 - \frac{1}{15^n})$ for $n \geq 1$. This latter result yields infinite linear classes. Again, as an example, with $n = 1$ it follows that for all $v \equiv 15, 45 \pmod{90}$ there exists an STS(v) with $f \geq (16/35)(14/15) = 32/75$. Thus the second Corollary hints at an asymptotic result for $\overline{f_v}$ as $v \to \infty$, and this is our third extension. We will prove that, if $F = \sup\{\overline{f_v} :$ v is admissible$\}$, then $\overline{f_v} \to F$ as $v \to \infty$. For this purpose we first prove the following Theorem.

Theorem 5
Suppose that $u = l^k v + w$ where l, w and $v + w$ are admissible, $v \geq w + 1$ and there exists a transversal design TD(l^k, v). Then

$$\overline{f_u} \geq \left(\frac{(u-w)(u-w-v)}{u(u-1)}\right)\overline{f_l}.$$

Proof
Since $v \geq w+1$, by the Doyen-Wilson Theorem [4] there exists an STS($v + w$) containing an STS(w) as a subsystem. We now take l^k copies of this STS($v + w$) intersecting in a common STS(w) subsystem; we may take the points of the i^{th} copy to be

$$1, 2, \ldots, w, 1_i, 2_i, \ldots, v_i.$$

Altogether there are $l^k v + w = u$ points and we may form an STS(u) on these points by taking as blocks all the blocks of all the STS($v + w$)s (the *horizontal* blocks) together with certain other blocks which we describe below (the *vertical* blocks). The horizontal blocks cover all pairs of the forms $\{a,b\},\{a,c_i\},\{c_i,d_i\}$ for $a,b=1,2,\ldots,w,\quad c,d=1,2,\ldots,v,\quad i=1,2,\ldots,l^k,\quad a\neq b$ and $c\neq d$. The vertical blocks must cover every pair of the form $\{c_i,d_j\}$ for $c,d=1,2,\ldots,v,\quad i,j=1,2,\ldots,l^k$ and $i\neq j$. To form the vertical blocks we take a TD(l^k,v) with groups $\{1_i,2_i,\ldots,v_i\}$ for $i=1,2,\ldots,l^k$. We then replace each block of size l^k with an STS(l^k) (on the same points) having the maximum value of f, namely $\overline{f_{l^k}}$. By Corollary 1 we have $\overline{f_{l^k}}\geq\overline{f_l}$. There are v^2 blocks in the TD and $l^k(l^k-1)/6$ blocks in each STS(l^k).

Ignoring contributions from the horizontal blocks, we have

$$\overline{f_u}\geq\left\{\frac{v^2l^k(l^k-1)}{6}\overline{f_l}\right\}\Big/\left\{\frac{u(u-1)}{6}\right\}=\frac{(u-w)(u-w-v)}{u(u-1)}\overline{f_l}.$$

□

Theorem 6
For any admissible $l\geq3$ and any $\epsilon>0$, there exists U_0 such that for all admissible $u>U_0$, $\quad\overline{f_u}>\overline{f_l}-\epsilon$.

Proof
There exists v_0 such that for all $v>v_0$, the number of MOLS (mutually orthogonal latin squares) of side v, say $N(v)$, satisfies $N(v)\geq v^{\frac{1}{14.8}}$ [1]. Hence, for $v>v_0$ and $m\leq v^{\frac{1}{14.8}}$, there exists a transversal design TD(m,v) [1]. We will assume that v_0 is so large that

$$14.8\frac{\log(v_0+4)}{\log v_0}<15.$$

Take $u\geq\max\{(v_0+4)^{\frac{16}{15}},l^{16}\}$ and admissible. Define $k=\lfloor(\log_l u)/16\rfloor$ so that $1\leq k\leq(\log_l u)/16<k+1$, and $l^{16k}\leq u<l^{16(k+1)}$. We may write $u=\sum_{i=0}^n u_il^i$, where $0\leq u_i<l$ and $u_n\neq0$. Since l is admissible, $l\equiv1$ or $3 \pmod 6$.

Next we will choose $\alpha\in\{0,1,2,3\}$ and define

$$\begin{aligned} v &= u_nl^{n-k}+u_{n-1}l^{n-k-1}+\cdots+u_k-\alpha,\\ w &= \alpha l^k+u_{k-1}l^{k-1}+u_{k-2}l^{k-2}+\cdots+u_0,\end{aligned}$$

so that $u=l^kv+w$. We specify the choice of α as follows to ensure that both w and $v+w$ are admissible.

Case (i) $\quad l\equiv1 \pmod 6$.

Note $w\equiv\alpha+u_{k-1}+u_{k-2}+\cdots+u_0 \pmod 6$, so simply choose α to ensure that w is admissible. Then observe that

$$v+w\equiv(u_n+u_{n-1}+\cdots+u_k-\alpha)+(\alpha+u_{k-1}+u_{k-2}+\cdots+u_0)\equiv u \pmod 6,$$

so that $v+w$ is admissible.

Case (ii) $l \equiv 3 \pmod 6$.

We have $w \equiv 3(\alpha + u_{k-1} + u_{k-2} + \cdots + u_1) + u_0 \pmod 6$. Note that $u_0 \not\equiv 2$ or $5 \pmod 6$ because $u \equiv 3(u_n + u_{n-1} + \cdots + u_1) + u_0$ (mod 6) is admissible. Hence we may first select the parity of α (odd or even) to ensure that w is admissible.
We also have

$$\begin{aligned} v + w &\equiv 3(u_n + u_{n-1} + \cdots + u_{k+1}) + (u_k - \alpha) \\ &\qquad + 3(\alpha + u_{k-1} + u_{k-2} + \cdots + u_1) + u_0 \\ &\equiv u + 2(\alpha - u_k) \pmod 6 \end{aligned}$$

If $u \equiv 1 \pmod 6$ then select α (with the previously chosen parity) as per Table 1 below.

$u_k \pmod 3$	0	1	2
α (even parity)	0	2	2
α (odd parity)	1	1	3

Table 1.

If $u \equiv 3 \pmod 6$ then select α (with the previously chosen parity) as per Table 2 below.

$u_k \pmod 3$	0	1	2
α (even parity)	0	0	2
α (odd parity)	3	1	1

Table 2.

For either residue class for u, w and $v + w$ are then both admissible.

By our choice of α, we have $0 \le w < 4.l^k$. Hence $l^k(v+4) > l^k v + w = u$ and so $v+4 > ul^{-k} \ge u^{\frac{15}{16}} \ge v_0+4$, giving $v > v_0$. Also, $l^k \le u^{\frac{1}{16}} < (l^k(v+4))^{\frac{1}{16}}$ and so $l^k < (v+4)^{\frac{1}{15}}$. But $v > v_0$ and so $(v+4)^{\frac{1}{15}} < v^{\frac{1}{14.8}}$, giving $l^k < v^{\frac{1}{14.8}}$. It follows that there is a TD(l^k, v).

Since $v+4 > u^{\frac{15}{16}}$, $w < 4.l^k \le 4.u^{\frac{1}{16}}$ and $u \ge l^{16}$, we have $v > u^{\frac{15}{16}} - 4 > 4u^{\frac{1}{16}} + 1 > w + 1$.

From the previous Theorem we now have

$$\overline{f_u} \ge \left(\frac{(u-w)(u-w-v)}{u(u-1)} \right) \overline{f_l}.$$

But $0 \le w < v$ and $0 < v \le ul^{-k} < u(lu^{-\frac{1}{16}}) = lu^{\frac{15}{16}}$.

$$\text{Hence} \quad \frac{(u-w)(u-w-v)}{u(u-1)} \to 1 \quad \text{as} \quad u \to \infty.$$

Consequently, for any $\epsilon > 0$ there exists U_0 such that for all admissible $u > U_0$, $\overline{f_u} > \overline{f_l} - \epsilon$. □

Corollary 6.1 *If* $F = \sup\{\overline{f_v} : v$ *is admissible*$\}$*, then* $\overline{f_v} \to F$ *as* $v \to \infty$*.* □

The precise value of F is unknown and may be difficult to determine. Gower [5] found for each $n \geq 2$ a *minimal* defining set for the point-line design of PG(n,2), the ratio of the cardinality of which to the number of lines in PG(n,2) has a limiting value of one. We conjecture that $F = 1$. The largest value for f so far established is $f = 16/35$ for the Steiner triple system formed by the point-line design of PG(3,2) [11, 12], and thus we may also state the following.

Corollary 6.2 *For every* $\epsilon > 0$ *there exists* U_0 *such that for all* $u > U_0$*,* $\overline{f_u} > \frac{16}{35} - \epsilon$*.* □

Discovery of any single Steiner triple system having a larger value for f would immediately improve this result, as indeed would an infinite family whose values for f monotonically increase to a limit greater than 16/35. It is known [6] that the values for f of the point-line design of PG(n,2) are non-decreasing as n increases, but it is not known whether they are strictly increasing, nor the limit of this sequence.

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